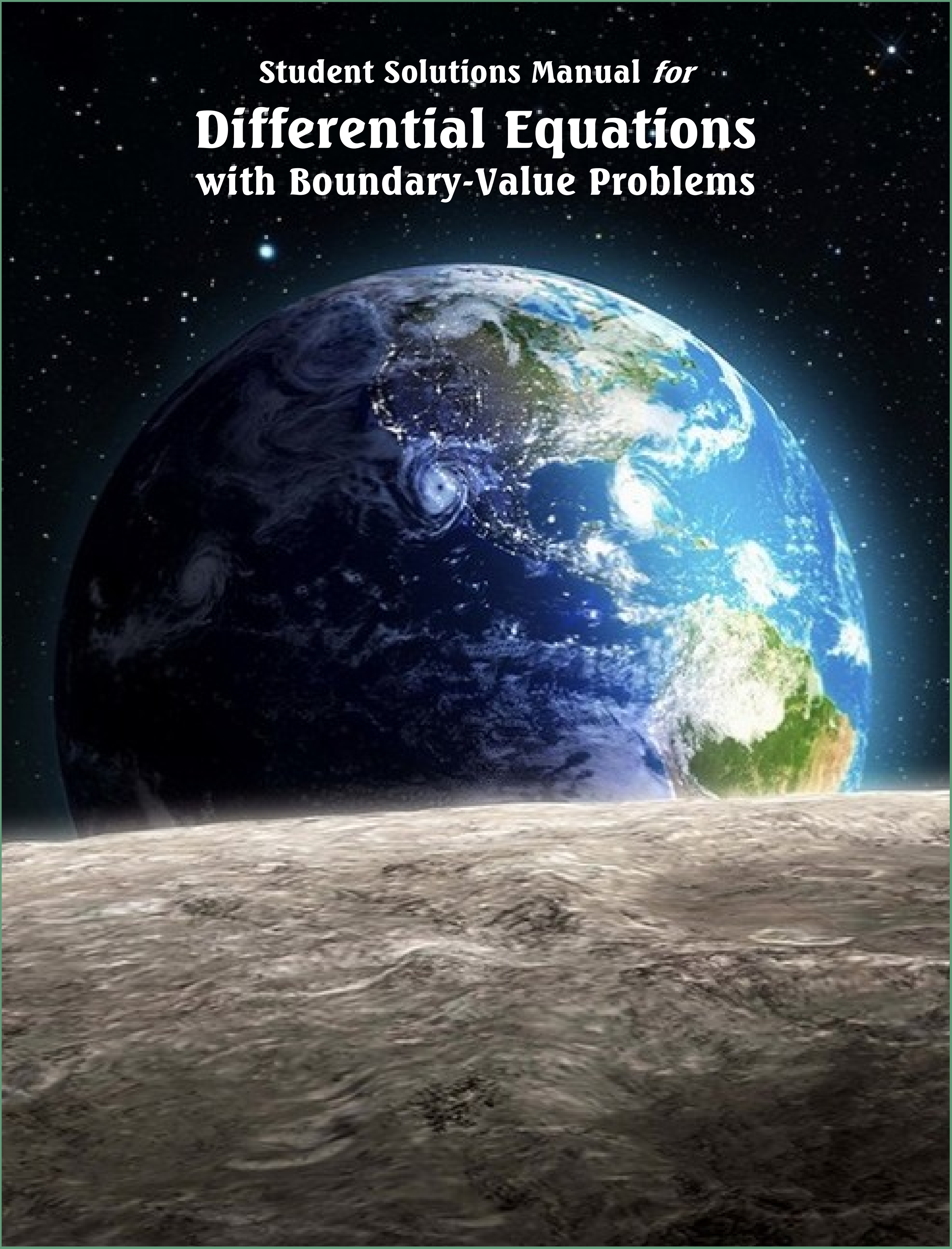
Student Solutions Manual *for*

# Differential Equations
## with Boundary-Value Problems



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# Chapter 1

# Introduction to Differential Equations

## 1.1 Definitions and Terminology

**1.** Second order; linear

**2.** Third order; nonlinear because of $(dy/dx)^4$

**3.** Fourth order; linear

**4.** Second order; nonlinear because of $\cos(r+u)$

**5.** Second order; nonlinear because of $(dy/dx)^2$ or $\sqrt{1+(dy/dx)^2}$

**6.** Second order; nonlinear because of $R^2$

**7.** Third order; linear

**8.** Second order; nonlinear because of $\dot{x}^2$

**9.** Writing the differential equation in the form $x(dy/dx)+y^2=1$, we see that it is nonlinear in $y$ because of $y^2$. However, writing it in the form $(y^2-1)(dx/dy)+x=0$, we see that it is linear in $x$.

**10.** Writing the differential equation in the form $u(dv/du)+(1+u)v=ue^u$ we see that it is linear in $v$. However, writing it in the form $(v+uv-ue^u)(du/dv)+u=0$, we see that it is nonlinear in $u$.

**11.** From $y=e^{-x/2}$ we obtain $y'=-\frac{1}{2}e^{-x/2}$. Then $2y'+y=-e^{-x/2}+e^{-x/2}=0$.

**12.** From $y=\frac{6}{5}-\frac{6}{5}e^{-20t}$ we obtain $dy/dt=24e^{-20t}$, so that

$$\frac{dy}{dt}+20y=24e^{-20t}+20\left(\frac{6}{5}-\frac{6}{5}e^{-20t}\right)=24.$$

**13.** From $y=e^{3x}\cos 2x$ we obtain $y'=3e^{3x}\cos 2x-2e^{3x}\sin 2x$ and $y''=5e^{3x}\cos 2x-12e^{3x}\sin 2x$, so that $y''-6y'+13y=0$.

**14.** From $y = -\cos x \ln(\sec x + \tan x)$ we obtain $y' = -1 + \sin x \ln(\sec x + \tan x)$ and $y'' = \tan x + \cos x \ln(\sec x + \tan x)$. Then $y'' + y = \tan x$.

**15.** The domain of the function, found by solving $x+2 \geq 0$, is $[-2, \infty)$. From $y' = 1+2(x+2)^{-1/2}$ we have

$$\begin{aligned}(y-x)y' &= (y-x)[1+(2(x+2)^{-1/2}]\\ &= y - x + 2(y-x)(x+2)^{-1/2}\\ &= y - x + 2[x + 4(x+2)^{1/2} - x](x+2)^{-1/2}\\ &= y - x + 8(x+2)^{1/2}(x+2)^{-1/2} = y - x + 8.\end{aligned}$$

An interval of definition for the solution of the differential equation is $(-2, \infty)$ because $y'$ is not defined at $x = -2$.

**16.** Since $\tan x$ is not defined for $x = \pi/2 + n\pi$, $n$ an integer, the domain of $y = 5\tan 5x$ is $\{x \mid 5x \neq \pi/2 + n\pi\}$
or $\{x \mid x \neq \pi/10 + n\pi/5\}$. From $y' = 25\sec^2 5x$ we have

$$y' = 25(1 + \tan^2 5x) = 25 + 25\tan^2 5x = 25 + y^2.$$

An interval of definition for the solution of the differential equation is $(-\pi/10, \pi/10)$. Another interval is $(\pi/10, 3\pi/10)$, and so on.

**17.** The domain of the function is $\{x \mid 4 - x^2 \neq 0\}$ or $\{x \mid x \neq -2 \text{ and } x \neq 2\}$. From $y' = 2x/(4-x^2)^2$ we have

$$y' = 2x\left(\frac{1}{4-x^2}\right)^2 = 2xy^2.$$

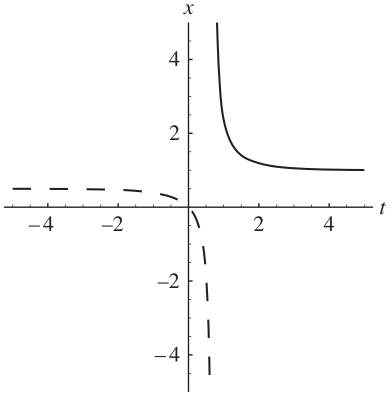
An interval of definition for the solution of the differential equation is $(-2, 2)$. Other intervals are $(-\infty, -2)$ and $(2, \infty)$.

**18.** The function is $y = 1/\sqrt{1 - \sin x}$, whose domain is obtained from $1 - \sin x \neq 0$ or $\sin x \neq 1$. Thus, the domain is $\{x \mid x \neq \pi/2 + 2n\pi\}$. From $y' = -\frac{1}{2}(1-\sin x)^{-3/2}(-\cos x)$ we have

$$2y' = (1 - \sin x)^{-3/2}\cos x = [(1-\sin x)^{-1/2}]^3 \cos x = y^3 \cos x.$$

An interval of definition for the solution of the differential equation is $(\pi/2, 5\pi/2)$. Another one is $(5\pi/2, 9\pi/2)$, and so on.

**19.** Writing $\ln(2X-1)-\ln(X-1)=t$ and differentiating implicitly we obtain



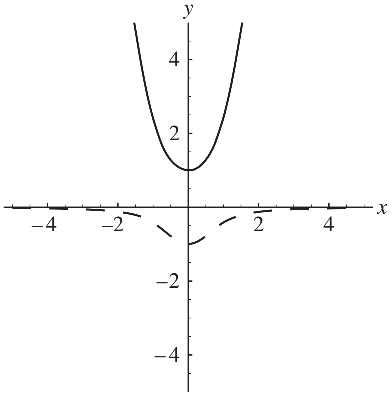
$$\frac{2}{2X-1}\frac{dX}{dt}-\frac{1}{X-1}\frac{dX}{dt}=1$$
$$\left(\frac{2}{2X-1}-\frac{1}{X-1}\right)\frac{dX}{dt}=1$$
$$\frac{2X-2-2X+1}{(2X-1)(X-1)}\frac{dX}{dt}=1$$
$$\frac{dX}{dt}=-(2X-1)(X-1)=(X-1)(1-2X).$$

Exponentiating both sides of the implicit solution we obtain

$$\frac{2X-1}{X-1}=e^t$$
$$2X-1=Xe^t-e^t$$
$$(e^t-1)=(e^t-2)X$$
$$X=\frac{e^t-1}{e^t-2}.$$

Solving $e^t-2=0$ we get $t=\ln 2$. Thus, the solution is defined on $(-\infty,\ln 2)$ or on $(\ln 2,\infty)$. The graph of the solution defined on $(-\infty,\ln 2)$ is dashed, and the graph of the solution defined on $(\ln 2,\infty)$ is solid.

**20.** Implicitly differentiating the solution, we obtain



$$-2x^2\frac{dy}{dx}-4xy+2y\frac{dy}{dx}=0$$
$$-x^2\,dy-2xy\,dx+y\,dy=0$$
$$2xy\,dx+(x^2-y)dy=0.$$

Using the quadratic formula to solve $y^2-2x^2y-1=0$ for $y$, we get $y=\left(2x^2\pm\sqrt{4x^4+4}\right)/2=x^2\pm\sqrt{x^4+1}$. Thus, two explicit solutions are $y_1=x^2+\sqrt{x^4+1}$ and $y_2=x^2-\sqrt{x^4+1}$. Both solutions are defined on $(-\infty,\infty)$. The graph of $y_1(x)$ is solid and the graph of $y_2$ is dashed.

**21.** Differentiating $P = c_1e^t/\left(1 + c_1e^t\right)$ we obtain

$$\frac{dP}{dt} = \frac{\left(1 + c_1e^t\right)c_1e^t - c_1e^t \cdot c_1e^t}{\left(1 + c_1e^t\right)^2} = \frac{c_1e^t}{1 + c_1e^t}\,\frac{\left[\left(1 + c_1e^t\right) - c_1e^t\right]}{1 + c_1e^t}$$

$$= \frac{c_1e^t}{1 + c_1e^t}\left[1 - \frac{c_1e^t}{1 + c_1e^t}\right] = P(1 - P).$$

**22.** Differentiating $y = 2x^2 - 1 + c_1e^{-2x^2}$ we obtain $\dfrac{dy}{dx} = 4x - 4xc_1e^{-2x^2}$, so that

$$\frac{dy}{dx} + 4xy = 4x - 4xc_1e^{-2x^2} + 8x^3 - 4x + 4c_1xe^{-x^2} = 8x^3$$

**23.** From $y = c_1e^{2x} + c_2xe^{2x}$ we obtain $\dfrac{dy}{dx} = (2c_1 + c_2)e^{2x} + 2c_2xe^{2x}$ and $\dfrac{d^2y}{dx^2} = (4c_1 + 4c_2)e^{2x} + 4c_2xe^{2x}$, so that

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = (4c_1 + 4c_2 - 8c_1 - 4c_2 + 4c_1)e^{2x} + (4c_2 - 8c_2 + 4c_2)xe^{2x} = 0.$$

**24.** From $y = c_1x^{-1} + c_2x + c_3x\ln x + 4x^2$ we obtain

$$\frac{dy}{dx} = -c_1x^{-2} + c_2 + c_3 + c_3\ln x + 8x,$$

$$\frac{d^2y}{dx^2} = 2c_1x^{-3} + c_3x^{-1} + 8,$$

and

$$\frac{d^3y}{dx^3} = -6c_1x^{-4} - c_3x^{-2},$$

so that

$$\begin{aligned} x^3\frac{d^3y}{dx^3} + 2x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y &= (-6c_1 + 4c_1 + c_1 + c_1)x^{-1} + (-c_3 + 2c_3 - c_2 - c_3 + c_2)x \\ &\quad + (-c_3 + c_3)x\ln x + (16 - 8 + 4)x^2 \\ &= 12x^2 \end{aligned}$$

*In Problems 25–28, we use the Product Rule and the derivative of an integral ((12) of this section):* $\dfrac{d}{dx}\displaystyle\int_a^x g(t)\,dt = g(x)$*.*

**25.** Differentiating $y = e^{3x}\displaystyle\int_1^x \frac{e^{-3t}}{t}\,dt$ we obtain $\dfrac{dy}{dx} = 3e^{3x}\displaystyle\int_1^x \frac{e^{-3t}}{t}\,dt + \frac{e^{-3x}}{x}\cdot e^{3x}$ or $\dfrac{dy}{dx} = 3e^{3x}\displaystyle\int_1^x \frac{e^{-3t}}{t}\,dt + \frac{1}{x}$, so that

$$\begin{aligned} x\frac{dy}{dx} - 3xy &= x\left(3e^{3x}\int_1^x \frac{e^{-3t}}{t}\,dt + \frac{1}{x}\right) - 3x\left(e^{3x}\int_1^x \frac{e^{-3t}}{t}\,dt\right) \\ &= 3xe^{3x}\int_1^x \frac{e^{-3t}}{t}\,dt + 1 - 3xe^{3x}\int_1^x \frac{e^{-3t}}{t}\,dt = 1 \end{aligned}$$

**26.** Differentiating $y = \sqrt{x}\int_4^x \frac{\cos t}{\sqrt{t}}\,dt$ we obtain $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}\int_4^x \frac{\cos t}{\sqrt{t}}\,dt + \frac{\cos x}{\sqrt{x}}\cdot\sqrt{x}$ or $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}\int_4^x \frac{\cos t}{\sqrt{t}}\,dt + \cos x$, so that

$$\begin{aligned} 2x\frac{dy}{dx} - y &= 2x\left(\frac{1}{2\sqrt{x}}\int_4^x \frac{\cos t}{\sqrt{t}}\,dt + \cos x\right) - \sqrt{x}\int_4^x \frac{\cos t}{\sqrt{t}}\,dt \\ &= \sqrt{x}\int_4^x \frac{\cos t}{\sqrt{t}}\,dt + 2x\cos x - \sqrt{x}\int_4^x \frac{\cos t}{\sqrt{t}}\,dt = 2x\cos x \end{aligned}$$

**27.** Differentiating $y = \frac{5}{x} + \frac{10}{x}\int_1^x \frac{\sin t}{t}\,dt$ we obtain $\frac{dy}{dx} = -\frac{5}{x^2} - \frac{10}{x^2}\int_1^x \frac{\sin t}{t}\,dt + \frac{\sin x}{x}\cdot\frac{10}{x}$ or $\frac{dy}{dx} = -\frac{5}{x^2} - \frac{10}{x^2}\int_1^x \frac{\sin t}{t}\,dt + \frac{10\sin x}{x^2}$, so that

$$\begin{aligned} x^2\frac{dy}{dx} + xy &= x^2\left(-\frac{5}{x^2} - \frac{10}{x^2}\int_1^x \frac{\sin t}{t}\,dt + \frac{10\sin x}{x^2}\right) + x\left(\frac{5}{x} + \frac{10}{x}\int_1^x \frac{\sin t}{t}\,dt\right) \\ &= -5 - 10\int_1^x \frac{\sin t}{t}\,dt + 10\sin x + 5 + 10\int_1^x \frac{\sin t}{t}\,dt = 10\sin x \end{aligned}$$

**28.** Differentiating $y = e^{-x^2} + e^{-x^2}\int_0^x e^{t^2}\,dt$ we obtain $\frac{dy}{dx} = -2xe^{-x^2} - 2xe^{-x^2}\int_0^x e^{t^2}\,dt + e^{x^2}\cdot e^{-x^2}$ or $\frac{dy}{dx} = -2xe^{-x^2} - 2xe^{-x^2}\int_0^x e^{t^2}\,dt + 1$, so that

$$\begin{aligned} \frac{dy}{dx} + 2xy &= \left(-2xe^{-x^2} - 2xe^{-x^2}\int_0^x e^{t^2}\,dt + 1\right) + 2x\left(e^{-x^2} + e^{-x^2}\int_0^x e^{t^2}\,dt\right) \\ &= -2xe^{-x^2} - 2xe^{-x^2}\int_0^x e^{t^2}\,dt + 1 + 2xe^{-x^2} + 2xe^{-x^2}\int_0^x e^{t^2}\,dt = 1 \end{aligned}$$

**29.** From

$$y = \begin{cases} -x^2, & x < 0 \\ x^2, & x \ge 0 \end{cases}$$

we obtain

$$y' = \begin{cases} -2x, & x < 0 \\ 2x, & x \ge 0 \end{cases}$$

so that $xy' - 2y = 0$.

**30.** The function $y(x)$ is not continuous at $x = 0$ since $\lim_{x\to 0^-} y(x) = 5$ and $\lim_{x\to 0^+} y(x) = -5$. Thus, $y'(x)$ does not exist at $x = 0$.

**31.** Substitute the function $y = e^{mx}$ into the equation $y' + 2y = 0$ to get

$$(e^{mx})' + 2(e^{mx}) = 0$$

$$me^{mx} + 2e^{mx} = 0$$

$$e^{mx}(m+2) = 0$$

Now since $e^{mx} > 0$ for all values of $x$, we must have $m = -2$ and so $y = e^{-2x}$ is a solution.

**32.** Substitute the function $y = e^{mx}$ into the equation $5y' - 2y = 0$ to get

$$\begin{aligned} 5(e^{mx})' - 2(e^{mx}) &= 0 \\ 5me^{mx} - 2e^{mx} &= 0 \\ e^{mx}(5m - 2) &= 0 \end{aligned}$$

Now since $e^{mx} > 0$ for all values of $x$, we must have $m = 2/5$ and so $y = e^{2x/5}$ is a solution.

**33.** Substitute the function $y = e^{mx}$ into the equation $y'' - 5y' + 6y = 0$ to get

$$\begin{aligned} (e^{mx})'' - 5(e^{mx})' + 6(e^{mx}) &= 0 \\ m^2e^{mx} - 5me^{mx} + 6e^{mx} &= 0 \\ e^{mx}(m^2 - 5m + 6) &= 0 \\ e^{mx}(m - 2)(m - 3) &= 0 \end{aligned}$$

Now since $e^{mx} > 0$ for all values of $x$, we must have $m = 2$ or $m = 3$ therefore $y = e^{2x}$ and $y = e^{3x}$ are solutions.

**34.** Substitute the function $y = e^{mx}$ into the equation $2y'' + 7y' - 4y = 0$ to get

$$\begin{aligned} 2(e^{mx})'' + 7(e^{mx})' - 4(e^{mx}) &= 0 \\ 2m^2e^{mx} + 7me^{mx} - 4e^{mx} &= 0 \\ e^{mx}(2m^2 + 7m - 4) &= 0 \\ e^{mx}(m + 4)(2m - 1) &= 0 \end{aligned}$$

Now since $e^{mx} > 0$ for all values of $x$ , we must have $m = -4$ or $m = 1/2$ therefore $y = e^{-4x}$ and $y = e^{x/2}$ are solutions.

**35.** Substitute the function $y = x^m$ into the equation $xy'' + 2y' = 0$ to get

$$\begin{aligned} x \cdot (x^m)'' + 2(x^m)' &= 0 \\ x \cdot m(m - 1)x^{m-2} + 2mx^{m-1} &= 0 \\ (m^2 - m)x^{m-1} + 2mx^{m-1} &= 0 \\ x^{m-1}[m^2 + m] &= 0 \\ x^{m-1}[m(m + 1)] &= 0 \end{aligned}$$

The last line implies that $m = 0$ or $m = -1$ therefore $y = x^0 = 1$ and $y = x^{-1}$ are solutions.

**36.** Substitute the function $y = x^m$ into the equation $x^2y'' - 7xy' + 15y = 0$ to get

$$\begin{aligned} x^2 \cdot (x^m)'' - 7x \cdot (x^m)' + 15(x^m) &= 0 \\ x^2 \cdot m(m-1)x^{m-2} - 7x \cdot mx^{m-1} + 15x^m &= 0 \\ (m^2 - m)x^m - 7mx^m + 15x^m &= 0 \\ x^m[m^2 - 8m + 15] &= 0 \\ x^m[(m-3)(m-5)] &= 0 \end{aligned}$$

The last line implies that $m = 3$ or $m = 5$ therefore $y = x^3$ and $y = x^5$ are solutions.

*In Problems 37–40, we substitute $y = c$ into the differential equations and use $y' = 0$ and $y'' = 0$*

**37.** Solving $5c = 10$ we see that $y = 2$ is a constant solution.

**38.** Solving $c^2 + 2c - 3 = (c+3)(c-1) = 0$ we see that $y = -3$ and $y = 1$ are constant solutions.

**39.** Since $1/(c-1) = 0$ has no solutions, the differential equation has no constant solutions.

**40.** Solving $6c = 10$ we see that $y = 5/3$ is a constant solution.

**41.** From $x = e^{-2t} + 3e^{6t}$ and $y = -e^{-2t} + 5e^{6t}$ we obtain

$$\frac{dx}{dt} = -2e^{-2t} + 18e^{6t} \quad \text{and} \quad \frac{dy}{dt} = 2e^{-2t} + 30e^{6t}.$$

Then

$$x + 3y = (e^{-2t} + 3e^{6t}) + 3(-e^{-2t} + 5e^{6t}) = -2e^{-2t} + 18e^{6t} = \frac{dx}{dt}$$

and

$$5x + 3y = 5(e^{-2t} + 3e^{6t}) + 3(-e^{-2t} + 5e^{6t}) = 2e^{-2t} + 30e^{6t} = \frac{dy}{dt}\,.$$

**42.** From $x = \cos 2t + \sin 2t + \frac{1}{5}e^t$ and $y = -\cos 2t - \sin 2t - \frac{1}{5}e^t$ we obtain

$$\frac{dx}{dt} = -2\sin 2t + 2\cos 2t + \frac{1}{5}e^t \quad \text{and} \quad \frac{dy}{dt} = 2\sin 2t - 2\cos 2t - \frac{1}{5}e^t$$

and

$$\frac{d^2x}{dt^2} = -4\cos 2t - 4\sin 2t + \frac{1}{5}e^t \quad \text{and} \quad \frac{d^2y}{dt^2} = 4\cos 2t + 4\sin 2t - \frac{1}{5}e^t.$$

Then

$$4y + e^t = 4(-\cos 2t - \sin 2t - \frac{1}{5}e^t) + e^t = -4\cos 2t - 4\sin 2t + \frac{1}{5}e^t = \frac{d^2x}{dt^2}$$

and

$$4x - e^t = 4(\cos 2t + \sin 2t + \frac{1}{5}e^t) - e^t = 4\cos 2t + 4\sin 2t - \frac{1}{5}e^t = \frac{d^2y}{dt^2}\,.$$

**43.** $(y')^2 + 1 = 0$ has no real solutions because $(y')^2 + 1$ is positive for all differentiable functions $y = \phi(x)$.

**44.** The only solution of $(y')^2 + y^2 = 0$ is $y = 0$, since if $y \neq 0$, $y^2 > 0$ and $(y')^2 + y^2 \geq y^2 > 0$.

**45.** The first derivative of $f(x) = e^x$ is $e^x$. The first derivative of $f(x) = e^{kx}$ is $ke^{kx}$. The differential equations are $y' = y$ and $y' = ky$, respectively.

**46.** Any function of the form $y = ce^x$ or $y = ce^{-x}$ is its own second derivative. The corresponding differential equation is $y'' - y = 0$. Functions of the form $y = c\sin x$ or $y = c\cos x$ have second derivatives that are the negatives of themselves. The differential equation is $y'' + y = 0$.

**47.** We first note that $\sqrt{1-y^2} = \sqrt{1-\sin^2 x} = \sqrt{\cos^2 x} = |\cos x|$. This prompts us to consider values of $x$ for which $\cos x < 0$, such as $x = \pi$. In this case

$$\left.\frac{dy}{dx}\right|_{x=\pi} = \left.\frac{d}{dx}(\sin x)\right|_{x=\pi} = \cos x\big|_{x=\pi} = \cos\pi = -1,$$

but

$$\sqrt{1-y^2}|_{x=\pi} = \sqrt{1-\sin^2\pi} = \sqrt{1} = 1.$$

Thus, $y = \sin x$ will only be a solution of $y' = \sqrt{1-y^2}$ when $\cos x > 0$. An interval of definition is then $(-\pi/2, \pi/2)$. Other intervals are $(3\pi/2, 5\pi/2)$, $(7\pi/2, 9\pi/2)$, and so on.

**48.** Since the first and second derivatives of $\sin t$ and $\cos t$ involve $\sin t$ and $\cos t$, it is plausible that a linear combination of these functions, $A\sin t + B\cos t$, could be a solution of the differential equation. Using $y' = A\cos t - B\sin t$ and $y'' = -A\sin t - B\cos t$ and substituting into the differential equation we get

$$\begin{aligned} y'' + 2y' + 4y &= -A\sin t - B\cos t + 2A\cos t - 2B\sin t + 4A\sin t + 4B\cos t \\ &= (3A - 2B)\sin t + (2A + 3B)\cos t = 5\sin t \end{aligned}$$

Thus $3A - 2B = 5$ and $2A + 3B = 0$. Solving these simultaneous equations we find $A = \frac{15}{13}$ and $B = -\frac{10}{13}$. A particular solution is $y = \frac{15}{13}\sin t - \frac{10}{13}\cos t$.

**49.** One solution is given by the upper portion of the graph with domain approximately $(0, 2.6)$. The other solution is given by the lower portion of the graph, also with domain approximately $(0, 2.6)$.

**50.** One solution, with domain approximately $(-\infty, 1.6)$ is the portion of the graph in the second quadrant together with the lower part of the graph in the first quadrant. A second solution, with domain approximately $(0, 1.6)$ is the upper part of the graph in the first quadrant. The third solution, with domain $(0, \infty)$, is the part of the graph in the fourth quadrant.

**51.** Differentiating $(x^3+y^3)/xy=3c$ we obtain

$$\frac{xy(3x^2+3y^2y')-(x^3+y^3)(xy'+y)}{x^2y^2}=0$$

$$3x^3y+3xy^3y'-x^4y'-x^3y-xy^3y'-y^4=0$$

$$(3xy^3-x^4-xy^3)y'=-3x^3y+x^3y+y^4$$

$$y'=\frac{y^4-2x^3y}{2xy^3-x^4}=\frac{y(y^3-2x^3)}{x(2y^3-x^3)}\,.$$

**52.** A tangent line will be vertical where $y'$ is undefined, or in this case, where $x(2y^3-x^3)=0$. This gives $x=0$ or $2y^3=x^3$. Substituting $y^3=x^3/2$ into $x^3+y^3=3xy$ we get

$$x^3+\frac{1}{2}x^3=3x\left(\frac{1}{2^{1/3}}\,x\right)$$

$$\frac{3}{2}x^3=\frac{3}{2^{1/3}}x^2$$

$$x^3=2^{2/3}x^2$$

$$x^2(x-2^{2/3})=0.$$

Thus, there are vertical tangent lines at $x=0$ and $x=2^{2/3}$, or at $(0,0)$ and $(2^{2/3},2^{1/3})$. Since $2^{2/3}\approx 1.59$, the estimates of the domains in Problem 50 were close.

**53.** The derivatives of the functions are $\phi_1'(x)=-x/\sqrt{25-x^2}$ and $\phi_2'(x)=x/\sqrt{25-x^2}$, neither of which is defined at $x=\pm 5$.

**54.** To determine if a solution curve passes through $(0,3)$ we let $t=0$ and $P=3$ in the equation $P=c_1e^t/(1+c_1e^t)$. This gives $3=c_1/(1+c_1)$ or $c_1=-\frac{3}{2}$. Thus, the solution curve

$$P=\frac{(-3/2)e^t}{1-(3/2)e^t}=\frac{-3e^t}{2-3e^t}$$

passes through the point $(0,3)$. Similarly, letting $t=0$ and $P=1$ in the equation for the one-parameter family of solutions gives $1=c_1/(1+c_1)$ or $c_1=1+c_1$. Since this equation has no solution, no solution curve passes through $(0,1)$.

**55.** For the first-order differential equation integrate $f(x)$. For the second-order differential equation integrate twice. In the latter case we get $y=\int(\int f(x)dx)\,dx+c_1x+c_2$.

**56.** Solving for $y'$ using the quadratic formula we obtain the two differential equations

$$y'=\frac{1}{x}\left(2+2\sqrt{1+3x^6}\right)\quad\text{and}\quad y'=\frac{1}{x}\left(2-2\sqrt{1+3x^6}\right),$$

so the differential equation cannot be put in the form $dy/dx=f(x,y)$.

**57.** The differential equation $yy' - xy = 0$ has normal form $dy/dx = x$. These are not equivalent because $y = 0$ is a solution of the first differential equation but not a solution of the second.

**58.** Differentiating we get $y' = c_1 + 2c_2x$ and $y'' = 2c_2$. Then $c_2 = y''/2$ and $c_1 = y' - xy''$, so

$$y = (y' - xy'')\,x + \left(\frac{y''}{2}\right)x^2 = xy' - \frac{1}{2}x^2y''$$

and the differential equation is $x^2y'' - 2xy' + 2y = 0$.

**59.** **(a)** Since $e^{-x^2}$ is positive for all values of $x$, $dy/dx > 0$ for all $x$, and a solution, $y(x)$, of the differential equation must be increasing on any interval.

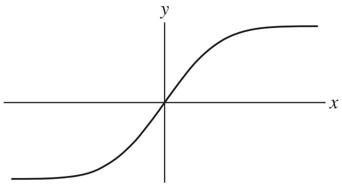
**(b)** $\displaystyle\lim_{x\to-\infty}\frac{dy}{dx} = \lim_{x\to-\infty}e^{-x^2} = 0$ and $\displaystyle\lim_{x\to\infty}\frac{dy}{dx} = \lim_{x\to\infty}e^{-x^2} = 0$. Since $dy/dx$ approaches 0 as $x$ approaches $-\infty$ and $\infty$, the solution curve has horizontal asymptotes to the left and to the right.

**(c)** To test concavity we consider the second derivative

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(e^{-x^2}\right) = -2xe^{-x^2}.$$

Since the second derivative is positive for $x < 0$ and negative for $x > 0$, the solution curve is concave up on $(-\infty, 0)$ and concave down on $(0, \infty)$.

**(d)**



**60.** **(a)** The derivative of a constant solution $y = c$ is 0, so solving $5 - c = 0$ we see that $c = 5$ and so $y = 5$ is a constant solution.

**(b)** A solution is increasing where $dy/dx = 5 - y > 0$ or $y < 5$. A solution is decreasing where $dy/dx = 5 - y < 0$ or $y > 5$.

**61.** **(a)** The derivative of a constant solution is 0, so solving $y(a - by) = 0$ we see that $y = 0$ and $y = a/b$ are constant solutions.

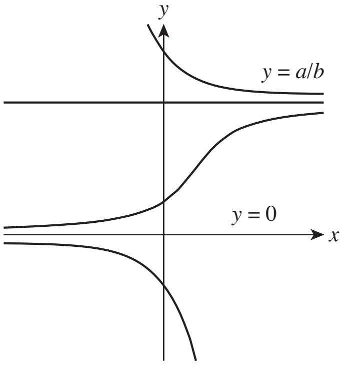
**(b)** A solution is increasing where $dy/dx = y(a - by) = by(a/b - y) > 0$ or $0 < y < a/b$. A solution is decreasing where $dy/dx = by(a/b - y) < 0$ or $y < 0$ or $y > a/b$.

**(c)** Using implicit differentiation we compute

$$\frac{d^2y}{dx^2} = y(-by') + y'(a - by) = y'(a - 2by).$$

Thus $d^2y/dx^2 = 0$ when $y = a/2b$. Since $d^2y/dx^2 > 0$ for $0 < y < a/2b$ and $d^2y/dx^2 < 0$ for $a/2b < y < a/b$, the graph of $y = \phi(x)$ has a point of inflection at $y = a/2b$.

**(d)**

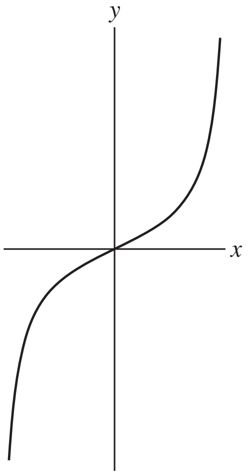


**62.** **(a)** If $y = c$ is a constant solution then $y' = 0$, but $c^2 + 4$ is never 0 for any real value of $c$.

**(b)** Since $y' = y^2 + 4 > 0$ for all $x$ where a solution $y = \phi(x)$ is defined, any solution must be increasing on any interval on which it is defined. Thus it cannot have any relative extrema.

**(c)** Using implicit differentiation we compute $d^2y/dx^2 = 2yy' = 2y(y^2 + 4)$. Setting $d^2y/dx^2 = 0$ we see that $y = 0$ corresponds to the only possible point of inflection. Since $d^2y/dx^2 < 0$ for $y < 0$ and $d^2y/dx^2 > 0$ for $y > 0$, there is a point of inflection where $y = 0$.

**(d)**



**63.** In *Mathematica* use

```
Clear[y]
y[x_]:= x Exp[5x] Cos[2x]
y[x]
y''''[x] – 20y'''[x] + 158y''[x] – 580y'[x] +841y[x]//Simplify
```

The output will show $y(x) = e^{5x}x\cos 2x$, which verifies that the correct function was entered, and 0, which verifies that this function is a solution of the differential equation.

**64.** In *Mathematica* use

```
Clear[y]
y[x_]:= 20Cos[5Log[x]]/x – 3Sin[5Log[x]]/x
y[x]
x^3 y'''[x] + 2x^2 y''[x] + 20x y'[x] – 78y[x]//Simplify
```

The output will show $y(x) = 20\cos(5\ln x)/x - 3\sin(5\ln x)/x$, which verifies that the correct function was entered, and 0, which verifies that this function is a solution of the differential equation.

## 1.2 Initial-Value Problems

**1.** Solving $-1/3 = 1/(1 + c_1)$ we get $c_1 = -4$. The solution is $y = 1/(1 - 4e^{-x})$.

**2.** Solving $2 = 1/(1 + c_1 e)$ we get $c_1 = -(1/2)e^{-1}$. The solution is $y = 2/(2 - e^{-(x+1)})$.

**3.** Letting $x = 2$ and solving $1/3 = 1/(4 + c)$ we get $c = -1$. The solution is $y = 1/(x^2 - 1)$. This solution is defined on the interval $(1, \infty)$.

**4.** Letting $x = -2$ and solving $1/2 = 1/(4 + c)$ we get $c = -2$. The solution is $y = 1/(x^2 - 2)$. This solution is defined on the interval $(-\infty, -\sqrt{2}\,)$.

**5.** Letting $x = 0$ and solving $1 = 1/c$ we get $c = 1$. The solution is $y = 1/(x^2 + 1)$. This solution is defined on the interval $(-\infty, \infty)$.

**6.** Letting $x = 1/2$ and solving $-4 = 1/(1/4 + c)$ we get $c = -1/2$. The solution is $y = 1/(x^2 - 1/2) = 2/(2x^2 - 1)$. This solution is defined on the interval $(-1/\sqrt{2}\,, 1/\sqrt{2}\,)$.

*In Problems 7–10, we use $x = c_1\cos t + c_2\sin t$ and $x' = -c_1\sin t + c_2\cos t$ to obtain a system of two equations in the two unknowns $c_1$ and $c_2$.*

**7.** From the initial conditions we obtain the system

$$c_1 = -1 c_2 = 8$$

The solution of the initial-value problem is $x = -\cos t + 8\sin t$.

**8.** From the initial conditions we obtain the system

$$c_2 = 0 - c_1 = 1$$

The solution of the initial-value problem is $x = -\cos t$.

**9.** From the initial conditions we obtain

$$\frac{\sqrt{3}}{2}c_1 + \frac{1}{2}c_2 = \frac{1}{2} - \frac{1}{2}c_2 + \frac{\sqrt{3}}{2} = 0$$

Solving, we find $c_1 = \sqrt{3}/4$ and $c_2 = 1/4$. The solution of the initial-value problem is

$$x = (\sqrt{3}/4)\cos t + (1/4)\sin t.$$

**10.** From the initial conditions we obtain

$$\frac{\sqrt{2}}{2}c_1 + \frac{\sqrt{2}}{2}c_2 = \sqrt{2}$$

$$[6pt] - \frac{\sqrt{2}}{2}c_1 + \frac{\sqrt{2}}{2}c_2 = 2\sqrt{2}.$$

Solving, we find $c_1 = -1$ and $c_2 = 3$. The solution of the initial-value problem is $x = -\cos t + 3\sin t$.

*In Problems 11–14, we use $y = c_1e^x + c_2e^{-x}$ and $y' = c_1e^x - c_2e^{-x}$ to obtain a system of two equations in the two unknowns $c_1$ and $c_2$.*

**11.** From the initial conditions we obtain

$$c_1 + c_2 = 1$$

$$c_1 - c_2 = 2.$$

Solving, we find $c_1 = \frac{3}{2}$ and $c_2 = -\frac{1}{2}$. The solution of the initial-value problem is $y = \frac{3}{2}e^x - \frac{1}{2}e^{-x}$.

**12.** From the initial conditions we obtain

$$ec_1 + e^{-1}c_2 = 0$$

$$ec_1 - e^{-1}c_2 = e.$$

Solving, we find $c_1 = \frac{1}{2}$ and $c_2 = -\frac{1}{2}e^2$. The solution of the initial-value problem is

$$y = \frac{1}{2}e^x - \frac{1}{2}e^2e^{-x} = \frac{1}{2}e^x - \frac{1}{2}e^{2-x}.$$

**13.** From the initial conditions we obtain

$$\begin{aligned} e^{-1}c_1 + ec_2 &= 5 \\ e^{-1}c_1 - ec_2 &= -5. \end{aligned}$$

Solving, we find $c_1 = 0$ and $c_2 = 5e^{-1}$. The solution of the initial-value problem is $y = 5e^{-1}e^{-x} = 5e^{-1-x}$.

**14.** From the initial conditions we obtain

$$\begin{aligned} c_1 + c_2 &= 0 \\ c_1 - c_2 &= 0. \end{aligned}$$

Solving, we find $c_1 = c_2 = 0$. The solution of the initial-value problem is $y = 0$.

**15.** Two solutions are $y = 0$ and $y = x^3$.

**16.** Two solutions are $y = 0$ and $y = x^2$. (Also, any constant multiple of $x^2$ is a solution.)

**17.** For $f(x, y) = y^{2/3}$ we have $\dfrac{\partial f}{\partial y} = \dfrac{2}{3}y^{-1/3}$. Thus, the differential equation will have a unique solution in any rectangular region of the plane where $y \neq 0$.

**18.** For $f(x, y) = \sqrt{xy}$ we have $\partial f/\partial y = \frac{1}{2}\sqrt{x/y}$. Thus, the differential equation will have a unique solution in any region where $x > 0$ and $y > 0$ or where $x < 0$ and $y < 0$.

**19.** For $f(x, y) = \dfrac{y}{x}$ we have $\dfrac{\partial f}{\partial y} = \dfrac{1}{x}$. Thus, the differential equation will have a unique solution in any region where $x \neq 0$.

**20.** For $f(x, y) = x + y$ we have $\dfrac{\partial f}{\partial y} = 1$. Thus, the differential equation will have a unique solution in the entire plane.

**21.** For $f(x, y) = x^2/(4 - y^2)$ we have $\partial f/\partial y = 2x^2y/(4 - y^2)^2$. Thus the differential equation will have a unique solution in any region where $y < -2$, $-2 < y < 2$, or $y > 2$.

**22.** For $f(x, y) = \dfrac{x^2}{1 + y^3}$ we have $\dfrac{\partial f}{\partial y} = \dfrac{-3x^2y^2}{(1 + y^3)^2}$. Thus, the differential equation will have a unique solution in any region where $y \neq -1$.

**23.** For $f(x, y) = \dfrac{y^2}{x^2 + y^2}$ we have $\dfrac{\partial f}{\partial y} = \dfrac{2x^2y}{(x^2 + y^2)^2}$. Thus, the differential equation will have a unique solution in any region not containing $(0, 0)$.

**24.** For $f(x,y) = (y+x)/(y-x)$ we have $\partial f/\partial y = -2x/(y-x)^2$. Thus the differential equation will have a unique solution in any region where $y < x$ or where $y > x$.

*In Problems 25–28, we identify $f(x,y) = \sqrt{y^2-9}$ and $\partial f/\partial y = y/\sqrt{y^2-9}$. We see that $f$ and $\partial f/\partial y$ are both continuous in the regions of the plane determined by $y < -3$ and $y > 3$ with no restrictions on $x$.*

**25.** Since $4 > 3$, $(1,4)$ is in the region defined by $y > 3$ and the differential equation has a unique solution through $(1,4)$.

**26.** Since $(5,3)$ is not in either of the regions defined by $y < -3$ or $y > 3$, there is no guarantee of a unique solution through $(5,3)$.

**27.** Since $(2,-3)$ is not in either of the regions defined by $y < -3$ or $y > 3$, there is no guarantee of a unique solution through $(2,-3)$.

**28.** Since $(-1,1)$ is not in either of the regions defined by $y < -3$ or $y > 3$, there is no guarantee of a unique solution through $(-1,1)$.

**29.** **(a)** A one-parameter family of solutions is $y = cx$. Since $y' = c$, $xy' = xc = y$ and $y(0) = c \cdot 0 = 0$.

**(b)** Writing the equation in the form $y' = y/x$, we see that $R$ cannot contain any point on the $y$-axis. Thus, any rectangular region disjoint from the $y$-axis and containing $(x_0, y_0)$ will determine an interval around $x_0$ and a unique solution through $(x_0, y_0)$. Since $x_0 = 0$ in part (a), we are not guaranteed a unique solution through $(0,0)$.

**(c)** The piecewise-defined function which satisfies $y(0) = 0$ is not a solution since it is not differentiable at $x = 0$.

**30.** **(a)** Since $\dfrac{d}{dx}\tan(x+c) = \sec^2(x+c) = 1 + \tan^2(x+c)$, we see that $y = \tan(x+c)$ satisfies the differential equation.

**(b)** Solving $y(0) = \tan c = 0$ we obtain $c = 0$ and $y = \tan x$. Since $\tan x$ is discontinuous at $x = \pm\pi/2$, the solution is not defined on $(-2,2)$ because it contains $\pm\pi/2$.

**(c)** The largest interval on which the solution can exist is $(-\pi/2, \pi/2)$.

**31.** **(a)** Since $\dfrac{d}{dx}\left(-\dfrac{1}{x+c}\right) = \dfrac{1}{(x+c)^2} = y^2$, we see that $y = -\dfrac{1}{x+c}$ is a solution of the differential equation.

**(b)** Solving $y(0) = -1/c = 1$ we obtain $c = -1$ and $y = 1/(1-x)$. Solving $y(0) = -1/c = -1$ we obtain $c = 1$ and $y = -1/(1+x)$. Being sure to include $x = 0$, we see that the interval of existence of $y = 1/(1-x)$ is $(-\infty, 1)$, while the interval of existence of $y = -1/(1+x)$ is $(-1, \infty)$.

**(c)** By inspection we see that $y = 0$ is a solution on $(-\infty, \infty)$.

**32. (a)** Applying $y(1) = 1$ to $y = -1/(x + c)$ gives

$$1 = -\frac{1}{1+c} \quad \text{or} \quad 1 + c = -1$$

Thus $c = -2$ and

$$y = -\frac{1}{x-2} = \frac{1}{2-x}.$$

**(b)** Applying $y(3) = -1$ to $y = -1/(x + c)$ gives

$$-1 = -\frac{1}{3+c} \quad \text{or} \quad 3 + c = 1.$$

Thus $c = -2$ and

$$y = -\frac{1}{x-2} = \frac{1}{2-x}.$$

**(c)** No, they are not the same solution. The interval $I$ of definition for the solution in part (a) is $(-\infty, 2)$; whereas the interval $I$ of definition for the solution in part (b) is $(2, \infty)$. See the figure.

**33. (a)** Differentiating $3x^2 - y^2 = c$ we get $6x - 2yy' = 0$ or $yy' = 3x$.

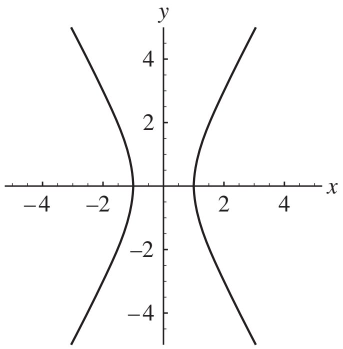
**(b)** Solving $3x^2 - y^2 = 3$ for $y$ we get

$$y = \phi_1(x) = \sqrt{3(x^2-1)}\,, \qquad 1 < x < \infty,$$

$$y = \phi_2(x) = -\sqrt{3(x^2-1)}\,, \qquad 1 < x < \infty,$$

$$y = \phi_3(x) = \sqrt{3(x^2-1)}\,, \qquad -\infty < x < -1,$$

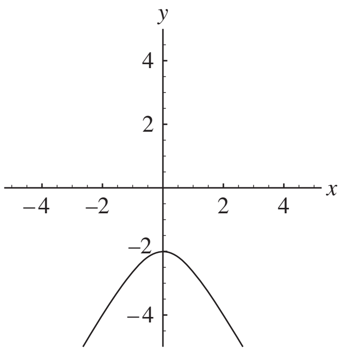
$$y = \phi_4(x) = -\sqrt{3(x^2-1)}\,, \qquad -\infty < x < -1.$$



**(c)** Only $y = \phi_3(x)$ satisfies $y(-2) = 3$.

**34. (a)** Setting $x = 2$ and $y = -4$ in $3x^2 - y^2 = c$ we get $12 - 16 = -4 = c$, so the explicit solution is

$$y = -\sqrt{3x^2+4}\,, \quad -\infty < x < \infty.$$



**(b)** Setting $c = 0$ we have $y = \sqrt{3}x$ and $y = -\sqrt{3}x$, both defined on $(-\infty, \infty)$.

*In Problems 35–38, we consider the points on the graphs with x-coordinates $x_0 = -1$, $x_0 = 0$, and $x_0 = 1$. The slopes of the tangent lines at these points are compared with the slopes given by $y'(x_0)$ in (a) through (f).*

**35.** The graph satisfies the conditions in (b) and (f).

**36.** The graph satisfies the conditions in (e).

**37.** The graph satisfies the conditions in (c) and (d).

**38.** The graph satisfies the conditions in (a).

*In Problems 39–44 $y = c_1 \cos 2x + c_2 \sin 2x$ is a two parameter family of solutions of the second-order differential equation $y'' + 4y = 0$. In some of the problems we will use the fact that $y' = -2c_1 \sin 2x + 2c_2 \cos 2x$.*

**39.** From the boundary conditions $y(0) = 0$ and $y\left(\frac{\pi}{4}\right) = 3$ we obtain

$$y(0) = c_1 = 0$$

$$y\left(\frac{\pi}{4}\right) = c_1 \cos\left(\frac{\pi}{2}\right) + c_2 \sin\left(\frac{\pi}{2}\right) = c_2 = 3.$$

Thus, $c_1 = 0$, $c_2 = 3$, and the solution of the boundary-value problem is $y = 3 \sin 2x$.

**40.** From the boundary conditions $y(0) = 0$ and $y(\pi) = 0$ we obtain

$$y(0) = c_1 = 0$$

$$y(\pi) = c_1 = 0.$$

Thus, $c_1 = 0$, $c_2$ is unrestricted, and the solution of the boundary-value problem is $y = c_2 \sin 2x$, where $c_2$ is any real number.

**41.** From the boundary conditions $y'(0) = 0$ and $y'\left(\frac{\pi}{6}\right) = 0$ we obtain

$$y'(0) = 2c_2 = 0$$

$$y'\left(\frac{\pi}{6}\right) = -2c_1 \sin\left(\frac{\pi}{3}\right) = -\sqrt{3}\, c_1 = 0.$$

Thus, $c_2 = 0$, $c_1 = 0$, and the solution of the boundary-value problem is $y = 0$.

**42.** From the boundary conditions $y(0) = 1$ and $y'(\pi) = 5$ we obtain

$$y(0) = c_1 = 1$$

$$y'(\pi) = 2c_2 = 5.$$

Thus, $c_1 = 1$, $c_2 = \frac{5}{2}$, and the solution of the boundary-value problem is $y = \cos 2x + \frac{5}{2} \sin 2x$.

**43.** From the boundary conditions $y(0) = 0$ and $y(\pi) = 2$ we obtain

$$y(0) = c_1 = 0$$

$$y(\pi) = c_1 = 2.$$

Since $0 \neq 2$, this is not possible and there is no solution.

**44.** From the boundary conditions $y'\left(\frac{\pi}{2}\right) = 1$ and $y'(\pi) = 0$ we obtain

$$y'\left(\frac{\pi}{2}\right) = 2c_2 = -1$$

$$y'(\pi) = 2c_2 = 0.$$

Since $0 \neq -1$, this is not possible and there is no solution.

**45.** Integrating $y' = 8e^{2x} + 6x$ we obtain

$$y = \int (8e^{2x} + 6x)\,dx = 4e^{2x} + 3x^2 + c.$$

Setting $x = 0$ and $y = 9$ we have $9 = 4 + c$ so $c = 5$ and $y = 4e^{2x} + 3x^2 + 5$.

**46.** Integrating $y'' = 12x - 2$ we obtain

$$y' = \int (12x - 2)\,dx = 6x^2 - 2x + c_1.$$

Then, integrating $y'$ we obtain

$$y = \int (6x^2 - 2x + c_1)\,dx = 2x^3 - x^2 + c_1x + c_2.$$

At $x = 1$ the $y$-coordinate of the point of tangency is $y = -1 + 5 = 4$. This gives the initial condition $y(1) = 4$. The slope of the tangent line at $x = 1$ is $y'(1) = -1$. From the initial conditions we obtain

$$2 - 1 + c_1 + c_2 = 4 \qquad \text{or} \qquad c_1 + c_2 = 3$$

and

$$6 - 2 + c_1 = -1 \qquad \text{or} \qquad c_1 = -5.$$

Thus, $c_1 = -5$ and $c_2 = 8$, so $y = 2x^3 - x^2 - 5x + 8$.

**47.** When $x = 0$ and $y = \frac{1}{2}$, $y' = -1$, so the only plausible solution curve is the one with negative slope at $(0, \frac{1}{2})$, or the red curve.

**48.** We note that the initial condition $y(0)=0$,

$$0=\int_0^y \frac{1}{\sqrt{t^3+1}}\,dt$$

is satisfied only when $y=0$. For any $y>0$, necessarily

$$\int_0^y \frac{1}{\sqrt{t^3+1}}\,dt>0$$

because the integrand is positive on the interval of integration. Then from (12) of Section 1.1 and the Chain Rule we have:

$$\begin{aligned}\frac{d}{dx}x&=\frac{d}{dx}\int_0^y \frac{1}{\sqrt{t^3+1}}\,dt\\ 1&=\frac{1}{\sqrt{y^3+1}}\frac{dy}{dx}\end{aligned}\qquad \text{and}\qquad \begin{aligned}\frac{dy}{dx}&=\sqrt{y^3+1}\\ y'(0)&=\left.\frac{dy}{dx}\right|_{x=0}=\sqrt{\left(y(0)\right)^3+1}=\sqrt{0+1}=1.\end{aligned}$$

Computing the second derivative, we see that:

$$\begin{aligned}\frac{d^2y}{dx^2}&=\frac{d}{dx}\sqrt{y^3+1}=\frac{3y^2}{2\sqrt{y^3+1}}\frac{dy}{dx}=\frac{3y^2}{2\sqrt{y^3+1}}\cdot\sqrt{y^3+1}=\frac{3}{2}y^2\\ \frac{d^2y}{dx^2}&=\frac{3}{2}y^2.\end{aligned}$$

This is equivalent to $2\dfrac{d^2y}{dx^2}-3y^2=0$.

**49.** If the solution is tangent to the $x$-axis at $(x_0,0)$, then $y'=0$ when $x=x_0$ and $y=0$. Substituting these values into $y'+2y=3x-6$ we get $0+0=3x_0-6$ or $x_0=2$.

**50.** The theorem guarantees a unique (meaning single) solution through any point. Thus, there cannot be two distinct solutions through any point.

**51.** When $y=\frac{1}{16}x^4$, $y'=\frac{1}{4}x^3=x(\frac{1}{4}x^2)=xy^{1/2}$, and $y(2)=\frac{1}{16}(16)=1$. When

$$y=\begin{cases}0, & x<0\\ \dfrac{1}{16}x^4, & x\ge 0\end{cases}$$

we have

$$y'=\begin{cases}0, & x<0\\ \frac{1}{4}x^3, & x\ge 0\end{cases}=x\begin{cases}0, & x<0\\ \frac{1}{4}x^2, & x\ge 0\end{cases}=xy^{1/2},$$

and $y(2)=\frac{1}{16}(16)=1$. The two different solutions are the same on the interval $(0,\infty)$, which is all that is required by Theorem 1.2.1.

## 1.3 Differential Equations as Mathematical Models

**1.** $\dfrac{dP}{dt} = kP + r; \qquad \dfrac{dP}{dt} = kP - r$

**2.** Let $b$ be the rate of births and $d$ the rate of deaths. Then $b = k_1P$ and $d = k_2P$. Since $dP/dt = b - d$, the differential equation is $dP/dt = k_1P - k_2P$.

**3.** Let $b$ be the rate of births and $d$ the rate of deaths. Then $b = k_1P$ and $d = k_2P^2$. Since $dP/dt = b - d$, the differential equation is $dP/dt = k_1P - k_2P^2$.

**4.** $\dfrac{dP}{dt} = k_1P - k_2P^2 - h, \;\; h > 0$

**5.** From the graph in the text we estimate $T_0 = 180°$ and $T_m = 75°$. We observe that when $T = 85$, $dT/dt \approx -1$. From the differential equation we then have

$$k = \frac{dT/dt}{T - T_m} = \frac{-1}{85 - 75} = -0.1.$$

**6.** By inspecting the graph in the text we take $T_m$ to be $T_m(t) = 80 - 30\cos(\pi t/12)$. Then the temperature of the body at time $t$ is determined by the differential equation

$$\frac{dT}{dt} = k\left[T - \left(80 - 30\cos\left(\frac{\pi}{12}t\right)\right)\right], \quad t > 0.$$

**7.** The number of students with the flu is $x$ and the number not infected is $1000 - x$, so $dx/dt = kx(1000 - x)$.

**8.** By analogy, with the differential equation modeling the spread of a disease, we assume that the rate at which the technological innovation is adopted is proportional to the number of people who have adopted the innovation and also to the number of people, $y(t)$, who have not yet adopted it. If one person who has adopted the innovation is introduced into the population, then $x + y = n + 1$ and

$$\frac{dx}{dt} = kx(n + 1 - x), \quad x(0) = 1.$$

**9.** The rate at which salt is leaving the tank is

$$R_{out}\ (3 \text{ gal/min}) \cdot \left(\frac{A}{300} \text{ lb/gal}\right) = \frac{A}{100} \text{ lb/min}.$$

Thus $dA/dt = A/100$. The initial amount is $A(0) = 50$.

**10.** The rate at which salt is entering the tank is

$$R_{in} = (3 \text{ gal/min}) \cdot (2 \text{ lb/gal}) = 6 \text{ lb/min}.$$

Since the solution is pumped out at a slower rate, it is accumulating at the rate of $(3 - 2)\text{gal/min} = 1$ gal/min. After $t$ minutes there are $300 + t$ gallons of brine in the tank. The rate at which salt is leaving is

$$R_{out} = (2 \text{ gal/min}) \cdot \left(\frac{A}{300+t} \text{ lb/gal}\right) = \frac{2A}{300+t} \text{ lb/min}.$$

The differential equation is

$$\frac{dA}{dt} = 6 - \frac{2A}{300+t}.$$

**11.** The rate at which salt is entering the tank is

$$R_{in} = (3 \text{ gal/min}) \cdot (2 \text{ lb/gal}) = 6 \text{ lb/min}.$$

Since the tank loses liquid at the net rate of

$$3 \text{ gal/min} - 3.5 \text{ gal/min} = -0.5 \text{ gal/min},$$

after $t$ minutes the number of gallons of brine in the tank is $300 - \frac{1}{2}t$ gallons. Thus the rate at which salt is leaving is

$$R_{out} = \left(\frac{A}{300 - t/2} \text{ lb/gal}\right) \cdot (3.5 \text{ gal/min}) = \frac{3.5A}{300 - t/2} \text{ lb/min} = \frac{7A}{600 - t} \text{ lb/min}.$$

The differential equation is

$$\frac{dA}{dt} = 6 - \frac{7A}{600 - t} \quad \text{or} \quad \frac{dA}{dt} + \frac{7}{600 - t} A = 6.$$

**12.** The rate at which salt is entering the tank is

$$R_{in} = (c_{in} \text{ lb/gal}) \cdot (r_{in} \text{ gal/min}) = c_{in} r_{in} \text{ lb/min}.$$

Now let $A(t)$ denote the number of pounds of salt and $N(t)$ the number of gallons of brine in the tank at time $t$. The concentration of salt in the tank as well as in the outflow is $c(t) = x(t)/N(t)$. But the number of gallons of brine in the tank remains steady, is increased, or is decreased depending on whether $r_{in} = r_{out}$, $r_{in} > r_{out}$, or $r_{in} < r_{out}$. In any case, the number of gallons of brine in the tank at time $t$ is $N(t) = N_0 + (r_{in} - r_{out})t$. The output rate of salt is then

$$R_{out} = \left(\frac{A}{N_0 + (r_{in} - r_{out})t} \text{ lb/gal}\right) \cdot (r_{out} \text{ gal/min}) = r_{out} \frac{A}{N_0 + (r_{in} - r_{out})t} \text{ lb/min}.$$

The differential equation for the amount of salt, $dA/dt = R_{in} - R_{out}$, is

$$\frac{dA}{dt} = c_{in} r_{in} - r_{out} \frac{A}{N_0 + (r_{in} - r_{out})t} \quad \text{or} \quad \frac{dA}{dt} + \frac{r_{out}}{N_0 + (r_{in} - r_{out})t} A = c_{in} r_{in}.$$

**13.** The volume of water in the tank at time $t$ is $V = A_w h$. The differential equation is then

$$\frac{dh}{dt} = \frac{1}{A_w}\frac{dV}{dt} = \frac{1}{A_w}\left(-cA_h\sqrt{2gh}\right) = -\frac{cA_h}{A_w}\sqrt{2gh}\,.$$

Using $A_h = \pi\left(\dfrac{2}{12}\right)^2 = \dfrac{\pi}{36}$, $A_w = 10^2 = 100$, and $g = 32$, this becomes

$$\frac{dh}{dt} = -\frac{c\pi/36}{100}\sqrt{64h} = -\frac{c\pi}{450}\sqrt{h}\,.$$

**14.** The volume of water in the tank at time $t$ is $V = \frac{1}{3}\pi r^2 h$ where $r$ is the radius of the tank at height $h$. From the figure in the text we see that $r/h = 8/20$ so that $r = \frac{2}{5}h$ and $V = \frac{1}{3}\pi\left(\frac{2}{5}h\right)^2 h = \frac{4}{75}\pi h^3$. Differentiating with respect to $t$ we have $dV/dt = \frac{4}{25}\pi h^2\,dh/dt$ or

$$\frac{dh}{dt} = \frac{25}{4\pi h^2}\frac{dV}{dt}\,.$$

From Problem 13 we have $dV/dt = -cA_h\sqrt{2gh}$ where $c = 0.6$, $A_h = \pi\left(\frac{2}{12}\right)^2$, and $g = 32$. Thus $dV/dt = -2\pi\sqrt{h}/15$ and

$$\frac{dh}{dt} = \frac{25}{4\pi h^2}\left(-\frac{2\pi\sqrt{h}}{15}\right) = -\frac{5}{6h^{3/2}}\,.$$

**15.** Since $i = dq/dt$ and $L\,d^2q/dt^2 + R\,dq/dt = E(t)$, we obtain $L\,di/dt + Ri = E(t)$.

**16.** By Kirchhoff's second law we obtain $R\dfrac{dq}{dt} + \dfrac{1}{C}q = E(t)$.

**17.** From Newton's second law we obtain $m\dfrac{dv}{dt} = -kv^2 + mg$.

**18.** Since the barrel in Figure 1.3.17(b) in the text is submerged an additional $y$ feet below its equilibrium position the number of cubic feet in the additional submerged portion is the volume of the circular cylinder: $\pi\times(\text{radius})^2\times\text{height}$ or $\pi(s/2)^2y$. Then we have from Archimedes' principle

$$\begin{aligned}\text{upward force of water on barrel} &= \text{weight of water displaced}\\ &= (62.4)\times(\text{volume of water displaced})\\ &= (62.4)\pi(s/2)^2y = 15.6\pi s^2 y.\end{aligned}$$

It then follows from Newton's second law that

$$\frac{w}{g}\frac{d^2y}{dt^2} = -15.6\pi s^2 y \qquad \text{or} \qquad \frac{d^2y}{dt^2} + \frac{15.6\pi s^2 g}{w}y = 0,$$

where $g = 32$ and $w$ is the weight of the barrel in pounds.

**19.** The net force acting on the mass is

$$F = ma = m\,\frac{d^2x}{dt^2} = -k(s+x) + mg = -kx + mg - ks.$$

Since the condition of equilibrium is $mg = ks$, the differential equation is

$$m\,\frac{d^2x}{dt^2} = -kx.$$

**20.** From Problem 19, without a damping force, the differential equation is $m\,d^2x/dt^2 = -kx$. With a damping force proportional to velocity, the differential equation becomes

$$m\,\frac{d^2x}{dt^2} = -kx - \beta\frac{dx}{dt} \qquad \text{or} \qquad m\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = 0.$$

**21.** As the rocket climbs (in the positive direction), it spends its amount of fuel and therefore the mass of the fuel changes with time. The air resistance acts in the opposite direction of the motion and the upward thrust $R$ works in the same direction. Using Newton's second law we get

$$\frac{d}{dt}(mv) = -mg - kv + R$$

Now because the mass is variable, we must use the product rule to expand the left side of the equation. Doing so gives us the following:

$$\begin{aligned} \frac{d}{dt}(mv) &= -mg - kv + R \\ v \times \frac{dm}{dt} + m \times \frac{dv}{dt} &= -mg - kv + R \end{aligned}$$

The last line is the differential equation we wanted to find.

**22.** **(a)** Since the mass of the rocket is $m(t) = m_p + m_v + m_f(t)$, take the time rate-of-change and get, by straight-forward calculation,

$$\frac{d}{dt}m(t) = \frac{d}{dt}(m_p + m_v + m_f(t)) = 0 + 0 + m_f'(t) = \frac{d}{dt}m_f(t)$$

Therefore the rate of change of the mass of the rocket is the same as the rate of change of the mass of the fuel which is what we wanted to show.

**(b)** The fuel is decreasing at the constant rate of $\lambda$ and so from part (a) we have

$$\begin{aligned} \frac{d}{dt}m(t) &= \frac{d}{dt}m_f(t) = -\lambda \\ m(t) &= -\lambda t + c \end{aligned}$$

Using the given condition to solve for $c$, $m(0) = 0 + c = m_0$ and so $m(t) = -\lambda t + m_0$. The differential equation in Problem 21 now becomes

$$\begin{aligned}
v\frac{dm}{dt} + m\frac{dv}{dt} + kv &= -mg + R \\
-\lambda v + (-\lambda t + m_0)\frac{dv}{dt} + kv &= -mg + R \\
(-\lambda t + m_0)\frac{dv}{dt} + (k-\lambda)v &= -mg + R \\
\frac{dv}{dt} + \frac{k-\lambda}{-\lambda t + m_0}v &= \frac{-mg}{-\lambda t + m_0} + \frac{R}{-\lambda t + m_0} \\
\frac{dv}{dt} + \frac{k-\lambda}{-\lambda t + m_0}v &= -g + \frac{R}{-\lambda t + m_0}
\end{aligned}$$

**(c)** From part (b) we have that $\frac{d}{dt}m_f(t) = -\lambda$ and so by integrating this result we get $m_f(t) = -\lambda t + c$. Now at time $t = 0$, $m_f(0) = 0 + c = c$ therefore $m_f(t) = -\lambda t + m_f(0)$ . At some later time $t_b$ we then have $m_f(t_b) = -\lambda t_b + m_f(0) = 0$ and solving this equation for that time we get $t_b = m_f(0)/\lambda$ which is what we wanted to show.

**23.** From $g = k/R^2$ we find $k = gR^2$. Using $a = d^2r/dt^2$ and the fact that the positive direction is upward we get

$$\frac{d^2r}{dt^2} = -a = -\frac{k}{r^2} = -\frac{gR^2}{r^2} \qquad \text{or} \qquad \frac{d^2r}{dt^2} + \frac{gR^2}{r^2} = 0.$$

**24.** The gravitational force on $m$ is $F = -kM_r m/r^2$. Since $M_r = 4\pi\delta r^3/3$ and $M = 4\pi\delta R^3/3$ we have $M_r = r^3M/R^3$ and

$$F = -k\,\frac{M_r m}{r^2} = -k\,\frac{r^3Mm/R^3}{r^2} = -k\,\frac{mM}{R^3}\,r.$$

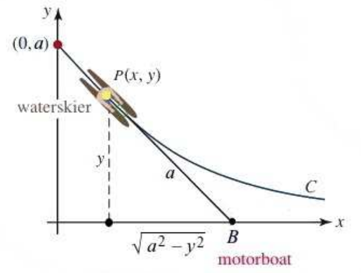
Now from $F = ma = d^2r/dt^2$ we have

$$m\,\frac{d^2r}{dt^2} = -k\,\frac{mM}{R^3}\,r \quad \text{or} \quad \frac{d^2r}{dt^2} = -\frac{kM}{R^3}\,r.$$

**25.** The differential equation is $\dfrac{dA}{dt} = k(M - A)$.

**26.** The differential equation is $\dfrac{dA}{dt} = k_1(M - A) - k_2A$.

**27.** The differential equation is $x'(t) = r - kx(t)$ where $k > 0$.



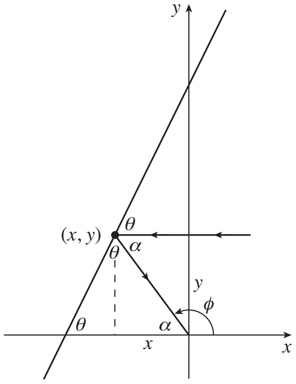
**28.** Consider the right triangle fromed by the waterskier ($P$), the boat ($B$), and the point on the $x$-axis directly below the waterskier. Using Pythagorean Theorem we have that the base of the triangle on the $x$-axis has length $\sqrt{a^2-y^2}$. Therefore the slope of the line tangent to curve $C$ is

$$y' = -\frac{y}{a^2-y^2}$$

Notice that the sign of the derivative is negative because as the boat proceeds along the positive $x$-axis, the $y$-coordinate decreases.

**29.** We see from the figure that $2\theta+\alpha=\pi$. Thus

$$\frac{y}{-x} = \tan\alpha = \tan(\pi-2\theta) = -\tan 2\theta = -\frac{2\tan\theta}{1-\tan^2\theta}\,.$$



Since the slope of the tangent line is $y'=\tan\theta$ we have $y/x = 2y'[1-(y')^2]$ or $y-y(y')^2 = 2xy'$, which is the quadratic equation $y(y')^2+2xy'-y=0$ in $y'$. Using the quadratic formula, we get

$$y' = \frac{-2x \pm \sqrt{4x^2+4y^2}}{2y} = \frac{-x \pm \sqrt{x^2+y^2}}{y}\,.$$

Since $dy/dx>0$, the differential equation is

$$\frac{dy}{dx} = \frac{-x+\sqrt{x^2+y^2}}{y} \qquad \text{or} \qquad y\,\frac{dy}{dx} - \sqrt{x^2+y^2} + x = 0.$$

**30.** The differential equation is $dP/dt = kP$, so from Problem 41 in Exercises 1.1, a one-parameter family of solutions is $P = ce^{kt}$.

**31.** The differential equation in (3) is $dT/dt = k(T-T_m)$. When the body is cooling, $T>T_m$, so $T-T_m>0$. Since $T$ is decreasing, $dT/dt<0$ and $k<0$. When the body is warming, $T<T_m$, so $T-T_m<0$. Since $T$ is increasing, $dT/dt>0$ and $k<0$.

**32.** The differential equation in (8) is $dA/dt = 6 - A/100$. If $A(t)$ attains a maximum, then $dA/dt = 0$ at this time and $A = 600$. If $A(t)$ continues to increase without reaching a maximum, then $A'(t)>0$ for $t>0$ and $A$ cannot exceed 600. In this case, if $A'(t)$ approaches 0 as $t$ increases to infinity, we see that $A(t)$ approaches 600 as $t$ increases to infinity.

**33.** This differential equation could describe a population that undergoes periodic fluctuations.

**34.** **(a)** As shown in Figure 1.3.23(a) in the text, the resultant of the reaction force of magnitude $F$ and the weight of magnitude $mg$ of the particle is the centripetal force of magnitude $m\omega^2 x$. The centripetal force points to the center of the circle of radius $x$ on which the particle rotates about the $y$-axis. Comparing parts of similar triangles gives

$$F\cos\theta = mg \quad \text{and} \quad F\sin\theta = m\omega^2 x.$$

**(b)** Using the equations in part (a) we find

$$\tan\theta = \frac{F\sin\theta}{F\cos\theta} = \frac{m\omega^2 x}{mg} = \frac{\omega^2 x}{g} \qquad \text{or} \qquad \frac{dy}{dx} = \frac{\omega^2 x}{g}.$$

**35.** From Problem 23, $d^2r/dt^2 = -gR^2/r^2$. Since $R$ is a constant, if $r = R + s$, then $d^2r/dt^2 = d^2s/dt^2$ and, using a Taylor series, we get

$$\frac{d^2s}{dt^2} = -g\frac{R^2}{(R+s)^2} = -gR^2(R+s)^{-2} \approx -gR^2[R^{-2} - 2sR^{-3} + \cdots] = -g + \frac{2gs}{R} + \cdots.$$

Thus, for $R$ much larger than $s$, the differential equation is approximated by $d^2s/dt^2 = -g$.

**36.** **(a)** If $\rho$ is the mass density of the raindrop, then $m = \rho V$ and

$$\frac{dm}{dt} = \rho\frac{dV}{dt} = \rho\frac{d}{dt}\left[\frac{4}{3}\pi r^3\right] = \rho\left(4\pi r^2\frac{dr}{dt}\right) = \rho S\frac{dr}{dt}.$$

If $dr/dt$ is a constant, then $dm/dt = kS$ where $\rho\, dr/dt = k$ or $dr/dt = k/\rho$. Since the radius is decreasing, $k < 0$. Solving $dr/dt = k/\rho$ we get $r = (k/\rho)t + c_0$. Since $r(0) = r_0$, $c_0 = r_0$ and $r = kt/\rho + r_0$.

**(b)** From Newton's second law, $\dfrac{d}{dt}[mv] = mg$, where $v$ is the velocity of the raindrop. Then

$$m\frac{dv}{dt} + v\frac{dm}{dt} = mg \qquad \text{or} \qquad \rho\left(\frac{4}{3}\pi r^3\right)\frac{dv}{dt} + v(k4\pi r^2) = \rho\left(\frac{4}{3}\pi r^3\right)g.$$

Dividing by $4\rho\pi r^3/3$ we get

$$\frac{dv}{dt} + \frac{3k}{\rho r}v = g \qquad \text{or} \qquad \frac{dv}{dt} + \frac{3k/\rho}{kt/\rho + r_0}v = g, \quad k < 0.$$

**37.** We assume that the plow clears snow at a constant rate of $k$ cubic miles per hour. Let $t$ be the time in hours after noon, $x(t)$ the depth in miles of the snow at time $t$, and $y(t)$ the distance the plow has moved in $t$ hours. Then $dy/dt$ is the velocity of the plow and the assumption gives

$$wx\frac{dy}{dt} = k,$$

where $w$ is the width of the plow. Each side of this equation simply represents the volume of snow plowed in one hour. Now let $t_0$ be the number of hours before noon when it started snowing and let $s$ be the constant rate in miles per hour at which $x$ increases. Then for $t > -t_0$, $x = s(t + t_0)$. The differential equation then becomes

$$\frac{dy}{dt} = \frac{k}{ws}\,\frac{1}{t + t_0}.$$

Integrating, we obtain

$$y = \frac{k}{ws}\,[\ln(t + t_0) + c]$$

where $c$ is a constant. Now when $t = 0$, $y = 0$ so $c = -\ln t_0$ and

$$y = \frac{k}{ws}\ln\left(1 + \frac{t}{t_0}\right).$$

Finally, from the fact that when $t = 1$, $y = 2$ and when $t = 2$, $y = 3$, we obtain

$$\left(1 + \frac{2}{t_0}\right)^2 = \left(1 + \frac{1}{t_0}\right)^3.$$

Expanding and simplifying gives $t_0^2 + t_0 - 1 = 0$. Since $t_0 > 0$, we find $t_0 \approx 0.618$ hours $\approx 37$ minutes. Thus it started snowing at about 11:23 in the morning.

**38.** At time $t$, when the population is 2 million cells, the differential equation $P'(t) = 0.15P(t)$ gives the rate of increase at time $t$. Thus, when $P(t) = 2$ (million cells), the rate of increase is $P'(t) = 0.15(2) = 0.3$ million cells per hour or 300,000 cells per hour.

**39.** Setting $A'(t) = -0.002$ and solving $A'(t) = -0.0004332A(t)$ for $A(t)$, we obtain

$$A(t) = \frac{A'(t)}{-0.0004332} = \frac{-0.002}{-0.0004332} \approx 4.6 \text{ grams.}$$

**40.** $(1):\ \dfrac{dP}{dt} = kP$ is linear

$(2):\ \dfrac{dA}{dt} = kA$ is linear

$(3):\ \dfrac{dT}{dt} = k\,(T - T_m)$ is linear

$(5):\ \dfrac{dx}{dt} = kx\,(n + 1 - x)$ is nonlinear

$(6):\ \dfrac{dX}{dt} = k\,(\alpha - X)\,(\beta - X)$ is nonlinear

$(8):\ \dfrac{dA}{dt} = 6 - \dfrac{A}{100}$ is linear

$(10):\ \dfrac{dh}{dt} = -\dfrac{A_h}{A_w}\sqrt{2gh}$ is nonlinear

$(11):\ L\dfrac{d^2q}{dt^2} + R\dfrac{dq}{dt} + \dfrac{1}{C}q = E(t)$ is linear

$(12):\ \dfrac{d^2s}{dt^2} = -g$ is linear

$(14):\ m\dfrac{dv}{dt} = mg - kv$ is linear

$(15):\ m\dfrac{d^2s}{dt^2} + k\dfrac{ds}{dt} = mg$ is linear

$(16):\ \dfrac{d^2x}{dt^2} - \dfrac{64}{L}x = 0$ is linear

$(17):$ linearity or nonlinearity is determined by the manner in which $W$ and $T_1$ involve $x$.

## Chapter 1 in Review

**1.** $\dfrac{d}{dx}c_1e^{10x} = 10\overbrace{c_1e^{10x}}^{y}; \qquad \dfrac{dy}{dx} = 10y$

**2.** $\dfrac{d}{dx}(5+c_1e^{-2x}) = -2c_1e^{-2x} = -2(\overbrace{5+c_1e^{-2x}}^{y}-5); \qquad \dfrac{dy}{dx} = -2(y-5) \quad \text{or} \quad \dfrac{dy}{dx} = -2y+10$

**3.** $\dfrac{d}{dx}(c_1\cos kx + c_2\sin kx) = -kc_1\sin kx + kc_2\cos kx;$

$$\frac{d^2}{dx^2}(c_1\cos kx + c_2\sin kx) = -k^2c_1\cos kx - k^2c_2\sin kx = -k^2(\overbrace{c_1\cos kx + c_2\sin kx}^{y});$$

$$\frac{d^2y}{dx^2} = -k^2y \quad \text{or} \quad \frac{d^2y}{dx^2} + k^2y = 0$$

**4.** $\dfrac{d}{dx}(c_1\cosh kx + c_2\sinh kx) = kc_1\sinh kx + kc_2\cosh kx;$

$$\frac{d^2}{dx^2}(c_1\cosh kx + c_2\sinh kx) = k^2c_1\cosh kx + k^2c_2\sinh kx = k^2(\overbrace{c_1\cosh kx + c_2\sinh kx}^{y});$$

$$\frac{d^2y}{dx^2} = k^2y \quad \text{or} \quad \frac{d^2y}{dx^2} - k^2y = 0$$

**5.** $y = c_1e^x + c_2xe^x; \qquad y' = c_1e^x + c_2xe^x + c_2e^x; \qquad y'' = c_1e^x + c_2xe^x + 2c_2e^x;$

$$y'' + y = 2(c_1e^x + c_2xe^x) + 2c_2e^x = 2(c_1e^x + c_2xe^x + c_2e^x) = 2y'; \qquad y'' - 2y' + y = 0$$

**6.** $y' = -c_1e^x\sin x + c_1e^x\cos x + c_2e^x\cos x + c_2e^x\sin x;$

$y'' = -c_1e^x\cos x - c_1e^x\sin x - c_1e^x\sin x + c_1e^x\cos x - c_2e^x\sin x + c_2e^x\cos x + c_2e^x\cos x + c_2e^x\sin x$

$$= -2c_1e^x\sin x + 2c_2e^x\cos x;$$

$$y'' - 2y' = -2c_1e^x\cos x - 2c_2e^x\sin x = -2y; \qquad y'' - 2y' + 2y = 0$$

**7.** a, d **8.** c **9.** b **10.** a, c **11.** b **12.** a, b, d

**13.** A few solutions are $y = 0$, $y = c$, and $y = e^x$.

**14.** Easy solutions to see are $y = 0$ and $y = 3$.

**15.** The slope of the tangent line at $(x, y)$ is $y'$, so the differential equation is $y' = x^2 + y^2$.

**16.** The rate at which the slope changes is $dy'/dx = y''$, so the differential equation is $y'' = -y'$ or $y'' + y' = 0$.

**17.** **(a)** The domain is all real numbers.

**(b)** Since $y' = 2/3x^{1/3}$, the solution $y = x^{2/3}$ is undefined at $x = 0$. This function is a solution of the differential equation on $(-\infty, 0)$ and also on $(0, \infty)$.

**18.** **(a)** Differentiating $y^2 - 2y = x^2 - x + c$ we obtain $2yy' - 2y' = 2x - 1$ or $(2y - 2)y' = 2x - 1$.

**(b)** Setting $x = 0$ and $y = 1$ in the solution we have $1 - 2 = 0 - 0 + c$ or $c = -1$. Thus, a solution of the initial-value problem is $y^2 - 2y = x^2 - x - 1$.

**(c)** Solving the equation $y^2 - 2y - (x^2 - x - 1) = 0$ by the quadratic formula we get $y = (2 \pm \sqrt{4 + 4(x^2 - x - 1)}\,)/2 = 1 \pm \sqrt{x^2 - x} = 1 \pm \sqrt{x(x-1)}$. Since $x(x-1) \geq 0$ for $x \leq 0$ or $x \geq 1$, we see that neither $y = 1 + \sqrt{x(x-1)}$ nor $y = 1 - \sqrt{x(x-1)}$ is differentiable at $x = 0$. Thus, both functions are solutions of the differential equation, but neither is a solution of the initial-value problem.

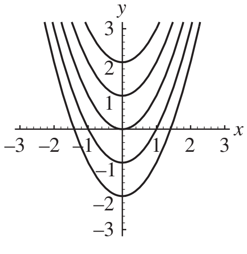
**19.** Setting $x = x_0$ and $y = 1$ in $y = -2/x + x$, we get

$$1 = -\frac{2}{x_0} + x_0 \qquad \text{or} \qquad x_0^2 - x_0 - 2 = (x_0 - 2)(x_0 + 1) = 0.$$

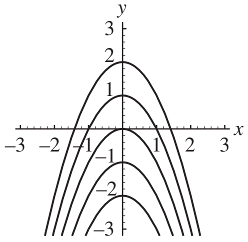
Thus, $x_0 = 2$ or $x_0 = -1$. Since $x \neq 0$ in $y = -2/x + x$, we see that $y = -2/x + x$ is a solution of the initial-value problem $xy' + y = 2x$, $y(-1) = 1$, on the interval $(-\infty, 0)$ because $-1 < 0$, and $y = -2/x + x$ is a solution of the initial-value problem $xy' + y = 2x$, $y(2) = 1$, on the interval $(0, \infty)$ because $2 > 0$.

**20.** From the differential equation, $y'(1) = 1^2 + [y(1)]^2 = 1 + (-1)^2 = 2 > 0$, so $y(x)$ is increasing in some neighborhood of $x = 1$. From $y'' = 2x + 2yy'$ we have $y''(1) = 2(1) + 2(-1)(2) = -2 < 0$, so $y(x)$ is concave down in some neighborhood of $x = 1$.

**21.** **(a)**



$y = x^2 + c_1$



$y = -x^2 + c_2$

**(b)** When $y = x^2 + c_1$, $y' = 2x$ and $(y')^2 = 4x^2$. When $y = -x^2 + c_2$, $y' = -2x$ and $(y')^2 = 4x^2$.

**(c)** Pasting together $x^2$, $x \geq 0$, and $-x^2$, $x \leq 0$, we get $y = \begin{cases} -x^2, & x \leq 0 \\ x^2, & x > 0\,. \end{cases}$

**22.** The slope of the tangent line is $y'\big|_{(-1,4)} = 6\sqrt{4} + 5(-1)^3 = 7$.

**23.** Differentiating $y = x\sin x + x\cos x$ we get

$$y' = x\cos x + \sin x - x\sin x + \cos x$$

and

$$\begin{aligned} y'' &= -x\sin x + \cos x + \cos x - x\cos x - \sin x - \sin x \\ &= -x\sin x - x\cos x + 2\cos x - 2\sin x. \end{aligned}$$

Thus

$$y'' + y = -x\sin x - x\cos x + 2\cos x - 2\sin x + x\sin x + x\cos x = 2\cos x - 2\sin x.$$

An interval of definition for the solution is $(-\infty, \infty)$.

**24.** Differentiating $y = x\sin x + (\cos x)\ln(\cos x)$ we get

$$\begin{aligned} y' &= x\cos x + \sin x + \cos x\left(\frac{-\sin x}{\cos x}\right) - (\sin x)\ln(\cos x) \\ &= x\cos x + \sin x - \sin x - (\sin x)\ln(\cos x) \\ &= x\cos x - (\sin x)\ln(\cos x) \end{aligned}$$

and

$$\begin{aligned} y'' &= -x\sin x + \cos x - \sin x\left(\frac{-\sin x}{\cos x}\right) - (\cos x)\ln(\cos x) \\ &= -x\sin x + \cos x + \frac{\sin^2 x}{\cos x} - (\cos x)\ln(\cos x) \\ &= -x\sin x + \cos x + \frac{1 - \cos^2 x}{\cos x} - (\cos x)\ln(\cos x) \\ &= -x\sin x + \cos x + \sec x - \cos x - (\cos x)\ln(\cos x) \\ &= -x\sin x + \sec x - (\cos x)\ln(\cos x). \end{aligned}$$

Thus

$$y'' + y = -x\sin x + \sec x - (\cos x)\ln(\cos x) + x\sin x + (\cos x)\ln(\cos x) = \sec x.$$

To obtain an interval of definition we note that the domain of $\ln x$ is $(0, \infty)$, so we must have $\cos x > 0$. Thus, an interval of definition is $(-\pi/2, \pi/2)$.

**25.** Differentiating $y = \sin(\ln x)$ we obtain $y' = \cos(\ln x)/x$ and $y'' = -[\sin(\ln x) + \cos(\ln x)]/x^2$. Then

$$x^2y'' + xy' + y = x^2\left(-\frac{\sin(\ln x) + \cos(\ln x)}{x^2}\right) + x\frac{\cos(\ln x)}{x} + \sin(\ln x) = 0.$$

An interval of definition for the solution is $(0, \infty)$.

**26.** Differentiating $y = \cos(\ln x)\ln(\cos(\ln x)) + (\ln x)\sin(\ln x)$ we obtain

$$y' = \cos(\ln x)\frac{1}{\cos(\ln x)}\left(-\frac{\sin(\ln x)}{x}\right) + \ln(\cos(\ln x))\left(-\frac{\sin(\ln x)}{x}\right) + \ln x\frac{\cos(\ln x)}{x} + \frac{\sin(\ln x)}{x}$$

$$= -\frac{\ln(\cos(\ln x))\sin(\ln x)}{x} + \frac{(\ln x)\cos(\ln x)}{x}$$

and

$$y'' = -x\left[\ln(\cos(\ln x))\frac{\cos(\ln x)}{x} + \sin(\ln x)\frac{1}{\cos(\ln x)}\left(-\frac{\sin(\ln x)}{x}\right)\right]\frac{1}{x^2}$$

$$+ \ln(\cos(\ln x))\sin(\ln x)\frac{1}{x^2} + x\left[(\ln x)\left(-\frac{\sin(\ln x)}{x}\right) + \frac{\cos(\ln x)}{x}\right]\frac{1}{x^2} - (\ln x)\cos(\ln x)\frac{1}{x^2}$$

$$= \frac{1}{x^2}\left[-\ln(\cos(\ln x))\cos(\ln x) + \frac{\sin^2(\ln x)}{\cos(\ln x)} + \ln(\cos(\ln x))\sin(\ln x)\right.$$

$$\left. - (\ln x)\sin(\ln x) + \cos(\ln x) - (\ln x)\cos(\ln x)\right].$$

Then

$$x^2y'' + xy' + y = -\ln(\cos(\ln x))\cos(\ln x) + \frac{\sin^2(\ln x)}{\cos(\ln x)} + \ln(\cos(\ln x))\sin(\ln x) - (\ln x)\sin(\ln x)$$

$$+ \cos(\ln x) - (\ln x)\cos(\ln x) - \ln(\cos(\ln x))\sin(\ln x)$$

$$+ (\ln x)\cos(\ln x) + \cos(\ln x)\ln(\cos(\ln x)) + (\ln x)\sin(\ln x)$$

$$= \frac{\sin^2(\ln x)}{\cos(\ln x)} + \cos(\ln x) = \frac{\sin^2(\ln x) + \cos^2(\ln x)}{\cos(\ln x)} = \frac{1}{\cos(\ln x)} = \sec(\ln x).$$

To obtain an interval of definition, we note that the domain of $\ln x$ is $(0, \infty)$, so we must have $\cos(\ln x) > 0$. Since $\cos x > 0$ when $-\pi/2 < x < \pi/2$, we require $-\pi/2 < \ln x < \pi/2$. Since $e^x$ is an increasing function, this is equivalent to $e^{-\pi/2} < x < e^{\pi/2}$. Thus, an interval of definition is $(e^{-\pi/2}, e^{\pi/2})$. (Much of this problem is more easily done using a computer algebra system such as *Mathematica* or *Maple*.)

*In Problems 27 - 30 we use (12) of Section 1.1 and the Product Rule.*

**27.**

$$y = e^{\cos x}\int_0^x te^{-\cos t}\,dt$$

$$\frac{dy}{dx} = e^{\cos x}\left(xe^{-\cos x}\right) - \sin xe^{\cos x}\int_0^x te^{-\cos t}\,dt$$

$$\frac{dy}{dx} + (\sin x)y = e^{\cos x}xe^{-\cos x} - \sin xe^{\cos x}\int_0^x te^{-\cos t}\,dt + \sin x\left(e^{\cos x}\int_0^x te^{-\cos t}\,dt\right)$$

$$= x - \sin xe^{\cos x}\int_0^x te^{-\cos t}\,dt + \sin xe^{\cos x}\int_0^x te^{-\cos t}\,dt = x$$

**28.**

$$y = e^{x^2}\int_0^x e^{t-t^2}\,dt$$

$$\frac{dy}{dx} = e^{x^2}e^{x-x^2} + 2xe^{x^2}\int_0^x e^{t-t^2}\,dt$$

$$\frac{dy}{dx} - 2xy = e^{x^2}e^{x-x^2} + 2xe^{x^2}\int_0^x e^{t-t^2}\,dt - 2x\left(e^{x^2}\int_0^x e^{t-t^2}\,dt\right) = e^x$$

**29.**

$$y = x\int_1^x \frac{e^{-t}}{t}\,dt$$

$$y' = x\frac{e^{-x}}{x} + \int_1^x \frac{e^{-t}}{t}\,dt = e^{-x} + \int_1^x \frac{e^{-t}}{t}\,dt$$

$$y'' = -e^{-x} + \frac{e^{-x}}{x}$$

$$\begin{aligned} x^2y'' + \left(x^2 - x\right)y' + (1-x)\,y &= \left(-x^2e^{-x} + xe^{-x}\right) \\ &\quad + \left(x^2e^{-x} + x^2\int_1^x \frac{e^{-t}}{t}\,dt - xe^{-x} - x\int_1^x \frac{e^{-t}}{t}\,dt\right) \\ &\quad + \left(x\int_1^x \frac{e^{-t}}{t}\,dt - x^2\int_1^x \frac{e^{-t}}{t}\,dt\right) = 0 \end{aligned}$$

**30.**

$$\begin{aligned} y &= \sin x\int_0^x e^{t^2}\cos t\,dt - \cos x\int_0^x e^{t^2}\sin t\,dt \\ y' &= \sin x\left(e^{x^2}\cos x\right) + \cos x\int_0^x e^{t^2}\cos t\,dt - \cos x\left(e^{x^2}\sin x\right) + \sin x\int_0^x e^{t^2}\sin t\,dt \\ &= \cos x\int_0^x e^{t^2}\cos t\,dt + \sin x\int_0^x e^{t^2}\sin t\,dt \\ y'' &= \cos x\left(e^{x^2}\cos x\right) - \sin x\int_0^x e^{t^2}\cos t\,dt + \sin x\left(e^{x^2}\sin x\right) + \cos x\int_0^x e^{t^2}\sin t\,dt \\ &= e^{x^2}\left(\cos^2 x + \sin^2 x\right) - \left(\overbrace{\sin x\int_0^x e^{t^2}\cos t\,dt - \cos x\int_0^x e^{t^2}\sin t\,dt}^{y}\right) \\ &= e^{x^2} - y \end{aligned}$$

$$y'' + y = e^{x^2} - y + y = e^{x^2}$$

**31.** Using implicit differentiation we get

$$\begin{aligned} x^3y^3 &= x^3+1 \\ 3x^2\cdot y^3 + x^3\cdot 3y^2\frac{dy}{dx} &= 3x^2 \\ \frac{3x^2y^3}{3x^2y^2} + \frac{x^3 3y^2}{3x^2y^2}\frac{dy}{dx} &= \frac{3x^2}{3x^2y^2} \\ y + x\frac{dy}{dx} &= \frac{1}{y^2} \end{aligned}$$

**32.** Using implicit differentiation we get

$$\begin{aligned} (x-5)^2 + y^2 &= 1 \\ 2(x-5) + 2y\frac{dy}{dx} &= 0 \\ 2y\frac{dy}{dx} &= -2(x-5) \\ y\frac{dy}{dx} &= -(x-5) \\ \left(y\frac{dy}{dx}\right)^2 &= (x-5)^2 \\ \left(\frac{dy}{dx}\right)^2 &= \frac{(x-5)^2}{y^2} \end{aligned}$$

Now from the original equation, isolating the first term leads to $(x-5)^2 = 1-y^2$. Continuing from the last line of our proof we now have

$$\left(\frac{dy}{dx}\right)^2 = \frac{(x-5)^2}{y^2} = \frac{1-y^2}{y^2} = \frac{1}{y^2} - 1$$

Adding 1 to both sides leads to the desired result.

**33.** Using implicit differentiation we get

$$\begin{aligned} y^3 + 3y &= 1 - 3x \\ 3y^2y' + 3y' &= -3 \\ y^2y' + y' &= -1 \\ (y^2+1)y' &= -1 \\ y' &= \frac{-1}{y^2+1} \end{aligned}$$

Differentiating the last line and remembering to use the quotient rule on the right side leads to

$$y'' = \frac{2yy'}{(y^2+1)^2}$$

Now since $y' = -1\big/\left(y^2+1\right)$ we can write the last equation as

$$y'' = \frac{2y}{(y^2+1)^2}y' = \frac{2y}{(y^2+1)^2}\frac{-1}{(y^2+1)} = 2y\left(\frac{-1}{y^2+1}\right)^3 = 2y(y')^3$$

which is what we wanted to show.

**34.** Using implicit differentiation we get

$$\begin{aligned} y &= e^{xy} \\ y' &= e^{xy}(y+xy') \\ y' &= ye^{xy} + xe^{xy}y' \\ (1-xe^{xy})y' &= ye^{xy} \end{aligned}$$

Now since $y = e^{xy}$, substitute this into the last line to get

$$(1-xy)y' = yy$$

or $(1-xy)y' = y^2$ which is what we wanted to show.

*In Problem 35–38, $y = c_1e^{3x} + c_2e^{-x} - 2x$ is given as a two-parameter family of solutions of the second-order differential equation $y'' - 2y' - 3y = 6x + 4$.*

**35.** If $y(0) = 0$ and $y'(0) = 0$, then

$$\begin{aligned} c_1 + c_2 &= 0 \\ 3c_1 - c_2 - 2 &= 0 \end{aligned}$$

so $c_1 = \frac{1}{2}$ and $c_2 = -\frac{1}{2}$. Thus $y = \frac{1}{2}\,e^{3x} - half\,e^{-x} - 2x$.

**36.** If $y(0) = 1$ and $y'(0) = -3$, then

$$\begin{aligned} c_1 + c_2 &= 1 \\ 3c_1 - c_2 - 2 &= -3 \end{aligned}$$

so $c_1 = 0$ and $c_2 = 1$. Thus $y = e^{-x} - 2x$.

**37.** If $y(1) = 4$ and $y'(1) = -2$, then

$$\begin{aligned} c_1e^3 + c_2e^{-1} - 2 &= 4 \\ 3c_1e^3 - c_2e^{-1} - 2 &= -2 \end{aligned}$$

so $c_1 = \frac{3}{2}\,e^{-3}$ and $c_2 = \frac{9}{2}\,e$. Thus $y = \frac{3}{2}\,e^{3x-3} + \frac{9}{2}\,e^{-x+1} - 2x$.

**38.** If $y(-1) = 0$ and $y'(-1) = 1$, then

$$\begin{aligned} c_1e^{-3} + c_2e + 2 &= 0 \\ 3c_1e^{-3} - c_2e - 2 &= 1 \end{aligned}$$

so $c_1 = \frac{1}{4}\,e^3$ and $c_2 = -\frac{9}{4}, e^{-1}$. Thus $y = \frac{1}{4}\,e^{3x+3} - \frac{9}{4}\,e^{-x-1} - 2x$.

**39.** From the graph we see that estimates for $y_0$ and $y_1$ are $y_0 = -3$ and $y_1 = 0$.

**40.** The differential equation is

$$\frac{dh}{dt} = -\frac{cA_0}{A_w}\sqrt{2gh}\,.$$

Using $A_0 = \pi(1/24)^2 = \pi/576$, $A_w = \pi(2)^2 = 4\pi$, and $g = 32$, this becomes

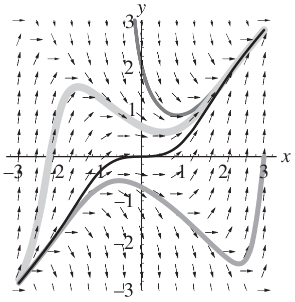
$$\frac{dh}{dt} = -\frac{c\pi/576}{4\pi}\sqrt{64h} = -\frac{c}{288}\sqrt{h}\,.$$

# Chapter 2

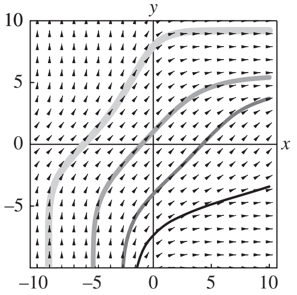
## First-Order Differential Equations

### 2.1 Solution Curves Without a Solution

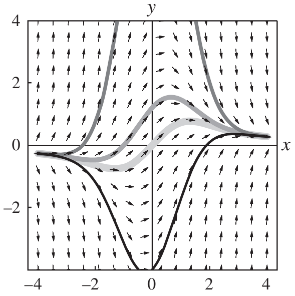
**1.**



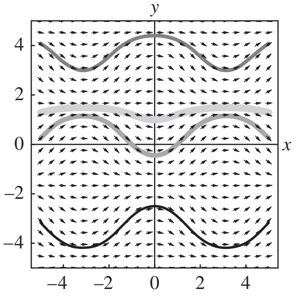
**2.**



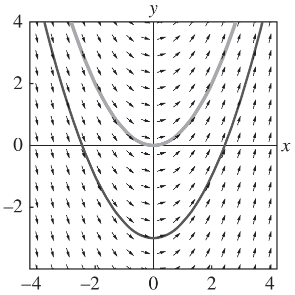
**3.**



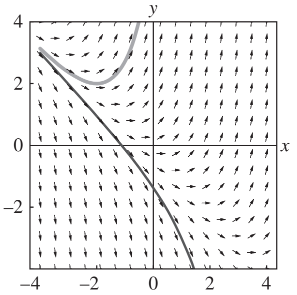
**4.**



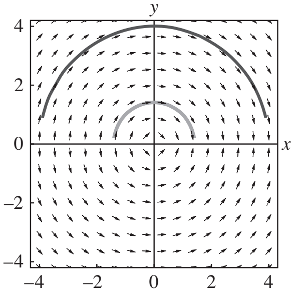
**5.**



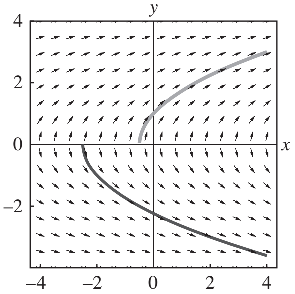
**6.**



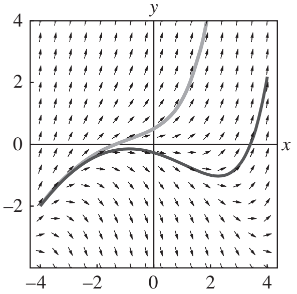
**7.**



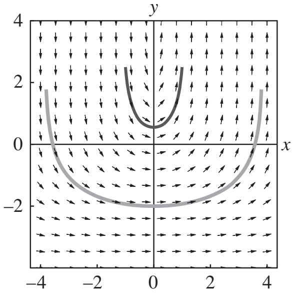
**8.**



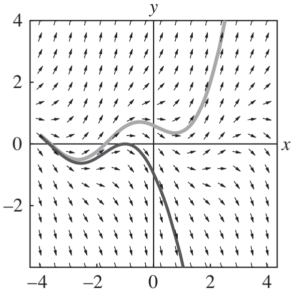
**9.**



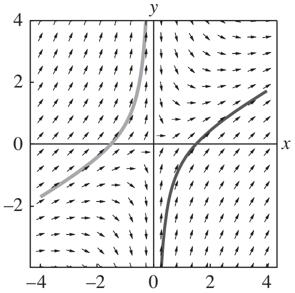
**10.**



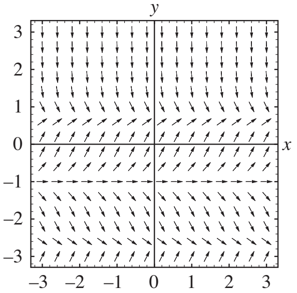
**11.**



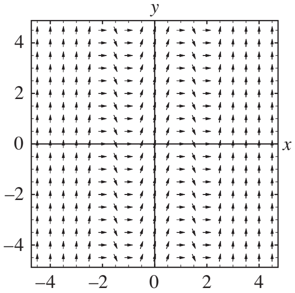
**12.**



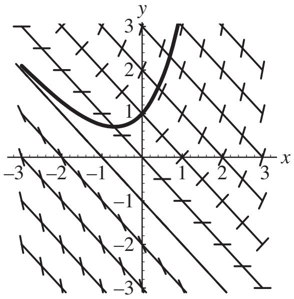
**13.**



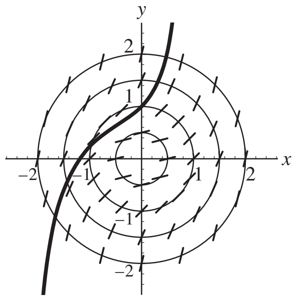
**14.**



**15. (a)** The isoclines have the form $y = -x + c$, which are straight lines with slope $-1$.



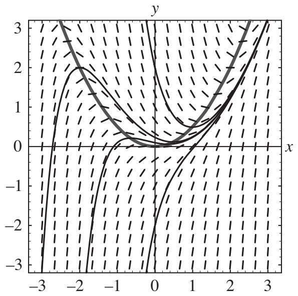
**(b)** The isoclines have the form $x^2 + y^2 = c$, which are circles centered at the origin.



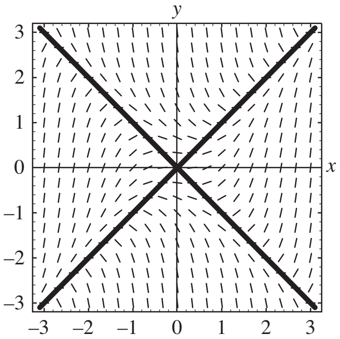
**16.** **(a)** When $x = 0$ or $y = 4$, $dy/dx = -2$ so the lineal elements have slope $-2$. When $y = 3$ or $y = 5$, $dy/dx = x - 2$, so the lineal elements at $(x, 3)$ and $(x, 5)$ have slopes $x - 2$.

**(b)** At $(0, y_0)$ the solution curve is headed down. If $y \to \infty$ as $x$ increases, the graph must eventually turn around and head up, but while heading up it can never cross $y = 4$ where a tangent line to a solution curve must have slope $-2$. Thus, $y$ cannot approach $\infty$ as $x$ approaches $\infty$.

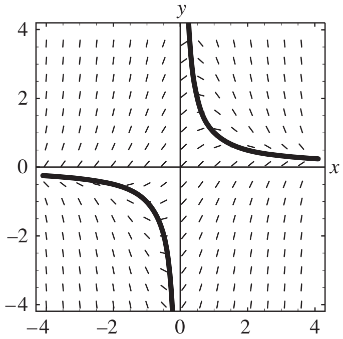
**17.** When $y < \frac{1}{2}x^2$, $y' = x^2 - 2y$ is positive and the portions of solution curves "outside" the nullcline parabola are increasing. When $y > \frac{1}{2}x^2$, $y' = x^2 - 2y$ is negative and the portions of the solution curves "inside" the nullcline parabola are decreasing.



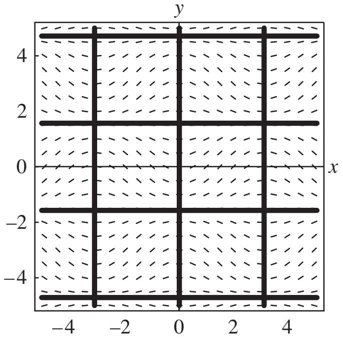
**18.** **(a)** Any horizontal lineal element should be at a point on a nullcline. In Problem 1 the nullclines are $x^2 - y^2 = 0$ or $y = \pm x$. In Problem 3 the nullclines are $1 - xy = 0$ or $y = 1/x$. In Problem 4 the nullclines are $(\sin x)\cos y = 0$ or $x = n\pi$ and $y = \pi/2 + n\pi$, where $n$ is an integer. The graphs on the next page show the nullclines for the equations in Problems 1, 3, and 4 superimposed on the corresponding direction field.



Problem 1



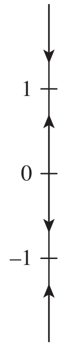
Problem 3



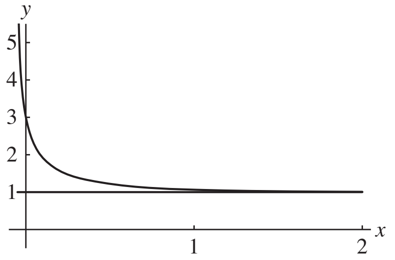
Problem 4

**(b)** An autonomous first-order differential equation has the form $y' = f(y)$. Nullclines have the form $y = c$ where $f(c) = 0$. These are the graphs of the equilibrium solutions of the differential equation.

**19.** Writing the differential equation in the form $dy/dx = y(1-y)(1+y)$ we see that critical points are $y = -1$, $y = 0$, and $y = 1$. The phase portrait is shown at the right.



**(a)**



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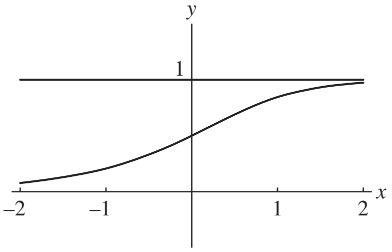
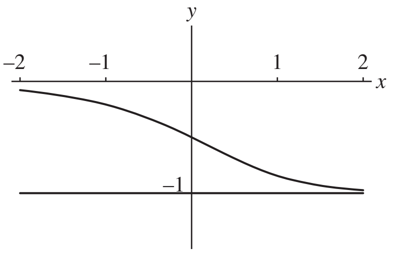
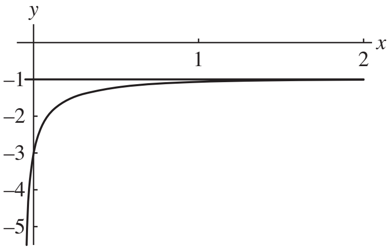


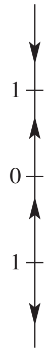
**(c)**



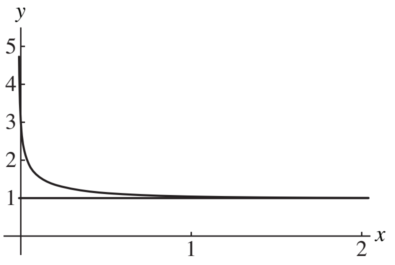
**(d)**



**20.** Writing the differential equation in the form $dy/dx = y^2(1-y)(1+y)$ we see that critical points are $y = -1$, $y = 0$, and $y = 1$. The phase portrait is shown at the right.



**(a)**



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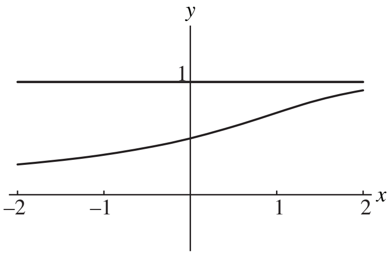
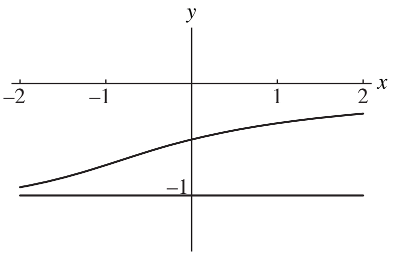
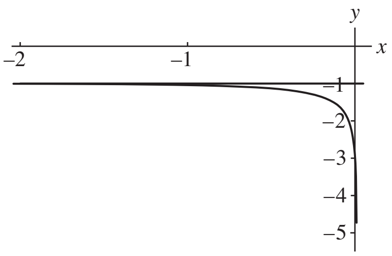


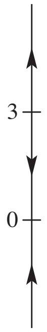
**(c)**

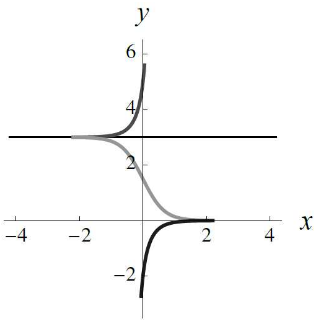


**(d)**

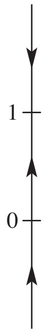


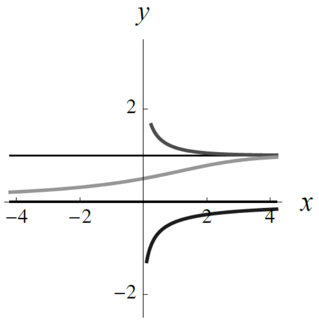
**21.** Solving $y^2 - 3y = y(y-3) = 0$ we obtain the critical points 0 and 3. From the phase portrait we see that 0 is asymptotically stable (attractor) and 3 is unstable (repeller).



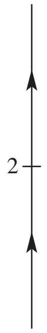


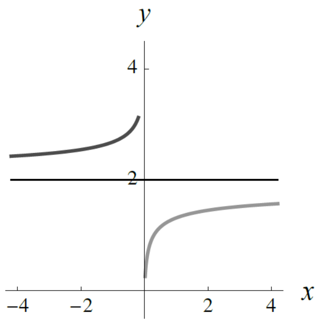
**22.** Solving $y^2 - y^3 = y^2(1-y) = 0$ we obtain the critical points 0 and 1. From the phase portrait we see that 1 is asymptotically stable (attractor) and 0 is semi-stable.



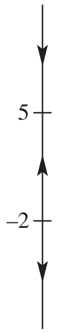


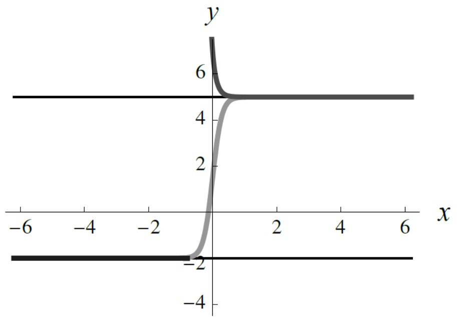
**23.** Solving $(y-2)^4 = 0$ we obtain the critical point 2. From the phase portrait we see that 2 is semi-stable.



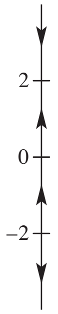


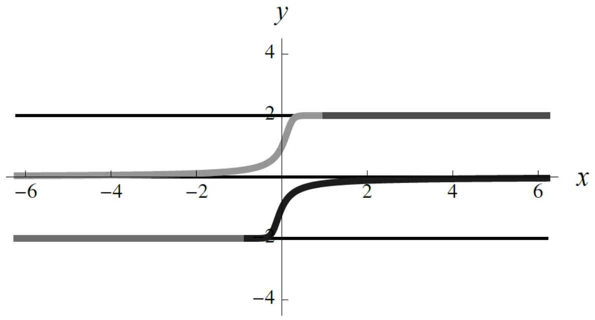
**24.** Solving $10 + 3y - y^2 = (5 - y)(2 + y) = 0$ we obtain the critical points $-2$ and $5$. From the phase portrait we see that 5 is asymptotically stable (attractor) and $-2$ is unstable (repeller).



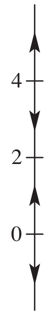


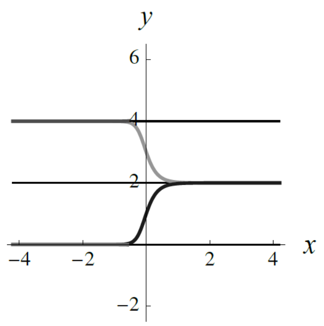
**25.** Solving $y^2(4 - y^2) = y^2(2 - y)(2 + y) = 0$ we obtain the critical points $-2$, $0$, and $2$. From the phase portrait we see that 2 is asymptotically stable (attractor), 0 is semi-stable, and $-2$ is unstable (repeller).



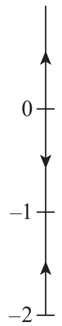


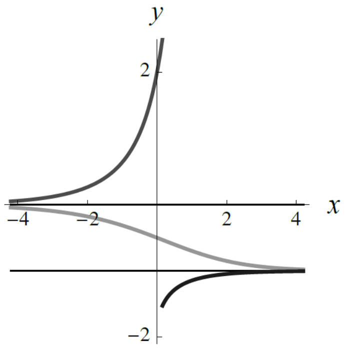
**26.** Solving $y(2 - y)(4 - y) = 0$ we obtain the critical points 0, 2, and 4. From the phase portrait we see that 2 is asymptotically stable (attractor) and 0 and 4 are unstable (repellers).



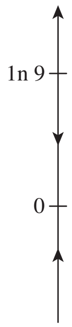


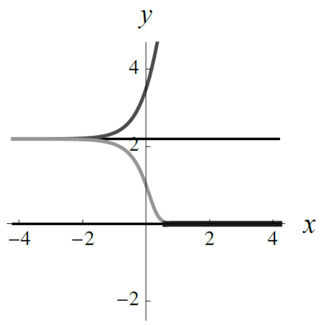
**27.** Solving $y\ln(y+2) = 0$ we obtain the critical points $-1$ and $0$. From the phase portrait we see that $-1$ is asymptotically stable (attractor) and $0$ is unstable (repeller).



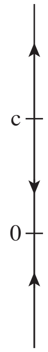


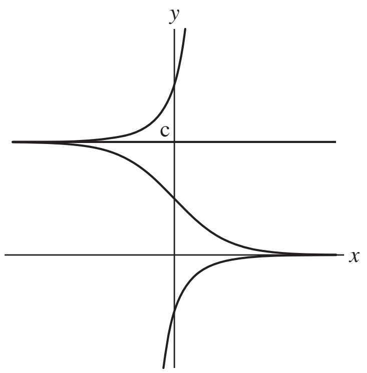
**28.** Solving $ye^y - 9y = y(e^y - 9) = 0$ (since $e^y$ is always positive) we obtain the critical points $0$ and $\ln 9$. From the phase portrait we see that $0$ is asymptotically stable (attractor) and $\ln 9$ is unstable (repeller).



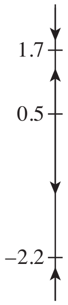


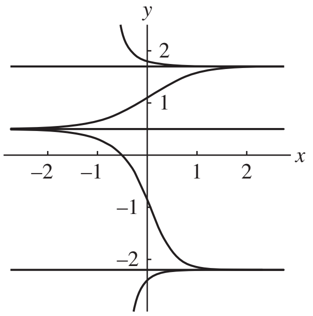
**29.** The critical points are $0$ and $c$ because the graph of $f(y)$ is 0 at these points. Since $f(y) > 0$ for $y < 0$ and $y > c$, the graph of the solution is increasing on the $y$-intervals $(-\infty, 0)$ and $(c, \infty)$. Since $f(y) < 0$ for $0 < y < c$, the graph of the solution is decreasing on the $y$-interval $(0, c)$.



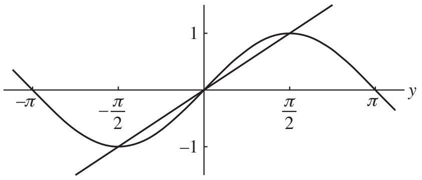


**30.** The critical points are approximately at $-2, 2$, $0.5$, and $1.7$. Since $f(y) > 0$ for $y < -2.2$ and $0.5 < y < 1.7$, the graph of the solution is increasing on the $y$-intervals $(-\infty, -2.2)$ and $(0.5, 1.7)$. Since $f(y) < 0$ for $-2.2 < y < 0.5$ and $y > 1.7$, the graph is decreasing on the $y$-interval $(-2.2, 0.5)$ and $(1.7, \infty)$.





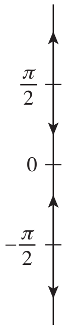
**31.** From the graphs of $z = \pi/2$ and $z = \sin y$ we see that $(2/\pi)y - \sin y = 0$ has only three solutions. By inspection we see that the critical points are $-\pi/2$, 0, and $\pi/2$.



From the graph at the right we see that

$$\frac{2}{\pi}y - \sin y \begin{cases} < 0 & \text{for} \quad y < -\pi/2 \\ > 0 & \text{for} \quad y > \pi/2 \end{cases}$$

$$\frac{2}{\pi}y - \sin y \begin{cases} > 0 & \text{for} \quad -\pi/2 < y < 0 \\ < 0 & \text{for} \quad 0 < y < \pi/2 \end{cases}$$



This enables us to construct the phase portrait shown at the right. From this portrait we see that $\pi/2$ and $-\pi/2$ are unstable (repellers), and 0 is asymptotically stable (attractor).

**32.** For $dy/dx = 0$ every real number is a critical point, and hence all critical points are nonisolated.

**33.** Recall that for $dy/dx = f(y)$ we are assuming that $f$ and $f'$ are continuous functions of $y$ on some interval $I$. Now suppose that the graph of a nonconstant solution of the differential equation crosses the line $y = c$. If the point of intersection is taken as an initial condition we have two distinct solutions of the initial-value problem. This violates uniqueness, so the graph of any nonconstant solution must lie entirely on one side of any equilibrium solution. Since $f$ is continuous it can only change signs at a point where it is 0. But this is a critical point. Thus, $f(y)$ is completely positive or completely negative in each region $R_i$. If $y(x)$ is oscillatory or has a relative extremum, then it must have a horizontal tangent line at some point $(x_0, y_0)$. In this case $y_0$ would be a critical point of the differential equation, but we saw above that the graph of a nonconstant solution cannot intersect the graph of the equilibrium solution $y = y_0$.

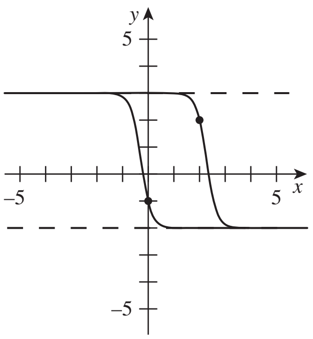
**34.** By Problem 33, a solution $y(x)$ of $dy/dx = f(y)$ cannot have relative extrema and hence must be monotone. Since $y'(x) = f(y) > 0$, $y(x)$ is monotone increasing, and since $y(x)$ is bounded above by $c_2$, $\lim_{x\to\infty} y(x) = L$, where $L \le c_2$. We want to show that $L = c_2$. Since $L$ is a horizontal asymptote of $y(x)$, $\lim_{x\to\infty} y'(x) = 0$. Using the fact that $f(y)$ is continuous we have

$$f(L) = f\left(\lim_{x\to\infty} y(x)\right) = \lim_{x\to\infty} f(y(x)) = \lim_{x\to\infty} y'(x) = 0.$$

But then $L$ is a critical point of $f$. Since $c_1 < L \le c_2$, and $f$ has no critical points between $c_1$ and $c_2$, $L = c_2$.

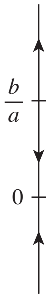
**35.** Assuming the existence of the second derivative, points of inflection of $y(x)$ occur where $y''(x) = 0$. From $dy/dx = f(y)$ we have $d^2y/dx^2 = f'(y)\,dy/dx$. Thus, the $y$-coordinate of a point of inflection can be located by solving $f'(y) = 0$. (Points where $dy/dx = 0$ correspond to constant solutions of the differential equation.)

**36.** Solving $y^2 - y - 6 = (y-3)(y+2) = 0$ we see that 3 and $-2$ are critical points. Now $d^2y/dx^2 = (2y-1)\,dy/dx = (2y-1)(y-3)(y+2)$, so the only possible point of inflection is at $y = \frac{1}{2}$, although the concavity of solutions can be different on either side of $y = -2$ and $y = 3$. Since $y''(x) < 0$ for $y < -2$ and $\frac{1}{2} < y < 3$, and $y''(x) > 0$ for $-2 < y < \frac{1}{2}$ and $y > 3$, we see that solution curves are concave down for $y < -2$ and $\frac{1}{2} < y < 3$ and concave up for $-2 < y < \frac{1}{2}$ and $y > 3$. Points of inflection of solutions of autonomous differential equations will have the same $y$-coordinates because between critical points they are horizontal translations of each other.



**37.** If (1) in the text has no critical points it has no constant solutions. The solutions have neither an upper nor lower bound. Since solutions are monotonic, every solution assumes all real values.

**38.** The critical points are 0 and $b/a$. From the phase portrait we see that 0 is an attractor and $b/a$ is a repeller. Thus, if an initial population satisfies $P_0 > b/a$, the population becomes unbounded as $t$ increases, most probably in finite time, i.e. $P(t) \to \infty$ as $t \to T$. If $0 < P_0 < b/a$, then the population eventually dies out, that is, $P(t) \to 0$ as $t \to \infty$. Since population $P > 0$ we do not consider the case $P_0 < 0$.



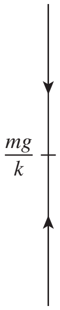
**39.** From the equation $dP/dt = k\,(P - h/k)$ we see that the only critical point of the autonomous differential equationis the positive number $h/k$. A phase portrait shows that this point is unstable, that is, $h/k$ is a repeller. For any initial condition $P(0) = P_0$ for which $0 < P_0 < h/k$, $dP/dt < 0$ which means $P(t)$ is monotonic decreasing and so the graph of $P(t)$ must cross the $t$-axis or the line $P - 0$ at some time $t_1 > 0$. But $P(t_1) = 0$ means the population is extinct at time $t_1$.

**40.** Writing the differential equation in the form

$$\frac{dv}{dt} = \frac{k}{m}\left(\frac{mg}{k} - v\right)$$

we see that a critical point is $mg/k$.

From the phase portrait we see that $mg/k$ is an asymptotically stable critical point. Thus, $\lim_{t\to\infty} v = mg/k$.

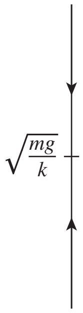


**41.** Writing the differential equation in the form

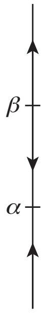
$$\frac{dv}{dt} = \frac{k}{m}\left(\frac{mg}{k} - v^2\right) = \frac{k}{m}\left(\sqrt{\frac{mg}{k}} - v\right)\left(\sqrt{\frac{mg}{k}} + v\right)$$

we see that the only physically meaningful critical point is $\sqrt{mg/k}$.

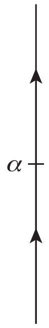
From the phase portrait we see that $\sqrt{mg/k}$ is an asymptotically stable critical point. Thus, $\lim_{t\to\infty} v = \sqrt{mg/k}$.



**42.** **(a)** From the phase portrait we see that critical points are $\alpha$ and $\beta$. Let $X(0) = X_0$. If $X_0 < \alpha$, we see that $X \to \alpha$ as $t \to \infty$. If $\alpha < X_0 < \beta$, we see that $X \to \alpha$ as $t \to \infty$. If $X_0 > \beta$, we see that $X(t)$ increases in an unbounded manner, but more specific behavior of $X(t)$ as $t \to \infty$ is not known.



**(b)** When $\alpha = \beta$ the phase portrait is as shown. If $X_0 < \alpha$, then $X(t) \to \alpha$ as $t \to \infty$. If $X_0 > \alpha$, then $X(t)$ increases in an unbounded manner. This could happen in a finite amount of time. That is, the phase portrait does not indicate that $X$ becomes unbounded as $t \to \infty$.



**(c)** When $k = 1$ and $\alpha = \beta$ the differential equation is $dX/dt = (\alpha - X)^2$. For $X(t) = \alpha - 1/(t+c)$ we have $dX/dt = 1/(t+c)^2$ and

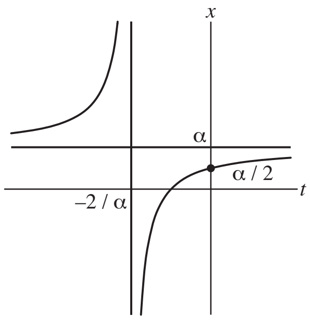
$$(\alpha - X)^2 = \left[\alpha - \left(\alpha - \frac{1}{t+c}\right)\right]^2 = \frac{1}{(t+c)^2} = \frac{dX}{dt}\,.$$

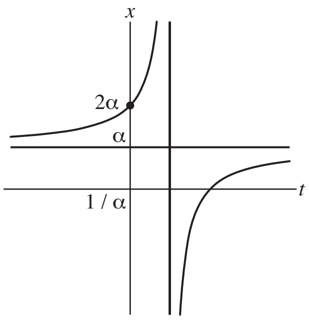
For $X(0) = \alpha/2$ we obtain

$$X(t) = \alpha - \frac{1}{t + 2/\alpha}\,.$$

For $X(0) = 2\alpha$ we obtain

$$X(t) = \alpha - \frac{1}{t - 1/\alpha}\,.$$





For $X_0 > \alpha$, $X(t)$ increases without bound up to $t = 1/\alpha$. For $t > 1/\alpha$, $X(t)$ increases but $X \to \alpha$ as $t \to \infty$.

## 2.2 Separable Variables

*In many of the following problems we will encounter an expression of the form* $\ln|g(y)| = f(x) + c$. *To solve for* $g(y)$ *we exponentiate both sides of the equation. This yields* $|g(y)| = e^{f(x)+c} = e^c e^{f(x)}$ *which implies* $g(y) = \pm e^c e^{f(x)}$. *Letting* $c_1 = \pm e^c$ *we obtain* $g(y) = c_1 e^{f(x)}$.

**1.** From $dy = \sin 5x\, dx$ we obtain $y = -\frac{1}{5}\cos 5x + c$.

**2.** From $dy = (x+1)^2\, dx$ we obtain $y = \frac{1}{3}(x+1)^3 + c$.

**3.** From $dy = -e^{-3x}\,dx$ we obtain $y = \frac{1}{3}e^{-3x} + c$.

**4.** From $\dfrac{1}{(y-1)^2}\,dy = dx$ we obtain $-\dfrac{1}{y-1} = x + c$ or $y = 1 - \dfrac{1}{x+c}$.

**5.** From $\dfrac{1}{y}\,dy = \dfrac{4}{x}\,dx$ we obtain $\ln|y| = 4\ln|x| + c$ or $y = c_1x^4$.

**6.** From $\dfrac{1}{y^2}\,dy = -2x\,dx$ we obtain $-\dfrac{1}{y} = -x^2 + c$ or $y = \dfrac{1}{x^2 + c_1}$.

**7.** From $e^{-2y}dy = e^{3x}dx$ we obtain $3e^{-2y} + 2e^{3x} = c$.

**8.** From $ye^y dy = \left(e^{-x} + e^{-3x}\right)dx$ we obtain $ye^y - e^y + e^{-x} + \dfrac{1}{3}e^{-3x} = c$.

**9.** From $\left(y + 2 + \dfrac{1}{y}\right)dy = x^2\ln x\,dx$ we obtain $\dfrac{y^2}{2} + 2y + \ln|y| = \dfrac{x^3}{3}\ln|x| - \dfrac{1}{9}x^3 + c$.

**10.** From $\dfrac{1}{(2y+3)^2}\,dy = \dfrac{1}{(4x+5)^2}\,dx$ we obtain $\dfrac{2}{2y+3} = \dfrac{1}{4x+5} + c$.

**11.** From $\dfrac{1}{\csc y}\,dy = -\dfrac{1}{\sec^2 x}\,dx$ or $\sin y\,dy = -\cos^2 x\,dx = -\frac{1}{2}(1 + \cos 2x)\,dx$ we obtain $-\cos y = -\frac{1}{2}x - \frac{1}{4}\sin 2x + c$ or $4\cos y = 2x + \sin 2x + c_1$.

**12.** From $2y\,dy = -\dfrac{\sin 3x}{\cos^3 3x}\,dx$ or $2y\,dy = -\tan 3x\sec^2 3x\,dx$ we obtain $y^2 = -\frac{1}{6}\sec^2 3x + c$.

**13.** From $\dfrac{e^y}{\left(e^y+1\right)^2}\,dy = \dfrac{-e^x}{\left(e^x+1\right)^3}\,dx$ we obtain $-\left(e^y+1\right)^{-1} = \frac{1}{2}\left(e^x+1\right)^{-2} + c$.

**14.** From $\dfrac{y}{\left(1+y^2\right)^{1/2}}\,dy = \dfrac{x}{\left(1+x^2\right)^{1/2}}\,dx$ we obtain $\left(1+y^2\right)^{1/2} = \left(1+x^2\right)^{1/2} + c$.

**15.** From $\dfrac{1}{S}\,dS = k\,dr$ we obtain $S = ce^{kr}$.

**16.** From $\dfrac{1}{Q-70}\,dQ = k\,dt$ we obtain $\ln|Q-70| = kt + c$ or $Q - 70 = c_1e^{kt}$.

**17.** From $\dfrac{1}{P-P^2}dP = \left(\dfrac{1}{P} + \dfrac{1}{1-P}\right)dP = dt$ we obtain $\ln|P| - \ln|1-P| = t + c$ so that $\ln\left|\dfrac{P}{1-P}\right| = t + c$ or $\dfrac{P}{1-P} = c_1e^t$. Solving for $P$ we have $P = \dfrac{c_1e^t}{1+c_1e^t}$.

**18.** From $\dfrac{1}{N}\,dN = \left(te^{t+2} - 1\right)dt$ we obtain $\ln|N| = te^{t+2} - e^{t+2} - t + c$ or $N = c_1e^{te^{t+2} - e^{t+2} - t}$.

**19.** From $\dfrac{y-2}{y+3}\,dy = \dfrac{x-1}{x+4}\,dx$ or $\left(1 - \dfrac{5}{y+3}\right)dy = \left(1 - \dfrac{5}{x+4}\right)dx$ we obtain

$y - 5\ln|y+3| = x - 5\ln|x+4| + c$ or $\left(\dfrac{x+4}{y+3}\right)^5 = c_1e^{x-y}$.

**20.** From $\dfrac{y+1}{y-1}\,dy = \dfrac{x+2}{x-3}\,dx$ or $\left(1+\dfrac{2}{y-1}\right)dy = \left(1+\dfrac{5}{x-3}\right)dx$ we obtain $y + 2\ln|y-1| = x + 5\ln|x-3| + c$ or $\dfrac{(y-1)^2}{(x-3)^5} = c_1e^{x-y}$.

**21.** From $x\,dx = \dfrac{1}{\sqrt{1-y^2}}\,dy$ we obtain $\frac{1}{2}x^2 = \sin^{-1}y + c$ or $y = \sin\left(\dfrac{x^2}{2}+c_1\right)$.

**22.** From $\dfrac{1}{y^2}\,dy = \dfrac{1}{e^x+e^{-x}}\,dx = \dfrac{e^x}{(e^x)^2+1}\,dx$ we obtain $-\dfrac{1}{y} = \tan^{-1}e^x + c$ or $y = -\dfrac{1}{\tan^{-1}e^x + c}$.

**23.** From $\dfrac{1}{x^2+1}\,dx = 4\,dt$ we obtain $\tan^{-1}x = 4t + c$. Using $x(\pi/4) = 1$ we find $c = -3\pi/4$. The solution of the initial-value problem is $\tan^{-1}x = 4t - \dfrac{3\pi}{4}$ or $x = \tan\left(4t - \dfrac{3\pi}{4}\right)$.

**24.** From $\dfrac{1}{y^2-1}\,dy = \dfrac{1}{x^2-1}\,dx$ or $\dfrac{1}{2}\left(\dfrac{1}{y-1} - \dfrac{1}{y+1}\right)dy = \dfrac{1}{2}\left(\dfrac{1}{x-1} - \dfrac{1}{x+1}\right)dx$ we obtain

$\ln|y-1| - \ln|y+1| = \ln|x-1| - \ln|x+1| + \ln c$ or $\dfrac{y-1}{y+1} = \dfrac{c(x-1)}{x+1}$. Using $y(2) = 2$ we find $c = 1$. A solution of the initial-value problem is $\dfrac{y-1}{y+1} = \dfrac{x-1}{x+1}$ or $y = x$.

**25.** From $\dfrac{1}{y}\,dy = \dfrac{1-x}{x^2}\,dx = \left(\dfrac{1}{x^2} - \dfrac{1}{x}\right)dx$ we obtain $\ln|y| = -\dfrac{1}{x} - \ln|x| = c$ or $xy = c_1e^{-1/x}$. Using $y(-1) = -1$ we find $c_1 = e^{-1}$. The solution of the initial-value problem is $xy = e^{-1-1/x}$ or $y = e^{-(1+1/x)}/x$.

**26.** From $\dfrac{1}{1-2y}\,dy = dt$ we obtain $-\frac{1}{2}\ln|1-2y| = t + c$ or $1 - 2y = c_1e^{-2t}$. Using $y(0) = 5/2$ we find $c_1 = -4$. The solution of the initial-value problem is $1 - 2y = -4e^{-2t}$ or $y = 2e^{-2t} + \frac{1}{2}$.

**27.** Separating variables and integrating we obtain

$$\frac{dx}{\sqrt{1-x^2}} - \frac{dy}{\sqrt{1-y^2}} = 0 \quad\text{and}\quad \sin^{-1}x - \sin^{-1}y = c.$$

Setting $x = 0$ and $y = \sqrt{3}/2$ we obtain $c = -\pi/3$. Thus, an implicit solution of the initial-value problem is $\sin^{-1}x - \sin^{-1}y = \pi/3$. Solving for $y$ and using an addition formula from trigonometry, we get

$$y = \sin\left(\sin^{-1}x + \frac{\pi}{3}\right) = x\cos\frac{\pi}{3} + \sqrt{1-x^2}\sin\frac{\pi}{3} = \frac{x}{2} + \frac{\sqrt{3}\sqrt{1-x^2}}{2}.$$

**28.** From $\dfrac{1}{1+(2y)^2}\,dy = \dfrac{-x}{1+\left(x^2\right)^2}\,dx$ we obtain

$$\frac{1}{2}\tan^{-1}2y = -\frac{1}{2}\tan^{-1}x^2 + c \quad \text{or} \quad \tan^{-1}2y + \tan^{-1}x^2 = c_1.$$

Using $y(1) = 0$ we find $c_1 = \pi/4$. Thus, an implicit solution of the initial-value problem is $\tan^{-1}2y + \tan^{-1}x^2 = \pi/4$. Solving for $y$ and using a trigonometric identity we get

$$\begin{aligned} 2y &= \tan\left(\frac{\pi}{4} - \tan^{-1}x^2\right) \\ y &= \frac{1}{2}\tan\left(\frac{\pi}{4} - \tan^{-1}x^2\right) \\ &= \frac{1}{2}\,\frac{\tan\frac{\pi}{4} - \tan\left(\tan^{-1}x^2\right)}{1 + \tan\frac{\pi}{4}\tan\left(\tan^{-1}x^2\right)} \\ &= \frac{1}{2}\,\frac{1-x^2}{1+x^2}\,. \end{aligned}$$

**29.** Separating variables and then proceeding as in Example 5 we get

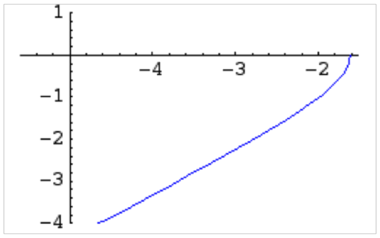
$$\begin{aligned} \frac{dy}{dx} &= ye^{-x^2} \\ \frac{1}{y}\,\frac{dy}{dx} &= e^{-x^2} \\ \int_4^x \frac{1}{y(t)}\,\frac{dy}{dt}\,dt &= \int_4^x e^{-t^2}\,dt \\ \ln y(t)\Big|_4^x &= \int_4^x e^{-t^2}\,dt \\ \ln y(x) - \ln y(4) &= \int_4^x e^{-t^2}\,dt \\ \ln y(x) &= \int_4^x e^{-t^2}\,dt \\ y(x) &= e^{\int_4^x e^{-t^2}\,dt} \end{aligned}$$

**30.** Separating variables and then proceeding as in Example 5 we get

$$\begin{aligned}\frac{dy}{dx} &= y^2\sin(x^2)\\ \frac{1}{y^2}\frac{dy}{dx} &= \sin(x^2)\\ \int_{-2}^{x}\frac{1}{y^2(t)}\frac{dy}{dt}\,dt &= \int_{-2}^{x}\sin(t^2)\,dt\\ \frac{-1}{y(t)}\bigg|_{-2}^{x} &= \int_{-2}^{x}\sin(t^2)\,dt\\ \frac{-1}{y(x)}+\frac{1}{y(-2)} &= \int_{-2}^{x}\sin(t^2)\,dt\\ \frac{-1}{y(x)}+3 &= \int_{-2}^{x}\sin(t^2)\,dt\\ y(x) &= \left[3-\int_{-2}^{x}\sin(t^2)\,dt\right]^{-1}\end{aligned}$$

**31.** Separating variables we get

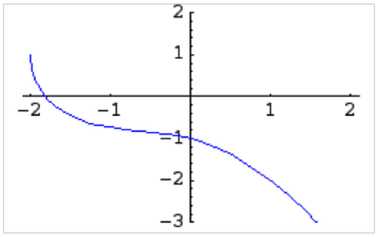
$$\begin{aligned}\frac{dy}{dx} &= \frac{2x+1}{2y}\\ 2y\,dy &= (2x+1)\,dx\\ \int 2y\,dy &= \int(2x+1)\,dx\\ y^2 &= x^2+x+c\end{aligned}$$



The condition $y(-2)=-1$ implies $c=-1$. Thus $y^2=x^2+x-1$ and $y=-\sqrt{x^2+x-1}$ in order for $y$ to be negative. Moreover for an interval containing $-2$ for values of $x$ such that $x^2+x-1>0$ we get $\left(-\infty,-\frac{1}{2}-\frac{\sqrt{5}}{2}\right)$.

**32.** Separating variables we get

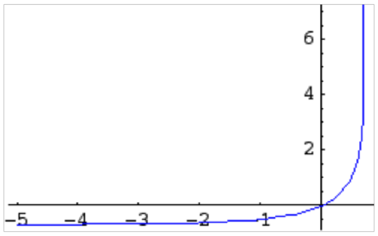
$$\begin{aligned}(2y-2)\,\frac{dy}{dx} &= 3x^2+4x+2\\ (2y-2)\;dy &= \left(3x^2+4x+2\right)\;dx\\ \int (2y-2)\;dy &= \int \left(3x^2+4x+2\right)\;dx\\ \int 2\,(y-1)\;dy &= \int \left(3x^2+4x+2\right)\;dx\\ (y-1)^2 &= x^3+2x^2+2x+c\end{aligned}$$



The condition $y(1)=-2$ implies $c=4$. Thus $y=1-\sqrt{x^3+2x^2+2x+4}$ where the minus sign is indicated by the initial condition. Now $x^3+2x^2+2x+4=(x+2)\left(x^2+1\right)>0$ implies $x>-2$, so the interval of definition is $(-2,\infty)$.

**33.** Separating variables we get

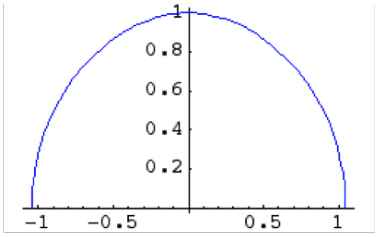
$$\begin{aligned}e^y\,dx-e^{-x}\,dy &= 0\\ e^y\,dx &= e^{-x}\,dy\\ e^x\,dx &= e^{-y}\,dy\\ \int e^x\,dx &= \int e^{-y}\,dy\\ e^x &= -e^{-y}+c\end{aligned}$$



The condition $y(0)=0$ implies $c=2$. Thus $e^{-y}=2-e^x$. Therefore $y=-\ln\left(2-e^x\right)$. Now we must have $2-e^x>0$ or $e^x<2$. Since $e^x$ is an increasing function this imples $x<\ln 2$ and so the interval of definition is $(-\infty,\ln 2)$.

**34.** Separating variables we get

$$\begin{aligned}\sin x\,dx+y\,dy &= 0\\ \int \sin x\,dx+\int y\,dy &= \int 0\,dx\\ -\cos x+\frac{1}{2}y^2 &= c\end{aligned}$$



The condition $y(0)=1$ implies $c=-\frac{1}{2}$. Thus $-\cos x+\frac{1}{2}y^2=-\frac{1}{2}$ or $y^2=2\cos x-1$. Therefore $y=\sqrt{2\cos x-1}$ where the positive root is indicated by the initial condition. Now we must have $2\cos x-1>0$ or $\cos x>\frac{1}{2}$. This means $-\pi/3<x<\pi/3$, so the the interval of definition is $(-\pi/3,\pi/3)$.

**35.** **(a)** The equilibrium solutions $y(x)=2$ and $y(x)=-2$ satisfy the initial conditions $y(0)=2$

and $y(0)=-2$, respectively. Setting $x=\frac{1}{4}$ and $y=1$ in $y=2(1+ce^{4x})/(1-ce^{4x})$ we obtain

$$1=2\frac{1+ce}{1-ce}, \quad 1-ce=2+2ce, \quad -1=3ce, \quad \text{and} \quad c=-\frac{1}{3e}.$$

The solution of the corresponding initial-value problem is

$$y=2\frac{1-\frac{1}{3}e^{4x-1}}{1+\frac{1}{3}e^{4x-1}}=2\frac{3-e^{4x-1}}{3+e^{4x-1}}.$$

**(b)** Separating variables and integrating yields

$$\begin{aligned}\frac{1}{4}\ln|y-2|-\frac{1}{4}\ln|y+2|+\ln c_1&=x\\ \ln|y-2|-\ln|y+2|+\ln c&=4x\\ \ln\left|\frac{c(y-2)}{y+2}\right|&=4x\\ c\frac{y-2}{y+2}&=e^{4x}.\end{aligned}$$

Solving for $y$ we get $y=2(c+e^{4x})/(c-e^{4x})$. The initial condition $y(0)=-2$ implies $2(c+1)/(c-1)=-2$ which yields $c=0$ and $y(x)=-2$. The initial condition $y(0)=2$ does not correspond to a value of $c$, and it must simply be recognized that $y(x)=2$ is a solution of the initial-value problem. Setting $x=\frac{1}{4}$ and $y=1$ in $y=2(c+e^{4x})/(c-e^{4x})$ leads to $c=-3e$. Thus, a solution of the initial-value problem is

$$y=2\frac{-3e+e^{4x}}{-3e-e^{4x}}=2\frac{3-e^{4x-1}}{3+e^{4x-1}}.$$

**36.** Separating variables, we have

$$\frac{dy}{y^2-y}=\frac{dx}{x} \qquad \text{or} \qquad \int\frac{dy}{y(y-1)}=\ln|x|+c.$$

Using partial fractions, we obtain

$$\begin{aligned}\int\left(\frac{1}{y-1}-\frac{1}{y}\right)dy&=\ln|x|+c\\ \ln|y-1|-\ln|y|&=\ln|x|+c\\ \ln\left|\frac{y-1}{xy}\right|&=c\\ \frac{y-1}{xy}&=e^c=c_1.\end{aligned}$$

Solving for $y$ we get $y=1/(1-c_1x)$. We note by inspection that $y=0$ is a singular solution of the differential equation.

**(a)** Setting $x = 0$ and $y = 1$ we have $1 = 1/(1 - 0)$, which is true for all values of $c_1$. Thus, solutions passing through $(0, 1)$ are $y = 1/(1 - c_1 x)$.

**(b)** Setting $x = 0$ and $y = 0$ in $y = 1/(1 - c_1 x)$ we get $0 = 1$. Thus, the only solution passing through $(0, 0)$ is $y = 0$.

**(c)** Setting $x = \frac{1}{2}$ and $y = \frac{1}{2}$ we have $\frac{1}{2} = 1/(1 - \frac{1}{2}c_1)$, so $c_1 = -2$ and $y = 1/(1 + 2x)$.

**(d)** Setting $x = 2$ and $y = \frac{1}{4}$ we have $\frac{1}{4} = 1/(1 - 2c_1)$, so $c_1 = -\frac{3}{2}$ and $y = 1/(1 + \frac{3}{2}x) = 2/(2 + 3x)$.

**37.** Singular solutions of $dy/dx = x\sqrt{1 - y^2}$ are $y = -1$ and $y = 1$. A singular solution of $(e^x + e^{-x})dy/dx = y^2$ is $y = 0$.

**38.** Differentiating $\ln(x^2 + 10) + \csc y = c$ we get

$$\frac{2x}{x^2 + 10} - \csc y \cot y \frac{dy}{dx} = 0,$$

$$\frac{2x}{x^2 + 10} - \frac{1}{\sin y} \cdot \frac{\cos y}{\sin y} \frac{dy}{dx} = 0,$$

or

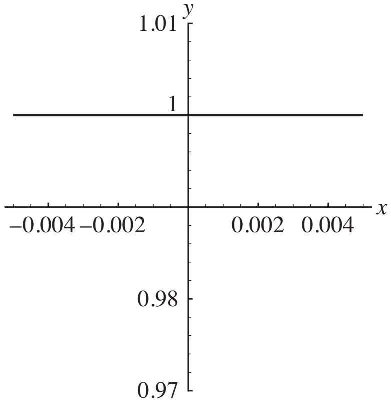
$$2x \sin^2 y \, dx - (x^2 + 10) \cos y \, dy = 0.$$

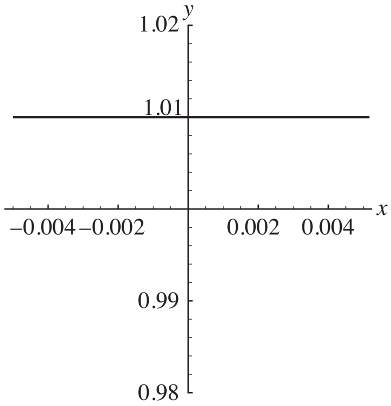
Writing the differential equation in the form

$$\frac{dy}{dx} = \frac{2x \sin^2 y}{(x^2 + 10) \cos y}$$

we see that singular solutions occur when $\sin^2 y = 0$, or $y = k\pi$, where $k$ is an integer.

**39.** The singular solution $y = 1$ satisfies the initial-value problem.



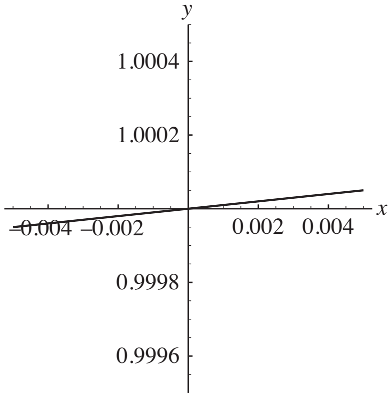


**40.** Separating variables we obtain $\dfrac{dy}{(y-1)^2} = dx$. Then

$$-\frac{1}{y-1} = x + c \quad \text{and} \quad y = \frac{x+c-1}{x+c}.$$

Setting $x = 0$ and $y = 1.01$ we obtain $c = -100$. The solution is

$$y = \frac{x-101}{x-100}.$$

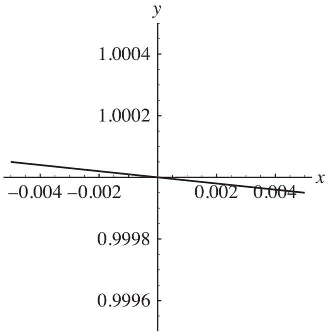


**41.** Separating variables we obtain $\dfrac{dy}{(y-1)^2 + 0.01} = dx$. Then

$$10\tan^{-1} 10(y-1) = x + c \quad \text{and} \quad y = 1 + \frac{1}{10}\tan\frac{x+c}{10}.$$

Setting $x = 0$ and $y = 1$ we obtain $c = 0$. The solution is

$$y = 1 + \frac{1}{10}\tan\frac{x}{10}.$$



**42.** Separating variables we obtain $\dfrac{dy}{(y-1)^2 - 0.01} = dx$. Then, with $u = y - 1$ and $a = \frac{1}{10}$, we get

$$5\ln\left|\frac{10y-11}{10y-9}\right| = x + c.$$

Setting $x = 0$ and $y = 1$ we obtain $c = 5\ln 1 = 0$. The solution is

$$5\ln\left|\frac{10y-11}{10y-9}\right| = x.$$

Solving for $y$ we obtain

$$y = \frac{11 + 9e^{x/5}}{10 + 10e^{x/5}}.$$

Alternatively, we can use the fact that

$$\int \frac{dy}{(y-1)^2 - 0.01} = -\frac{1}{0.1}\tanh^{-1}\frac{y-1}{0.1} = -10\tanh^{-1} 10(y-1).$$

(We use the inverse hyperbolic tangent because $|y-1| < 0.1$ or $0.9 < y < 1.1$. This follows from the initial condition $y(0) = 1$.) Solving the above equation for $y$ we get $y = 1 + 0.1\tanh(x/10)$.

**43.** Separating variables, we have

$$\frac{dy}{y-y^3} = \frac{dy}{y(1-y)(1+y)} = \left(\frac{1}{y} + \frac{1/2}{1-y} - \frac{1/2}{1+y}\right) dy = dx.$$

Integrating, we get

$$\ln|y| - \frac{1}{2}\ln|1-y| - \frac{1}{2}\ln|1+y| = x + c.$$

When $y > 1$, this becomes

$$\ln y - \frac{1}{2}\ln(y-1) - \frac{1}{2}\ln(y+1) = \ln\frac{y}{\sqrt{y^2-1}} = x + c.$$

Letting $x = 0$ and $y = 2$ we find $c = \ln(2/\sqrt{3}\,)$. Solving for $y$ we get $y_1(x) = 2e^x/\sqrt{4e^{2x}-3}\,$, where $x > \ln(\sqrt{3}/2)$.

When $0 < y < 1$ we have

$$\ln y - \frac{1}{2}\ln(1-y) - \frac{1}{2}\ln(1+y) = \ln\frac{y}{\sqrt{1-y^2}} = x + c.$$

Letting $x = 0$ and $y = \frac{1}{2}$ we find $c = \ln(1/\sqrt{3}\,)$. Solving for $y$ we get $y_2(x) = e^x/\sqrt{e^{2x}+3}\,$, where $-\infty < x < \infty$.

When $-1 < y < 0$ we have

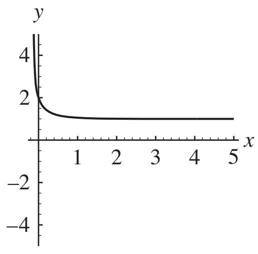
$$\ln(-y) - \frac{1}{2}\ln(1-y) - \frac{1}{2}\ln(1+y) = \ln\frac{-y}{\sqrt{1-y^2}} = x + c.$$

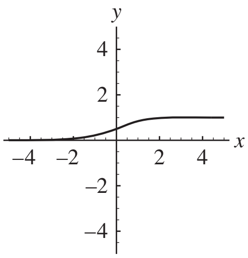
Letting $x = 0$ and $y = -\frac{1}{2}$ we find $c = \ln(1/\sqrt{3}\,)$. Solving for $y$ we get $y_3(x) = -e^x/\sqrt{e^{2x}+3}\,$, where $-\infty < x < \infty$.

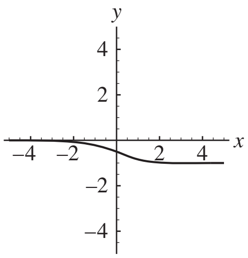
When $y < -1$ we have

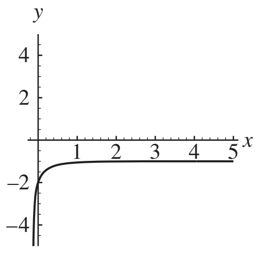
$$\ln(-y) - \frac{1}{2}\ln(1-y) - \frac{1}{2}\ln(-1-y) = \ln\frac{-y}{\sqrt{y^2-1}} = x + c.$$

Letting $x = 0$ and $y = -2$ we find $c = \ln(2/\sqrt{3}\,)$. Solving for $y$ we get $y_4(x) = -2e^x/\sqrt{4e^{2x}-3}\,$, where $x > \ln(\sqrt{3}/2)$.





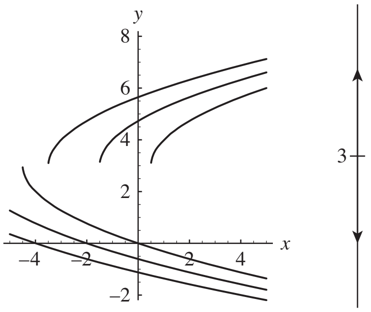




**44.** **(a)** The second derivative of $y$ is

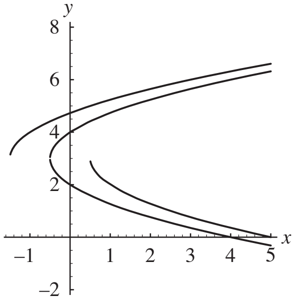
$$\frac{d^2y}{dx^2} = -\frac{dy/dx}{(y-1)^2} = -\frac{1/(y-3)}{(y-3)^2} = -\frac{1}{(y-3)^3}\,.$$

The solution curve is concave down when $d^2y/dx^2 < 0$ or $y > 3$, and concave up when $d^2y/dx^2 > 0$ or $y < 3$. From the phase portrait we see that the solution curve is decreasing when $y < 3$ and increasing when $y > 3$.



**(b)** Separating variables and integrating we obtain

$$\begin{aligned}(y-3)\,dy &= dx\\ \frac{1}{2}y^2 - 3y &= x + c\\ y^2 - 6y + 9 &= 2x + c_1\\ (y-3)^2 &= 2x + c_1\\ y &= 3 \pm \sqrt{2x + c_1}\,.\end{aligned}$$



The initial condition dictates whether to use the plus or minus sign.

When $y_1(0) = 4$ we have $c_1 = 1$ and $y_1(x) = 3 + \sqrt{2x+1}$ where $(-1/2, \infty)$.

When $y_2(0) = 2$ we have $c_1 = 1$ and $y_2(x) = 3 - \sqrt{2x+1}$ where $(-1/2, \infty)$.

When $y_3(1) = 2$ we have $c_1 = -1$ and $y_3(x) = 3 - \sqrt{2x-1}$ where $(1/2, \infty)$.

When $y_4(-1) = 4$ we have $c_1 = 3$ and $y_4(x) = 3 + \sqrt{2x+3}$ where $(-3/2, \infty)$.

**45.** We separate variables and rationalize the denominator. Then

$$\begin{aligned}dy &= \frac{1}{1+\sin x}\cdot\frac{1-\sin x}{1-\sin x}\,dx = \frac{1-\sin x}{1-\sin^2 x}\,dx = \frac{1-\sin x}{\cos^2 x}\,dx\\ &= \left(\sec^2 x - \tan x \sec x\right)\,dx.\end{aligned}$$

Integrating, we have $y = \tan x - \sec x + C$.

**46.** Separating variables we have $\sqrt{y}\,dy = \sin\sqrt{x}\,dx$. Then

$$\int \sqrt{y}\,dy = \int \sin\sqrt{x}\,dx \qquad \text{and} \qquad \frac{2}{3}y^{3/2} = \int \sin\sqrt{x}\,dx.$$

To integrate $\sin\sqrt{x}$ we first make the substitution $u = \sqrt{x}$. Then $du = \dfrac{1}{2\sqrt{x}}\,dx = \frac{1}{2u}\,du$ and

$$\int \sin\sqrt{x}\,dx = \int (\sin u)\,(2u)\;du = 2\int u \sin u\,du.$$

Using integration by parts we find

$$\int u\sin u\,du = -u\cos u + \sin u = -\sqrt{x}\cos\sqrt{x} + \sin\sqrt{x}.$$

Thus

$$\frac{2}{3}y = \int \sin\sqrt{x}\,dx = -2\sqrt{x}\,\cos\sqrt{x} + 2\sin\sqrt{x} + C$$

and

$$y = 3^{2/3}\left(-\sqrt{x}\,\cos\sqrt{x} + \sin\sqrt{x} + C\right).$$

**47.** Separating variables we have $dy/\left(\sqrt{y}+y\right) = dx/\left(\sqrt{x}+x\right)$. To integrate $\int dx/\left(\sqrt{x}+x\right)$ we substitute $u^2 = x$ and get

$$\int \frac{2u}{u+u^2}\,du = \int \frac{2}{1+u}\,du = 2\ln|1+u| + c = 2\ln\left(1+\sqrt{x}\right) + c.$$

Integrating the separated differential equation we have

$$2\ln(1+\sqrt{y}) = 2\ln\left(1+\sqrt{x}\right) + c \quad \text{or} \quad \ln(1+\sqrt{y}) = \ln\left(1+\sqrt{x}\right) + \ln c_1.$$

Solving for $y$ we get $y = \left[c_1\left(1+\sqrt{x}\right) - 1\right]^2$.

**48.** Separating variables and integrating we have

$$\int \frac{dy}{y^{2/3}\left(1-y^{1/3}\right)} = \int dx$$

$$\int \frac{y^{2/3}}{1-y^{1/3}}\,dy = x + c_1$$

$$-3\ln\left|1-y^{1/3}\right| = x + c_1$$

$$\ln\left|1-y^{1/3}\right| = -\frac{x}{3} + c_2$$

$$\left|1-y^{1/3}\right| = c_3e^{-x/3}$$

$$1-y^{1/3} = c_4e^{-x/3}$$

$$y^{1/3} = 1 + c_5e^{-x/3}$$

$$y = \left(1 + c_5e^{-x/3}\right)^3.$$

**49.** Separating variables we have $y\,dy = e^{\sqrt{x}}\,dx$. If $u = \sqrt{x}$, then $u^2 = x$ and $2u\,du = dx$. Thus, $\displaystyle\int e^{\sqrt{x}}\,dx = \int 2ue^u\,du$ and, using integration by parts, we find

$$\int y\,dy = \int e^{\sqrt{x}}\,dx \qquad \text{so} \qquad \frac{1}{2}\,y^2 = \int 2ue^u\,du = -2e^u + C = 2\sqrt{x}\,e^{\sqrt{x}} - 2e^{\sqrt{x}} + C,$$

and

$$y = 2\sqrt{\sqrt{x}\,e^{\sqrt{x}} - e^{\sqrt{x}} + C}\,.$$

To find $C$ we solve $y(1) = 4$.

$$y(1) = 2\sqrt{\sqrt{1}\,e^{\sqrt{1}} - e^{\sqrt{1}} + C} = 2\sqrt{C} = 4 \qquad \text{so} \qquad C = 4.$$

and the solution of the intial-value problem is $y = 2\sqrt{\sqrt{x}\,e^{\sqrt{x}} - e^{\sqrt{x}} + 4}\,$.

**50.** Seperating variables we have $y\,dy = x\tan^{-1}x\,dx$. Integrating both sides and using integration by parts with $u = \tan^{-1}x$ and $dv = x\,dx$ we have

$$\int y\,dy = x\tan^{-1}x\,dx$$

$$\frac{1}{2}\,y^2 = \frac{1}{2}\,x^2\tan^{-1}x - \frac{1}{2}\,x + \frac{1}{2}\,\tan^{-1}x + C$$

$$y^2 = x^2\tan^{-1}x - x + \tan^{-1}x + C_1$$

$$y = \sqrt{x^2\tan^{-1}x - x + \tan^{-1}x + C_1}$$

To find $C_1$ we solve $y(0) = 3$.

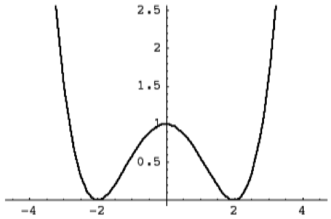
$$y(0) = \sqrt{0^2\tan^{-1}0 - 0 + \tan^{-1}0 + C_1} = \sqrt{C_1} = 3 \qquad \text{so} \qquad C_1 = 9,$$

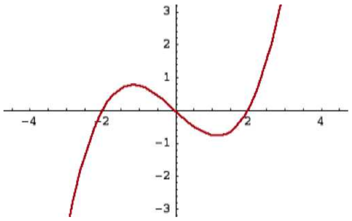
and the solution of the initial-value problem is $y = \sqrt{x^2\tan^{-1}x - x + \tan^{-1}x + 9}\,$.

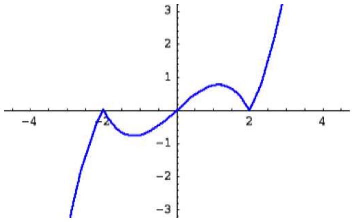
**51.** **(a)** While $y_2(x) = -\sqrt{25 - x^2}$ is defined at $x = -5$ and $x = 5$, $y_2'(x)$ is not defined at these values, and so the interval of definition is the open interval $(-5, 5)$.

**(b)** At any point on the $x$-axis the derivative of $y(x)$ is undefined, so no solution curve can cross the $x$-axis. Since $-x/y$ is not defined when $y = 0$, the initial-value problem has no solution.

**52.** The derivative of $y = \left(\frac{1}{4}x^2 - 1\right)^2$ is $dy/dx = x\left(\frac{1}{4}x^2 - 1\right)$. We note that $xy^{1/2} = x\left|\frac{1}{4}x^2 - 1\right|$. We see from the graphs of $y$ (black), $dy/dx$ (red), and $xy^{1/2}$ (blue), below that $dy/dx = xy^{1/2}$ on $(-\infty, 2]$ and $[2, \infty)$.





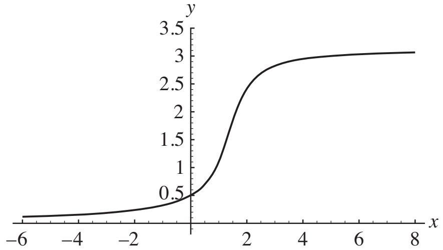


Alternatively, because $\sqrt{X^2} = |X|$ we can write

$$xy^{1/2} = x\sqrt{y} = x\sqrt{\left(\frac{1}{4}x^2 - 1\right)^2} = x\left|\frac{1}{4}x^2 - 1\right| = \begin{cases} x\left(\frac{1}{4}x^2 - 1\right), & -\infty < x \le -2 \\ -x\left(\frac{1}{4}x^2 - 1\right), & -2 < x < 2 \\ x\left(\frac{1}{4}x^2 - 1\right), & 2 \le x < \infty\,. \end{cases}$$

From this we see that $dy/dx = xy^{1/2}$ on $(-\infty, -2]$ and on $[2, \infty)$.

**53.** Separating variables we have $dy/\left(\sqrt{1+y^2}\,\sin^2 y\right) = dx$ which is not readily integrated (even by a CAS). We note that $dy/dx \ge 0$ for all values of $x$ and $y$ and that $dy/dx = 0$ when $y = 0$ and $y = \pi$, which are equilibrium solutions.



**54.** **(a)** The solution of $y' = y$, $y(0) = 1$, is $y = e^x$. Using separation of variables we find that the solution of $y' = y\,[1 + 1/(x \ln x)]$, $y(e) = 1$, is $y = e^{x-e}\ln x$. Solving the two solutions simultaneously we obtain

$$e^x = e^{x-e}\ln x, \qquad \text{so} \qquad e^e = \ln x \qquad \text{and} \qquad x = e^{e^e}.$$

**(b)** Since $y = e^{\left(e^{e^e}\right)} \approx 2.33 \times 10^{1{,}656{,}520}$, the $y$-coordinate of the point of intersection of the two solution curves has over 1.65 million digits.

**55.** We are looking for a function $y(x)$ such that

$$y^2 + \left(\frac{dy}{dx}\right)^2 = 1.$$

Using the positive square root gives

$$\frac{dy}{dx} = \sqrt{1-y^2}$$

$$\frac{dy}{\sqrt{1-y^2}} = dx$$

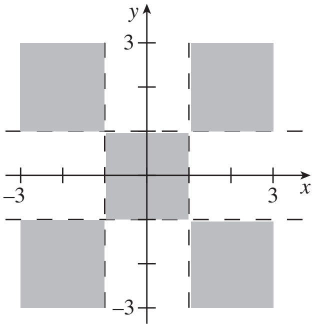
$$\sin^{-1} y = x + c.$$

Thus a solution is $y = \sin(x + c)$. If we use the negative square root we obtain

$$y = \sin(c - x) = -\sin(x - c) = -\sin(x + c_1).$$

Note that when $c = c_1 = 0$ and when $c = c_1 = \pi/2$ we obtain the well known particular solutions $y = \sin x$, $y = -\sin x$, $y = \cos x$, and $y = -\cos x$. Note also that $y = 1$ and $y = -1$ are singular solutions.

**56.** **(a)**



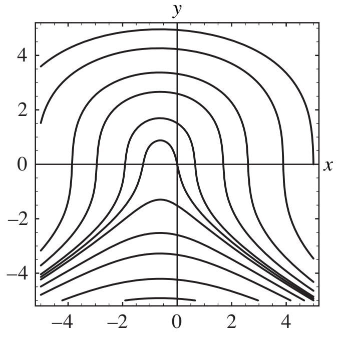
**(b)** For $|x| > 1$ and $|y| > 1$ the differential equation is $dy/dx = \sqrt{y^2 - 1}\,/\sqrt{x^2 - 1}$. Separating variables and integrating, we obtain

$$\frac{dy}{\sqrt{y^2 - 1}} = \frac{dx}{\sqrt{x^2 - 1}} \quad \text{and} \quad \cosh^{-1} y = \cosh^{-1} x + c.$$

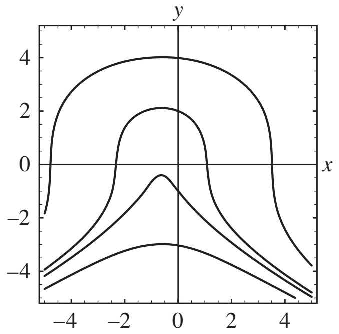
Setting $x = 2$ and $y = 2$ we find $c = \cosh^{-1} 2 - \cosh^{-1} 2 = 0$ and $\cosh^{-1} y = \cosh^{-1} x$. An explicit solution is $y = x$.

**57.** Since the tension $T_1$ (or magnitude $T_1$) acts at the lowest point of the cable, we use symmetry to solve the problem on the interval $[0, L/2]$. The assumption that the roadbed is uniform (that is, weighs a constant $\rho$ pounds per horizontal foot) implies $W = \rho x$, where $x$ is measured in feet and $0 \le x \le L/2$. Therefore (10) becomes $dy/dx = (\rho/T_1)x$. This last equation is a separable equation of the form given in (1) of Section 2.2 in the text. Integrating and using the initial condition $y(0) = a$ shows that the shape of the cable is a parabola: $y(x) = (\rho/2T_1)x^2 + a$. In terms of the sag $h$ of the cable and the span $L$, we see from Figure 2.2.5 in the text that $y(L/2) = h + a$. By applying this last condition to $y(x) = (\rho/2T_1)x^2 + a$ enables us to express $\rho/2T_1$ in terms of $h$ and $L$: $y(x) = (4h/L^2)x^2 + a$. Since $y(x)$ is an even function of $x$, the solution is valid on $-L/2 \le x \le L/2$.

**58.** **(a)** Separating variables and integrating, we have $(3y^2 + 1)\,dy = -(8x + 5)\,dx$ and $y^3 + y = -4x^2 - 5x + c$. Using a CAS we show various contours of $f(x, y) = y^3 + y + 4x^2 + 5x$. The plots shown on $[-5, 5] \times [-5, 5]$ correspond to $c$-values of 0, ±5, ±20, ±40, ±80, and ±125.



**(b)** The value of $c$ corresponding to $y(0) = -1$ is $f(0,-1) = -2$; to $y(0) = 2$ is $f(0,2) = 10$; to $y(-1) = 4$ is $f(-1,4) = 67$; and to $y(-1) = -3$ is $-31$.



**59. (a)** An implicit solution of the differential equation $(2y+2)dy - (4x^3 + 6x)\,dx = 0$ is

$$y^2 + 2y - x^4 - 3x^2 + c = 0.$$

The condition $y(0) = -3$ implies that $c = -3$. Therefore $y^2 + 2y - x^4 - 3x^2 - 3 = 0$.

**(b)** Using the quadratic formula we can solve for $y$ in terms of $x$:

$$y = \frac{-2 \pm \sqrt{4 + 4(x^4 + 3x^2 + 3)}}{2}\,.$$

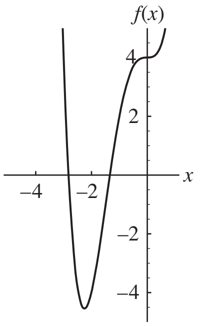
The explicit solution that satisfies the initial condition is then

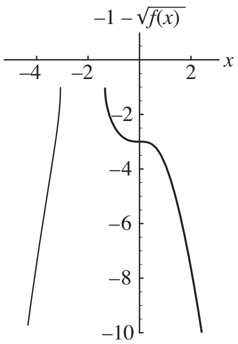
$$y = -1 - \sqrt{x^4 + 3x^3 + 4}\,.$$

**(c)** From the graph of the function $f(x) = x^4 + 3x^3 + 4$ below we see that $f(x) \le 0$ on the approximate interval $-2.8 \le x \le -1.3$. Thus the approximate domain of the function

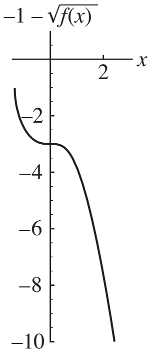
$$y = -1 - \sqrt{x^4 + 3x^3 + 4} = -1 - \sqrt{f(x)}$$

is $x \le -2.8$ or $x \ge -1.3$. The graph of this function is shown below.





**(d)** Using the root finding capabilities of a CAS, the zeros of $f$ are found to be $-2.82202$ and $-1.3409$. The domain of definition of the solution $y(x)$ is then $x > -1.3409$. The equality has been removed since the derivative $dy/dx$ does not exist at the points where $f(x) = 0$. The graph of the solution $y = \phi(x)$ is given on the right.



**60. (a)** Separating variables and integrating, we have

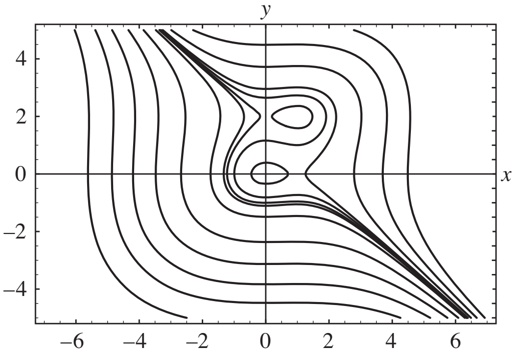
$$(-2y + y^2)\,dy = (x - x^2)\,dx$$

and

$$-y^2 + \frac{1}{3}y^3 = \frac{1}{2}x^2 - \frac{1}{3}x^3 + c$$

Using a CAS we show some contours of

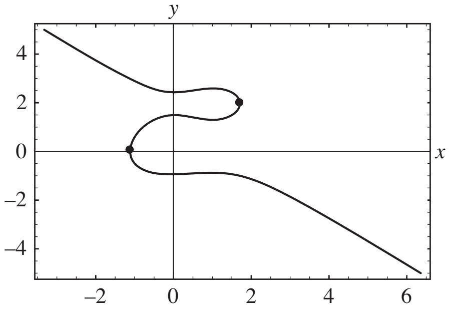
$$f(x, y) = 2y^3 - 6y^2 + 2x^3 - 3x^2.$$



The plots shown on $[-7, 7] \times [-5, 5]$ correspond to $c$-values of $-450$, $-300$, $-200$, $-120$, $-60$, $-20$, $-10$, $-8.1$, $-5$, $-0.8$, $20$, $60$, and $120$.

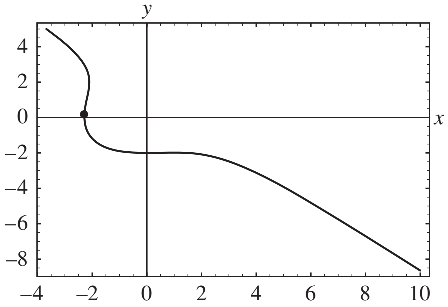
**(b)** The value of $c$ corresponding to $y(0) = \frac{3}{2}$ is $f\left(0, \frac{3}{2}\right) = -\frac{27}{4}$. The portion of the graph between the dots corresponds to the solution curve satisfying the intial condition. To determine the interval of definition we find $dy/dx$ for

$$2y^3 - 6y^2 + 2x^3 - 3x^2 = -\frac{27}{4}.$$



Using implicit differentiation we get $y' = (x - x^2)/(y^2 - 2y)$, which is infinite when $y = 0$ and $y = 2$. Letting $y = 0$ in $2y^3 - 6y^2 + 2x^3 - 3x^2 = -\frac{27}{4}$ and using a CAS to solve for $x$ we get $x = -1.13232$. Similarly, letting $y = 2$, we find $x = 1.71299$. The largest interval of definition is approximately $(-1.13232, 1.71299)$.

**(c)** The value of $c$ corresponding to $y(0) = -2$ is $f(0,-2) = -40$. The portion of the graph to the right of the dot corresponds to the solution curve satisfying the initial condition. To determine the interval of definition we find $dy/dx$ for



$$2y^3 - 6y^2 + 2x^3 - 3x^2 = -40.$$

Using implicit differentiation we get $y' = (x - x^2)/(y^2 - 2y)$, which is infinite when $y = 0$ and $y = 2$. Letting $y = 0$ in $2y^3 - 6y^2 + 2x^3 - 3x^2 = -40$ and using a CAS to solve for $x$ we get $x = -2.29551$. The largest interval of definition is approximately $(-2.29551, \infty)$.

## 2.3 Linear Equations

**1.** For $y' - 5y = 0$ an integrating factor is $e^{-\int 5\,dx} = e^{-5x}$ so that $\dfrac{d}{dx}\left[e^{-5x}y\right] = 0$ and $y = ce^{5x}$ for $-\infty < x < \infty$.

**2.** For $y' + 2y = 0$ an integrating factor is $e^{\int 2\,dx} = e^{2x}$ so that $\dfrac{d}{dx}\left[e^{2x}y\right] = 0$ and $y = ce^{-2x}$ for $-\infty < x < \infty$. The transient term is $ce^{-2x}$.

**3.** For $y' + y = e^{3x}$ an integrating factor is $e^{\int dx} = e^x$ so that $\dfrac{d}{dx}[e^x y] = e^{4x}$ and $y = \frac{1}{4}e^{3x} + ce^{-x}$ for $-\infty < x < \infty$. The transient term is $ce^{-x}$.

**4.** For $y' + 4y = \frac{4}{3}$ an integrating factor is $e^{\int 4\,dx} = e^{4x}$ so that $\dfrac{d}{dx}\left[e^{4x}y\right] = \frac{4}{3}e^{4x}$ and $y = \frac{1}{3} + ce^{-4x}$ for $-\infty < x < \infty$. The transient term is $ce^{-4x}$.

**5.** For $y' + 3x^2 y = x^2$ an integrating factor is $e^{\int 3x^2\,dx} = e^{x^3}$ so that $\dfrac{d}{dx}\left[e^{x^3}y\right] = x^2 e^{x^3}$ and $y = \frac{1}{3} + ce^{-x^3}$ for $-\infty < x < \infty$. The transient term is $ce^{-x^3}$.

**6.** For $y' + 2xy = x^3$ an integrating factor is $e^{\int 2x\,dx} = e^{x^2}$ so that $\dfrac{d}{dx}\left[e^{x^2}y\right] = x^3 e^{x^2}$ and $y = \frac{1}{2}x^2 - \frac{1}{2} + ce^{-x^2}$ for $-\infty < x < \infty$. The transient term is $ce^{-x^2}$.

**7.** For $y' + \dfrac{1}{x}\,y = \dfrac{1}{x^2}$ an integrating factor is $e^{\int (1/x)\,dx} = x$ so that $\dfrac{d}{dx}[xy] = \dfrac{1}{x}$ and $y = \dfrac{1}{x}\ln x + \dfrac{c}{x}$ for $0 < x < \infty$. The entire solution is transient.

**8.** For $y'-2y = x^2+5$ an integrating factor is $e^{-\int 2\,dx} = e^{-2x}$ so that $\frac{d}{dx}\left[e^{-2x}y\right] = x^2e^{-2x}+5e^{-2x}$ and $y = -\frac{1}{2}x^2 - \frac{1}{2}x - \frac{11}{4} + ce^{2x}$ for $-\infty < x < \infty$. There is no transient term.

**9.** For $y' - \frac{1}{x}y = x\sin x$ an integrating factor is $e^{-\int (1/x)\,dx} = \frac{1}{x}$ so that $\frac{d}{dx}\left[\frac{1}{x}y\right] = \sin x$ and $y = cx - x\cos x$ for $0 < x < \infty$. There is no transient term.

**10.** For $y'+\frac{2}{x}y = \frac{3}{x}$ an integrating factor is $e^{\int(2/x)dx} = x^2$ so that $\frac{d}{dx}\left[x^2y\right] = 3x$ and $y = \frac{3}{2}+cx^{-2}$ for $0 < x < \infty$. The trancient term is $cx^{-2}$.

**11.** For $y' + \frac{4}{x}y = x^2 - 1$ an integrating factor is $e^{\int(4/x)dx} = x^4$ so that $\frac{d}{dx}\left[x^4y\right] = x^6 - x^4$ and $y = \frac{1}{7}x^3 - \frac{1}{5}x + cx^{-4}$ for $0 < x < \infty$. The transient term is $cx^{-4}$.

**12.** For $y' - \frac{x}{(1+x)}y = x$ an integrating factor is $e^{-\int[x/(1+x)]dx} = (x+1)e^{-x}$ so that $\frac{d}{dx}\left[(x+1)e^{-x}y\right] = x(x+1)e^{-x}$ and $y = -x - \frac{2x+3}{x+1} + \frac{ce^x}{x+1}$ for $-1 < x < \infty$. There is no transient term.

**13.** For $y' + \left(1+\frac{2}{x}\right)y = \frac{e^x}{x^2}$ an integrating factor is $e^{\int[1+(2/x)]dx} = x^2e^x$ so that $\frac{d}{dx}[x^2e^xy] = e^{2x}$ and $y = \frac{1}{2}\frac{e^x}{x^2} + \frac{ce^{-x}}{x^2}$ for $0 < x < \infty$. The transient term is $\frac{ce^{-x}}{x^2}$.

**14.** For $y' + \left(1+\frac{1}{x}\right)y = \frac{1}{x}e^{-x}\sin 2x$ an integrating factor is $e^{\int[1+(1/x)]dx} = xe^x$ so that $\frac{d}{dx}\left[xe^xy\right] = \sin 2x$ and $y = -\frac{1}{2x}e^{-x}\cos 2x + \frac{ce^{-x}}{x}$ for $0 < x < \infty$. The entire solution is transient.

**15.** For $\frac{dx}{dy} - \frac{4}{y}x = 4y^5$ an integrating factor is $e^{-\int(4/y)\,dy} = e^{\ln y^{-4}} = y^{-4}$ so that $\frac{d}{dy}\left[y^{-4}x\right] = 4y$ and $x = 2y^6 + cy^4$ for $0 < y < \infty$. There is no transient term.

**16.** For $\frac{dx}{dy} + \frac{2}{y}x = e^y$ an integrating factor is $e^{\int(2/y)\,dy} = y^2$ so that $\frac{d}{dy}\left[y^2x\right] = y^2e^y$ and $x = e^y - \frac{2}{y}e^y + \frac{2}{y^2}e^y + \frac{c}{y^2}$ for $0 < y < \infty$. The transient term is $\frac{c}{y^2}$.

**17.** For $y' + (\tan x)y = \sec x$ an integrating factor is $e^{\int \tan x\,dx} = \sec x$ so that $\dfrac{d}{dx}[(\sec x)y] = \sec^2 x$ and $y = \sin x + c\cos x$ for $-\pi/2 < x < \pi/2$. There is no transient term.

**18.** For $y' + (\cot x)y = \sec^2 x \csc x$ an integrating factor is $e^{\int \cot x\,dx} = e^{\ln|\sin x|} = \sin x$ so that $\dfrac{d}{dx}[(\sin x)\,y] = \sec^2 x$ and $y = \sec x + c\csc x$ for $0 < x < \pi/2$. There is no transient term.

**19.** For $y' + \dfrac{x+2}{x+1}y = \dfrac{2xe^{-x}}{x+1}$ an integrating factor is $e^{\int[(x+2)/(x+1)]dx} = (x+1)e^x$, so $\dfrac{d}{dx}[(x+1)e^x y] = 2x$ and $y = \dfrac{x^2}{x+1}e^{-x} + \dfrac{c}{x+1}e^{-x}$ for $-1 < x < \infty$. The entire solution is transient.

**20.** For $y' + \dfrac{4}{x+2}y = \dfrac{5}{(x+2)^2}$ an integrating factor is $e^{\int[4/(x+2)]\,dx} = (x+2)^4$ so that $\dfrac{d}{dx}\left[(x+2)^4 y\right] = 5(x+2)^2$ and $y = \dfrac{5}{3}(x+2)^{-1} + c(x+2)^{-4}$ for $-2 < x < \infty$. The entire solution is transient.

**21.** For $\dfrac{dr}{d\theta} + r\sec\theta = \cos\theta$ an integrating factor is $e^{\int \sec\theta\,d\theta} = e^{\ln|\sec x + \tan x|} = \sec\theta + \tan\theta$ so that $\dfrac{d}{d\theta}[(\sec\theta + \tan\theta)r] = 1 + \sin\theta$ and $(\sec\theta + \tan\theta)r = \theta - \cos\theta + c$ for $-\pi/2 < \theta < \pi/2$. There is no transient term.

**22.** For $\dfrac{dP}{dt} + (2t-1)P = 4t - 2$ an integrating factor is $e^{\int(2t-1)\,dt} = e^{t^2-t}$ so that $\dfrac{d}{dt}\left[e^{t^2-t}P\right] = (4t-2)e^{t^2-t}$ and $P = 2 + ce^{t-t^2}$ for $-\infty < t < \infty$. The transient term is $ce^{t-t^2}$.

**23.** For $y' + \left(3 + \dfrac{1}{x}\right)y = \dfrac{e^{-3x}}{x}$ an integrating factor is $e^{\int[3+(1/x)]dx} = xe^{3x}$ so that $\dfrac{d}{dx}\left[xe^{3x}y\right] = 1$ and $y = e^{-3x} + \dfrac{ce^{-3x}}{x}$ for $0 < x < \infty$. The transient term is $ce^{-3x}/x$.

**24.** For $y' + \dfrac{2}{x^2-1}y = \dfrac{x+1}{x-1}$ an integrating factor is $e^{\int[2/(x^2-1)]dx} = \dfrac{x-1}{x+1}$ so that $\dfrac{d}{dx}\left[\dfrac{x-1}{x+1}y\right] = 1$ and $(x-1)y = x(x+1) + c(x+1)$ for $-1 < x < 1$. There is no transient term.

**25.** For $y' - 5y = x$ an integrating factor is $e^{\int -5\,dx} = e^{-5x}$ so that $\dfrac{d}{dx}\left[e^{-5x}y\right] = xe^{-5x}$ and

$$y = e^{5x}\int xe^{-5x}\,dx = e^{5x}\left(-\frac{1}{5}\,xe^{-5x} - \frac{1}{25}\,e^{-5x} + c\right) = -\frac{1}{5}\,x - \frac{1}{25} + ce^{5x}.$$

If $y(0) = 3$ then $c = \dfrac{1}{25}$ and $y = -\dfrac{1}{5}\,x - \dfrac{1}{25} + \dfrac{76}{25}\,e^{5x}$. The solution is defined on $I = (-\infty, \infty)$.

**26.** For $y' + 3y = 2x$ and integrating factor is $e^{\int 3\,dx} = e^{3x}$ so that $\dfrac{d}{dx}\left[e^{3x}y\right] = 2xe^{3x}$ and

$$y = e^{-3x}\int 2xe^{3x}\,dx = e^{-3x}\left(\frac{2}{3}\,xe^{3x} - \frac{2}{9}\,e^{3x} + c\right) = \frac{2}{3}\,x - \frac{2}{9} + ce^{-3x}.$$

If $y(0) = \dfrac{1}{3}$ then $c = \dfrac{5}{9}$ and $y = \dfrac{2}{3}\,x - \dfrac{2}{9} + \dfrac{5}{9}\,e^{-3x}$. THe solution is defined on $I = (-\infty, \infty)$.

**27.** For $y' + \dfrac{1}{x}\,y = \dfrac{1}{x}\,e^x$ an integrating factor is $e^{\int (1/x)dx} = x$ so that $\dfrac{d}{dx}\,[xy] = e^x$ and $y = \dfrac{1}{x}e^x + \dfrac{c}{x}$ for $0 < x < \infty$. If $y(1) = 2$ then $c = 2 - e$ and $y = \dfrac{1}{x}e^x + \dfrac{2-e}{x}$. The solution is defined on $I = (0, \infty)$.

**28.** For $\dfrac{dx}{dy} - \dfrac{1}{y}\,x = 2y$ an integrating factor is $e^{-\int (1/y)dy} = \dfrac{1}{y}$ so that $\dfrac{d}{dy}\left[\dfrac{1}{y}x\right] = 2$ and $x = 2y^2 + cy$ for $0 < y < \infty$. If $y(1) = 5$ then $c = -49/5$ and $x = 2y^2 - \dfrac{49}{5}y$. The solution is defined on $I = (0, \infty)$.

**29.** For $\dfrac{di}{dt} + \dfrac{R}{L}\,i = \dfrac{E}{L}$ an integrating factor is $e^{\int (R/L)\,dt} = e^{Rt/L}$ so that $\dfrac{d}{dt}\left[e^{Rt/L}\,i\right] = \dfrac{E}{L}e^{Rt/L}$ and $i = \dfrac{E}{R} + ce^{-Rt/L}$ for $-\infty < t < \infty$. If $i(0) = i_0$ then $c = i_0 - E/R$ and $i = \dfrac{E}{R} + \left(i_0 - \dfrac{E}{R}\right)e^{-Rt/L}$. The solution is defined on $I = (-\infty, \infty)$

**30.** For $\dfrac{dT}{dt} - kT = -T_m k$ an integrating factor is $e^{\int (-k)\,dt} = e^{-kt}$ so that $\dfrac{d}{dt}\,[e^{-kt}T] = -T_m ke^{-kt}$ and $T = T_m + ce^{kt}$ for $-\infty < t < \infty$. If $T(0) = T_0$ then $c = T_0 - T_m$ and $T = T_m + (T_0 - T_m)e^{kt}$. The solution is defined on $I = (-\infty, \infty)$

**31.** For $y' + \dfrac{1}{x}\,y = 4 + \dfrac{1}{x}$ an integrating factor is $e^{\int (1/x)\,dx} = x$ so that $\dfrac{d}{dx}\,[xy] = 4x + 1$ and

$$y = \frac{1}{x}\int (4x+1)\,dx = \frac{1}{x}\left(2x^2 + x + c\right) = 2x + 1 + \frac{c}{x}.$$

If $y(1) = 8$ then $c = 5$ and $y = 2x + 1 + \dfrac{5}{x}$. The solution is defined on $I = (0, \infty)$.

**32.** For $y' + 4xy = x^3 e^{x^2}$ an integrating factor is $e^{\int 4x\,dx} = e^{2x^2}$ so that $\dfrac{d}{dx}\left[e^{2x^2} y\right] = x^3 e^{3x^2}$ and

$$y = e^{-2x^2} \int x^3 e^{3x^2}\,dx = e^{-2x^2}\left(\frac{1}{6}x^2 e^{3x^2} - \frac{1}{18}e^{3x^2} + c\right) = \frac{1}{6}x^2 e^{x^2} - \frac{1}{18}e^{x^2} + ce^{-2x^2}.$$

If $y(0) = -1$ then $c = -\dfrac{17}{18}$ and $y = \dfrac{1}{6}x^2 e^{x^2} - \dfrac{1}{18}e^{x^2} - \dfrac{17}{18}e^{-2x^2}$. The solution is defined on $I = (-\infty, \infty)$.

**33.** For $y' + \dfrac{1}{x+1}y = \dfrac{\ln x}{x+1}$ an integrating factor is $e^{\int [1/(x+1)]\,dx} = x+1$ so that $\dfrac{d}{dx}[(x+1)y] = \ln x$ and

$$y = \frac{x}{x+1}\ln x - \frac{x}{x+1} + \frac{c}{x+1} \qquad \text{for} \quad 0 < x < \infty.$$

If $y(1) = 10$ then $c = 21$ and $y = \dfrac{x}{x+1}\ln x - \dfrac{x}{x+1} + \dfrac{21}{x+1}$. The solution is defined on $I = (0, \infty)$.

**34.** For $y' + \dfrac{1}{x+1}y = \dfrac{1}{x\,(x+1)}$ an integrating factor is $e^{\int [1/(x+1)]\,dx} = x + 1$ so that $\dfrac{d}{dx}\left[(x+1)\,y\right] = \dfrac{1}{x}$ and

$$y = \frac{1}{x+1}\int \frac{1}{x}\,dx = \frac{1}{x+1}(\ln x + c) = \frac{\ln x}{x+1} + \frac{c}{x+1}.$$

If $y(e) = 1$ then $c = e$ and $y = \dfrac{\ln x}{x+1} + \dfrac{e}{x+1}$. The solution is defined on $I = (0, \infty)$.

**35.** For $y' - (\sin x)\,y = 2\sin x$ an integrating factor is $e^{\int (-\sin x)\,dx} = e^{\cos x}$ so that $\dfrac{d}{dx}\left[e^{\cos x} y\right] = 2\,(\sin x)\,e^{\cos x}$ and

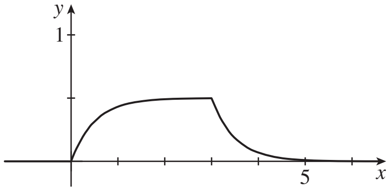
$$y = e^{-\cos x}\int 2\,(\sin x)\,e^{\cos x}\,dx = e^{-\cos x}\left(-2e^{\cos x} + c\right) = -2 + ce^{-\cos x}.$$

If $y(\pi/2) = 1$ then $c = 3$ and $y = -2 + 3e^{-\cos x}$. The solution is defined on $I = (-\infty, \infty)$.

**36.** For $y' + (\tan x)y = \cos^2 x$ an integrating factor is $e^{\int \tan x\, dx} = e^{\ln|\sec x|} = \sec x$ so that $\dfrac{d}{dx}[(\sec x)\, y] = \cos x$ and $y = \sin x \cos x + c\cos x$ for $-\pi/2 < x < \pi/2$. If $y(0) = -1$ then $c = -1$ and $y = \sin x \cos x - \cos x$. The solution is defined on $I = (-\pi/2, \pi/2)$.

**37.** For $y' + 2y = f(x)$ an integrating factor is $e^{2x}$ so that

$$ye^{2x} = \begin{cases} \dfrac{1}{2}e^{2x} + c_1, & 0 \le x \le 3 \\ c_2, & x > 3. \end{cases}$$

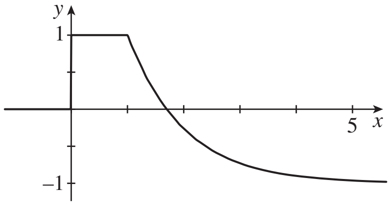


If $y(0) = 0$ then $c_1 = -1/2$ and for continuity we must have $c_2 = \frac{1}{2}e^6 - \frac{1}{2}$ so that

$$y = \begin{cases} \dfrac{1}{2}(1 - e^{-2x}), & 0 \le x \le 3 \\ \dfrac{1}{2}(e^6 - 1)e^{-2x}, & x > 3. \end{cases}$$

**38.** For $y' + y = f(x)$ an integrating factor is $e^x$ so that

$$ye^x = \begin{cases} e^x + c_1, & 0 \le x \le 1 \\ -e^x + c_2, & x > 1. \end{cases}$$

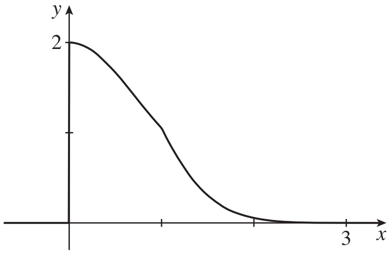


If $y(0) = 1$ then $c_1 = 0$ and for continuity we must have $c_2 = 2e$ so that

$$y = \begin{cases} 1, & 0 \le x \le 1 \\ 2e^{1-x} - 1, & x > 1. \end{cases}$$

**39.** For $y' + 2xy = f(x)$ an integrating factor is $e^{x^2}$ so that

$$ye^{x^2} = \begin{cases} \dfrac{1}{2}e^{x^2} + c_1, & 0 \le x \le 1 \\ c_2, & x > 1. \end{cases}$$



If $y(0) = 2$ then $c_1 = 3/2$ and for continuity we must have $c_2 = \frac{1}{2}e + \frac{3}{2}$ so that

$$y = \begin{cases} \dfrac{1}{2} + \dfrac{3}{2}e^{-x^2}, & 0 \le x \le 1 \\ \left(\dfrac{1}{2}e + \dfrac{3}{2}\right)e^{-x^2}, & x > 1. \end{cases}$$

**40.** For

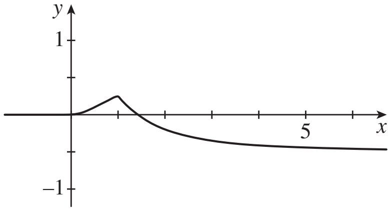
$$y' + \frac{2x}{1+x^2}y = \begin{cases} \dfrac{x}{1+x^2}, & 0 \le x \le 1 \\ \dfrac{-x}{1+x^2}, & x > 1, \end{cases}$$

an integrating factor is $1 + x^2$ so that

$$\left(1+x^2\right)y = \begin{cases} \dfrac{1}{2}x^2 + c_1, & 0 \le x \le 1 \\ -\dfrac{1}{2}x^2 + c_2, & x > 1. \end{cases}$$

If $y(0) = 0$ then $c_1 = 0$ and for continuity we must have $c_2 = 1$ so that

$$y = \begin{cases} \dfrac{1}{2} - \dfrac{1}{2\left(1+x^2\right)}, & 0 \le x \le 1 \\ \dfrac{3}{2\left(1+x^2\right)} - \dfrac{1}{2}, & x > 1. \end{cases}$$

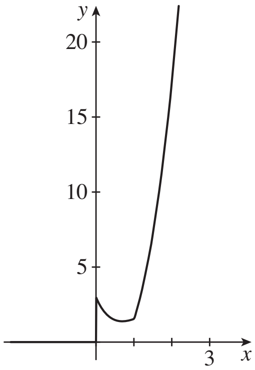


**41.** We first solve the initial-value problem $y' + 2y = 4x$, $y(0) = 3$ on the interval $[0, 1]$. The integrating factor is $e^{\int 2\,dx} = e^{2x}$, so

$$\frac{d}{dx}[e^{2x}y] = 4xe^{2x}$$

$$e^{2x}y = \int 4xe^{2x}\,dx = 2xe^{2x} - e^{2x} + c_1$$

$$y = 2x - 1 + c_1e^{-2x}.$$



Using the initial condition, we find $y(0) = -1 + c_1 = 3$, so $c_1 = 4$ and $y = 2x - 1 + 4e^{-2x}$, $0 \le x \le 1$. Now, since $y(1) = 2 - 1 + 4e^{-2} = 1 + 4e^{-2}$, we solve the initial-value problem $y' - (2/x)y = 4x$, $y(1) = 1 + 4e^{-2}$ on the interval $(1, \infty)$. The integrating factor is $e^{\int (-2/x)\,dx} = e^{-2\ln x} = x^{-2}$, so

$$\frac{d}{dx}[x^{-2}y] = 4xx^{-2} = \frac{4}{x}$$

$$x^{-2}y = \int \frac{4}{x}\,dx = 4\ln x + c_2$$

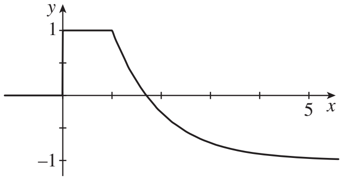
$$y = 4x^2\ln x + c_2x^2.$$

(We use $\ln x$ instead of $\ln|x|$ because $x > 1$.) Using the initial condition we find

$y(1) = c_2 = 1 + 4e^{-2}$, so $y = 4x^2 \ln x + (1 + 4e^{-2})x^2$, $x > 1$. Thus,

$$y = \begin{cases} 2x - 1 + 4e^{-2x}, & 0 \le x \le 1 \\ \\ 4x^2 \ln x + \left(1 + 4e^{-2}\right) x^2, & x > 1. \end{cases}$$

**42.** We first solve the initial-value problem $y' + y = 0$, $y(0) = 4$ on the interval $[0, 2]$. The integrating factor is $e^{\int 1\, dx} = e^x$, so



$$\frac{d}{dx}[e^x y] = 0$$

$$e^x y = \int 0\, dx = c_1$$

$$y = c_1 e^{-x}.$$

Using the initial condition, we find $y(0) = c_1 = 4$, so $c_1 = 4$ and $y = 4e^{-x}$, $0 \le x \le 2$. Now, since $y(2) = 4e^{-2}$, we solve the initial-value problem $y' + 5y = 0$, $y(1) = 4e^{-2}$ on the interval $(2, \infty)$. The integrating factor is $e^{\int 5\, dx} = e^{5x}$, so

$$\frac{d}{dx}\left[e^{5x} y\right] = 0$$

$$e^{5x} y = \int 0\, dx = c_2$$

$$y = c_2 e^{-5x}.$$

Using the initial condition we find $y(2) = c_2 e^{-10} = 4e^{-2}$, so $c_2 = 4e^8$ and $y = 4e^8 e^{-5x} = 4e^{8-5x}$, $x > 2$. Thus, the solution of the original initial-value problem is

$$y = \begin{cases} 4e^{-x}, & 0 \le x \le 2 \\ \\ 4x^{8-5x}, & x > 2. \end{cases}$$

**43.** An integrating factor for $y' - 2xy = 1$ is $e^{-x^2}$. Thus

$$\begin{aligned} \frac{d}{dx}[e^{-x^2}y] &= e^{-x^2} \\ e^{-x^2}y &= \int_0^x e^{-t^2}\,dt = \frac{\sqrt{\pi}}{2}\operatorname{erf}(x) + c \\ y &= \frac{\sqrt{\pi}}{2}e^{x^2}\operatorname{erf}(x) + ce^{x^2}. \end{aligned}$$

From $y(1) = (\sqrt{\pi}/2)e\operatorname{erf}(1) + ce = 1$ we get $c = e^{-1} - \frac{\sqrt{\pi}}{2}\operatorname{erf}(1)$. The solution of the initial-value problem is

$$\begin{aligned} y &= \frac{\sqrt{\pi}}{2}\,e^{x^2}\operatorname{erf}(x) + \left(e^{-1} - \frac{\sqrt{\pi}}{2}\operatorname{erf}(1)\right)e^{x^2} \\ &= e^{x^2-1} + \frac{\sqrt{\pi}}{2}\,e^{x^2}(\operatorname{erf}(x) - \operatorname{erf}(1)). \end{aligned}$$

**44.** An integrating factor for $y' - 2xy = -1$ is $e^{-x^2}$. Thus

$$\begin{aligned} \frac{d}{dx}[e^{-x^2}y] &= -e^{-x^2} \\ e^{-x^2}y &= -\int_0^x e^{-t^2}\,dt = -\frac{\sqrt{\pi}}{2}\operatorname{erf}(x) + c. \end{aligned}$$

From $y(0) = \sqrt{\pi}/2$, and noting that $\operatorname{erf}(0) = 0$, we get $c = \sqrt{\pi}/2$. Thus

$$y = e^{x^2}\left(-\frac{\sqrt{\pi}}{2}\operatorname{erf}(x) + \frac{\sqrt{\pi}}{2}\right) = \frac{\sqrt{\pi}}{2}\,e^{x^2}(1 - \operatorname{erf}(x)) = \frac{\sqrt{\pi}}{2}\,e^{x^2}\operatorname{erfc}(x).$$

**45.** For $y' + e^x y = 1$ an integrating factor is $e^{e^x}$. Thus

$$\frac{d}{dx}\left[e^{e^x}y\right] = e^{e^x} \quad \text{and} \quad e^{e^x}y = \int_0^x e^{e^t}\,dt + c.$$

From $y(0) = 1$ we get $c = e$, so $y = e^{-e^x}\int_0^x e^{e^t}\,dt + e^{1-e^x}$.

**46.** Dividing by $x^2$ we have $y' - \dfrac{1}{x^2}y = x$. An integrating factor is $e^{1/x}$. Thus

$$\frac{d}{dx}\left[e^{1/x}y\right] = xe^{1/x} \quad \text{and} \quad e^{1/x}y = \int_1^x te^{1/t}\,dt + c.$$

From $y(1) = 0$ we get $c = 0$, so $y = e^{-1/x}\int_1^x te^{1/t}\,dt$.

**47.** An integrating factor for

$$y' + \frac{2}{x}y = \frac{10\sin x}{x^3}$$

is $x^2$. Thus

$$\begin{aligned}\frac{d}{dx}\left[x^2y\right] &= 10\frac{\sin x}{x}\\ x^2y &= 10\int_0^x \frac{\sin t}{t}\,dt + c\\ y &= 10x^{-2}\,\mathrm{Si}\,(x) + cx^{-2}.\end{aligned}$$

From $y(1) = 0$ we get $c = -10\,\mathrm{Si}\,(1)$. Thus

$$y = 10x^{-2}\,\mathrm{Si}\,(x) - 10x^{-2}\,\mathrm{Si}\,(1) = 10x^{-2}\left(\mathrm{Si}\,(x) - \mathrm{Si}\,(1)\right).$$

**48.** The integrating factor for $y' - \left(\sin x^2\right)y = 0$ is $e^{-\int_0^x \sin t^2\,dt}$. Then

$$\begin{aligned}\frac{d}{dx}\left[e^{-\int_0^x \sin t^2\,dt}y\right] &= 0\\ e^{-\int_0^x \sin t^2\,dt}y &= c_1\\ y &= c_1e^{\int_0^x \sin t^2\,dt}\end{aligned}$$

Letting $t = \sqrt{\pi/2}\,u$ we have $dt = \sqrt{\pi/2}\,du$ and

$$\int_0^x \sin t^2\,dt = \sqrt{\frac{\pi}{2}}\int_0^{\sqrt{2/\pi}\,x}\sin\left(\frac{\pi}{2}u^2\right)du = \sqrt{\frac{\pi}{2}}\,S\left(\sqrt{\frac{2}{\pi}}\,x\right)$$

so $y = c_1e^{\sqrt{\pi/2}\,S\left(\sqrt{2/\pi}\,x\right)}$. Using $S(0) = 0$ and $y(0) = c_1 = 5$ we have $y = 5e^{\sqrt{\pi/2}\,S\left(\sqrt{2/\pi}\,x\right)}$.

**49.** We want 4 to be a critical point, so we use $y' = 4 - y$.

**50.** **(a)** All solutions of the form $y = x^5e^x - x^4e^x + cx^4$ satisfy the initial condition. In this case, since $4/x$ is discontinuous at $x = 0$, the hypotheses of Theorem 1.2.1 are not satisfied and the initial-value problem does not have a unique solution.

**(b)** The differential equation has no solution satisfying $y(0) = y_0$, $y_0 > 0$.

**(c)** In this case, since $x_0 > 0$, Theorem 1.2.1 applies and the initial-value problem has a unique solution given by $y = x^5e^x - x^4e^x + cx^4$ where $c = y_0/x_0^4 - x_0e^{x_0} + e^{x_0}$.

**51.** On the interval $(-3, 3)$ the integrating factor is

$$e^{\int x\,dx/(x^2-9)} = e^{-\int x\,dx/(9-x^2)} = e^{\frac{1}{2}\ln(9-x^2)} = \sqrt{9-x^2}$$

and so

$$\frac{d}{dx}\left[\sqrt{9-x^2}\,y\right] = 0 \quad \text{and} \quad y = \frac{c}{\sqrt{9-x^2}}\,.$$

**52.** We want the general solution to be $y = 3x - 5 + ce^{-x}$. (Rather than $e^{-x}$, any function that approaches 0 as $x \to \infty$ could be used.) Differentiating we get

$$y' = 3 - ce^{-x} = 3 - (y - 3x + 5) = -y + 3x - 2,$$

so the differential equation $y' + y = 3x - 2$ has solutions asymptotic to the line $y = 3x - 5$.

**53.** The left-hand derivative of the function at $x = 1$ is $1/e$ and the right-hand derivative at $x = 1$ is $1 - 1/e$. Thus, $y$ is not differentiable at $x = 1$.

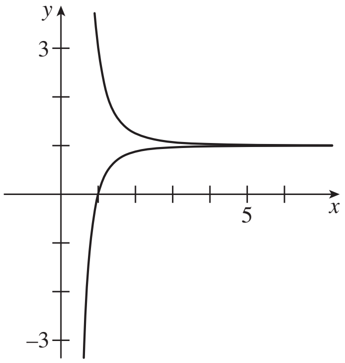
**54.** **(a)** Differentiating $y_c = c/x^3$ we get

$$y_c' = -\frac{3c}{x^4} = -\frac{3}{x}\,\frac{c}{x^3} = -\frac{3}{x}\,y_c$$

so a differential equation with general solution $y_c = c/x^3$ is $xy' + 3y = 0$. Now using $y_p = x^3$

$$xy_p' + 3y_p = x(3x^2) + 3(x^3) = 6x^3$$

so a differential equation with general solution $y = c/x^3 + x^3$ is $xy' + 3y = 6x^3$. This will be a general solution on $(0, \infty)$.



**(b)** Since $y(1) = 1^3 - 1/1^3 = 0$, an initial condition is $y(1) = 0$. Since $y(1) = 1^3 + 2/1^3 = 3$, an initial condition is $y(1) = 3$. In each case the interval of definition is $(0, \infty)$. The initial-value problem $xy' + 3y = 6x^3$, $y(0) = 0$ has solution $y = x^3$ for $-\infty < x < \infty$. In the figure the lower curve is the graph of $y(x) = x^3 - 1/x^3$, while the upper curve is the graph of $y = x^3 - 2/x^3$.

**(c)** The first two initial-value problems in part (b) are not unique. For example, setting $y(2) = 2^3 - 1/2^3 = 63/8$, we see that $y(2) = 63/8$ is also an initial condition leading to the solution $y = x^3 - 1/x^3$.

**55.** Since $e^{\int P(x)\,dx + c} = e^c e^{\int P(x)\,dx} = c_1 e^{\int P(x)\,dx}$, we would have

$$c_1 e^{\int P(x)\,dx} y = c_2 + \int c_1 e^{\int P(x)\,dx} f(x)\,dx \quad \text{and} \quad e^{\int P(x)\,dx} y = c_3 + \int e^{\int P(x)\,dx} f(x)\,dx,$$

which is the same as (4) in the text.

**56.** We see by inspection that $y = 0$ is a solution.

**57.** The solution of the first equation is $x = c_1 e^{-\lambda_1 t}$. From $x(0) = x_0$ we obtain $c_1 = x_0$ and so $x = x_0 e^{-\lambda_1 t}$. The second equation then becomes

$$\frac{dy}{dt} = x_0 \lambda_1 e^{-\lambda_1 t} - \lambda_2 y \qquad \text{or} \qquad \frac{dy}{dt} + \lambda_2 y = x_0 \lambda_1 e^{-\lambda_1 t}$$

which is linear. An integrating factor is $e^{\lambda_2 t}$. Thus

$$\frac{d}{dt}\left[e^{\lambda_2 t} y\right] = x_0 \lambda_1 e^{-\lambda_1 t} e^{\lambda_2 t} = x_0 \lambda_1 e^{(\lambda_2 - \lambda_1)t}$$

$$e^{\lambda_2 t} y = \frac{x_0 \lambda_1}{\lambda_2 - \lambda_1} e^{(\lambda_2 - \lambda_1)t} + c_2$$

$$y = \frac{x_0 \lambda_1}{\lambda_2 - \lambda_1} e^{-\lambda_1 t} + c_2 e^{-\lambda_2 t}.$$

From $y(0) = y_0$ we obtain $c_2 = (y_0\lambda_2 - y_0\lambda_1 - x_0\lambda_1)\,/\,(\lambda_2 - \lambda_1)$. The solution is

$$y = \frac{x_0\lambda_1}{\lambda_2 - \lambda_1}\,e^{-\lambda_1 t} + \frac{y_0\lambda_2 - y_0\lambda_1 - x_0\lambda_1}{\lambda_2 - \lambda_1}\,e^{-\lambda_2 t}.$$

**58.** Writing the differential equation as $\dfrac{dE}{dt} + \dfrac{1}{RC}E = 0$ we see that an integrating factor is $e^{t/RC}$. Then

$$\frac{d}{dt}\left[e^{t/RC}E\right] = 0$$

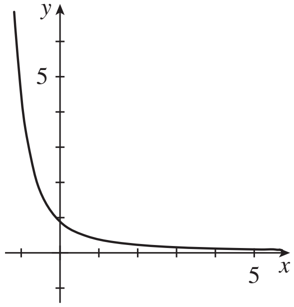
$$e^{t/RC}E = c$$

$$E = ce^{-t/RC}$$

From $E(4) = ce^{-4/RC} = E_0$ we find $c = E_0e^{4/RC}$. Thus, the solution of the initial-value problem is

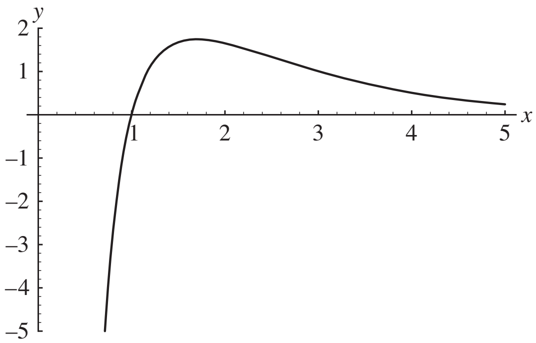
$$E = E_0e^{4/RC}e^{-t/RC} = E_0e^{-(t-4)/RC}.$$

**59. (a)**



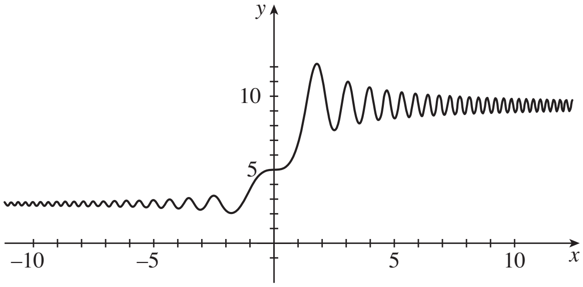
**(b)** Using a CAS we find $y(2) \approx 0.226339$.

**60. (a)**



**(b)** From the graph in part (b) we see that the absolute maximum occurs around $x = 1.7$. Using the root-finding capability of a CAS and solving $y'(x) = 0$ for $x$ we see that the absolute maximum is $(1.688, 1.742)$.

**61. (a)**



**(b)** From the graph we see that as $x \to \infty$, $y(x)$ oscillates with decreasing amplitudes approaching 9.35672. Since $\lim_{x\to\infty} S(x) = \dfrac{1}{2}$, we have $\lim_{x\to\infty} y(x) = 5e^{\sqrt{\pi/8}} \approx 9.357$, and since $\lim_{x\to-\infty} S(x) = -\dfrac{1}{2}$, we have $\lim_{x\to-\infty} y(x) = 5e^{-\sqrt{\pi/8}} \approx 2.672$.

**(c)** From the graph in part (b) we see that the absolute maximum occurs around $x = 1.7$ and the absolute minimum occurs around $x = -1.8$. Using the root-finding capability of a CAS and solving $y'(x) = 0$ for $x$, we see that the absolute maximum is $(1.772, 12.235)$ and the absolute minimum is $(-1.772, 2.044)$.

## 2.4 Exact Equations

**1.** Let $M = 2x - 1$ and $N = 3y + 7$ so that $M_y = 0 = N_x$. From $f_x = 2x - 1$ we obtain $f = x^2 - x + h(y)$, $h'(y) = 3y + 7$, and $h(y) = \frac{3}{2}y^2 + 7y$. A solution is $x^2 - x + \frac{3}{2}y^2 + 7y = c$.

**2.** Let $M = 2x + y$ and $N = -x - 6y$. Then $M_y = 1$ and $N_x = -1$, so the equation is not exact.

**3.** Let $M = 5x + 4y$ and $N = 4x - 8y^3$ so that $M_y = 4 = N_x$. From $f_x = 5x + 4y$ we obtain $f = \frac{5}{2}x^2 + 4xy + h(y)$, $h'(y) = -8y^3$, and $h(y) = -2y^4$. A solution is $\frac{5}{2}x^2 + 4xy - 2y^4 = c$.

**4.** Let $M = \sin y - y\sin x$ and $N = \cos x + x\cos y - y$ so that $M_y = \cos y - \sin x = N_x$. From $f_x = \sin y - y\sin x$ we obtain $f = x\sin y + y\cos x + h(y)$, $h'(y) = -y$, and $h(y) = -\frac{1}{2}y^2$. A solution is $x\sin y + y\cos x - \frac{1}{2}y^2 = c$.

**5.** Let $M = 2y^2x - 3$ and $N = 2yx^2 + 4$ so that $M_y = 4xy = N_x$. From $f_x = 2y^2x - 3$ we obtain $f = x^2y^2 - 3x + h(y)$, $h'(y) = 4$, and $h(y) = 4y$. A solution is $x^2y^2 - 3x + 4y = c$.

**6.** Let $M = 4x^3 - 3y\sin 3x - y/x^2$ and $N = 2y - 1/x + \cos 3x$ so that $M_y = -3\sin 3x - 1/x^2$ and $N_x = 1/x^2 - 3\sin 3x$. The equation is not exact.

**7.** Let $M = x^2 - y^2$ and $N = x^2 - 2xy$ so that $M_y = -2y$ and $N_x = 2x - 2y$. The equation is not exact.

**8.** Let $M = 1 + \ln x + y/x$ and $N = -1 + \ln x$ so that $M_y = 1/x = N_x$. From $f_y = -1 + \ln x$ we obtain $f = -y + y\ln x + h(x)$, $h'(x) = 1 + \ln x$, and $h(x) = x\ln x$. A solution is $-y + y\ln x + x\ln x = c$.

**9.** Let $M = y^3 - y^2\sin x - x$ and $N = 3xy^2 + 2y\cos x$ so that $M_y = 3y^2 - 2y\sin x = N_x$. From $f_x = y^3 - y^2\sin x - x$ we obtain $f = xy^3 + y^2\cos x - \frac{1}{2}x^2 + h(y)$, $h'(y) = 0$, and $h(y) = 0$. A solution is $xy^3 + y^2\cos x - \frac{1}{2}x^2 = c$.

**10.** Let $M = x^3 + y^3$ and $N = 3xy^2$ so that $M_y = 3y^2 = N_x$. From $f_x = x^3 + y^3$ we obtain $f = \frac{1}{4}x^4 + xy^3 + h(y)$, $h'(y) = 0$, and $h(y) = 0$. A solution is $\frac{1}{4}x^4 + xy^3 = c$.

**11.** Let $M = y\ln y - e^{-xy}$ and $N = 1/y + x\ln y$ so that $M_y = 1 + \ln y + xe^{-xy}$ and $N_x = \ln y$. The equation is not exact.

**12.** Let $M = 3x^2y + e^y$ and $N = x^3 + xe^y - 2y$ so that $M_y = 3x^2 + e^y = N_x$. From $f_x = 3x^2y + e^y$ we obtain $f = x^3y + xe^y + h(y)$, $h'(y) = -2y$, and $h(y) = -y^2$. A solution is $x^3y + xe^y - y^2 = c$.

**13.** Let $M = y - 6x^2 - 2xe^x$ and $N = x$ so that $M_y = 1 = N_x$. From $f_x = y - 6x^2 - 2xe^x$ we obtain $f = xy - 2x^3 - 2xe^x + 2e^x + h(y)$, $h'(y) = 0$, and $h(y) = 0$. A solution is $xy - 2x^3 - 2xe^x + 2e^x = c$.

**14.** Let $M = 1 - 3/x + y$ and $N = 1 - 3/y + x$ so that $M_y = 1 = N_x$. From $f_x = 1 - 3/x + y$ we obtain $f = x - 3\ln|x| + xy + h(y)$, $h'(y) = 1 - \dfrac{3}{y}$, and $h(y) = y - 3\ln|y|$. A solution is $x + y + xy - 3\ln|xy| = c$.

**15.** Let $M = x^2y^3 - 1/\left(1 + 9x^2\right)$ and $N = x^3y^2$ so that $M_y = 3x^2y^2 = N_x$. From $f_x = x^2y^3 - 1/\left(1 + 9x^2\right)$ we obtain $f = \frac{1}{3}x^3y^3 - \frac{1}{3}\arctan(3x) + h(y)$, $h'(y) = 0$, and $h(y) = 0$. A solution is $x^3y^3 - \arctan(3x) = c$.

**16.** Let $M = -2y$ and $N = 5y - 2x$ so that $M_y = -2 = N_x$. From $f_x = -2y$ we obtain $f = -2xy + h(y)$, $h'(y) = 5y$, and $h(y) = \frac{5}{2}y^2$. A solution is $-2xy + \frac{5}{2}y^2 = c$.

**17.** Let $M = \tan x - \sin x\sin y$ and $N = \cos x\cos y$ so that $M_y = -\sin x\cos y = N_x$. From $f_x = \tan x - \sin x\sin y$ we obtain $f = \ln|\sec x| + \cos x\sin y + h(y)$, $h'(y) = 0$, and $h(y) = 0$. A solution is $\ln|\sec x| + \cos x\sin y = c$.

**18.** Let $M = 2y\sin x\cos x - y + 2y^2e^{xy^2}$ and $N = -x + \sin^2 x + 4xye^{xy^2}$ so that

$$M_y = 2\sin x\cos x - 1 + 4xy^3e^{xy^2} + 4ye^{xy^2} = N_x.$$

From $f_x = 2y\sin x\cos x - y + 2y^2e^{xy^2}$ we obtain $f = y\sin^2 x - xy + 2e^{xy^2} + h(y)$, $h'(y) = 0$, and $h(y) = 0$. A solution is $y\sin^2 x - xy + 2e^{xy^2} = c$.

**19.** Let $M = 4t^3y - 15t^2 - y$ and $N = t^4 + 3y^2 - t$ so that $M_y = 4t^3 - 1 = N_t$. From $f_t = 4t^3y - 15t^2 - y$ we obtain $f = t^4y - 5t^3 - ty + h(y)$, $h'(y) = 3y^2$, and $h(y) = y^3$. A solution is $t^4y - 5t^3 - ty + y^3 = c$.

**20.** Let $M = 1/t + 1/t^2 - y/\left(t^2+y^2\right)$ and $N = ye^y + t/\left(t^2+y^2\right)$ so that $M_y = \left(y^2-t^2\right)/\left(t^2+y^2\right)^2 = N_t$. From $f_t = 1/t + 1/t^2 - y/\left(t^2+y^2\right)$ we obtain $f = \ln|t| - \dfrac{1}{t} - \arctan\left(\dfrac{t}{y}\right) + h(y)$, $h'(y) = ye^y$, and $h(y) = ye^y - e^y$. A solution is

$$\ln|t| - \frac{1}{t} - \arctan\left(\frac{t}{y}\right) + ye^y - e^y = c.$$

**21.** Let $M = x^2 + 2xy + y^2$ and $N = 2xy + x^2 - 1$ so that $M_y = 2(x+y) = N_x$. From $f_x = x^2 + 2xy + y^2$ we obtain $f = \frac{1}{3}x^3 + x^2y + xy^2 + h(y)$, $h'(y) = -1$, and $h(y) = -y$. The solution is $\frac{1}{3}x^3 + x^2y + xy^2 - y = c$. If $y(1) = 1$ then $c = 4/3$ and a solution of the initial-value problem is $\frac{1}{3}x^3 + x^2y + xy^2 - y = \frac{4}{3}$.

**22.** Let $M = e^x + y$ and $N = 2 + x + ye^y$ so that $M_y = 1 = N_x$. From $f_x = e^x + y$ we obtain $f = e^x + xy + h(y)$, $h'(y) = 2 + ye^y$, and $h(y) = 2y + ye^y - e^y$. The solution is $e^x + xy + 2y + ye^y - e^y = c$. If $y(0) = 1$ then $c = 3$ and a solution of the initial-value problem is $e^x + xy + 2y + ye^y - e^y = 3$.

**23.** Let $M = 4y + 2t - 5$ and $N = 6y + 4t - 1$ so that $M_y = 4 = N_t$. From $f_t = 4y + 2t - 5$ we obtain $f = 4ty + t^2 - 5t + h(y)$, $h'(y) = 6y - 1$, and $h(y) = 3y^2 - y$. The solution is $4ty + t^2 - 5t + 3y^2 - y = c$. If $y(-1) = 2$ then $c = 8$ and a solution of the initial-value problem is $4ty + t^2 - 5t + 3y^2 - y = 8$.

**24.** Let $M = t/2y^4$ and $N = \left(3y^2 - t^2\right)/y^5$ so that $M_y = -2t/y^5 = N_t$. From $f_t = t/2y^4$ we obtain $f = \dfrac{t^2}{4y^4} + h(y)$, $h'(y) = \dfrac{3}{y^3}$, and $h(y) = -\dfrac{3}{2y^2}$. The solution is $\dfrac{t^2}{4y^4} - \dfrac{3}{2y^2} = c$. If $y(1) = 1$ then $c = -5/4$ and a solution of the initial-value problem is $\dfrac{t^2}{4y^4} - \dfrac{3}{2y^2} = -\dfrac{5}{4}$.

**25.** Let $M = y^2\cos x - 3x^2y - 2x$ and $N = 2y\sin x - x^3 + \ln y$ so that $M_y = 2y\cos x - 3x^2 = N_x$. From $f_x = y^2\cos x - 3x^2y - 2x$ we obtain $f = y^2\sin x - x^3y - x^2 + h(y)$, $h'(y) = \ln y$, and $h(y) = y\ln y - y$. The solution is $y^2\sin x - x^3y - x^2 + y\ln y - y = c$. If $y(0) = e$ then $c = 0$ and a solution of the initial-value problem is $y^2\sin x - x^3y - x^2 + y\ln y - y = 0$.

**26.** Let $M = y^2 + y\sin x$ and $N = 2xy - \cos x - 1/\left(1+y^2\right)$ so that $M_y = 2y + \sin x = N_x$. From $f_x = y^2 + y\sin x$ we obtain $f = xy^2 - y\cos x + h(y)$, $h'(y) = \dfrac{-1}{1+y^2}$, and $h(y) = -\tan^{-1} y$.

The solution is $xy^2 - y\cos x - \tan^{-1} y = c$. If $y(0) = 1$ then $c = -1 - \pi/4$ and a solution of the initial-value problem is $xy^2 - y\cos x - \tan^{-1} y = -1 - \dfrac{\pi}{4}$.

**27.** Equating $M_y = 3y^2 + 4kxy^3$ and $N_x = 3y^2 + 40xy^3$ we obtain $k = 10$.

**28.** Equating $M_y = 18xy^2 - \sin y$ and $N_x = 4kxy^2 - \sin y$ we obtain $k = 9/2$.

**29.** Let $M = -x^2y^2\sin x + 2xy^2\cos x$ and $N = 2x^2y\cos x$ so that $M_y = -2x^2y\sin x + 4xy\cos x = N_x$. From $f_y = 2x^2y\cos x$ we obtain $f = x^2y^2\cos x + h(y)$, $h'(y) = 0$, and $h(y) = 0$. A solution of the differential equation is $x^2y^2\cos x = c$.

**30.** Let $M = (x^2 + 2xy - y^2)/(x^2 + 2xy + y^2)$ and $N = (y^2 + 2xy - x^2)/(y^2 + 2xy + x^2)$ so that $M_y = -4xy/(x + y)^3 = N_x$. From $f_x = \left(x^2 + 2xy + y^2 - 2y^2\right)/(x + y)^2$ we obtain $f = x + \dfrac{2y^2}{x + y} + h(y)$, $h'(y) = -1$, and $h(y) = -y$. A solution of the differential equation is $x^2 + y^2 = c(x + y)$.

**31.** We note that $(M_y - N_x)/N = 1/x$, so an integrating factor is $e^{\int dx/x} = x$. Let $M = 2xy^2 + 3x^2$ and $N = 2x^2y$ so that $M_y = 4xy = N_x$. From $f_x = 2xy^2 + 3x^2$ we obtain $f = x^2y^2 + x^3 + h(y)$, $h'(y) = 0$, and $h(y) = 0$. A solution of the differential equation is $x^2y^2 + x^3 = c$.

**32.** We note that $(M_y - N_x)/N = 1$, so an integrating factor is $e^{\int dx} = e^x$. Let $M = xye^x + y^2e^x + ye^x$ and $N = xe^x + 2ye^x$ so that $M_y = xe^x + 2ye^x + e^x = N_x$. From $f_y = xe^x + 2ye^x$ we obtain $f = xye^x + y^2e^x + h(x)$, $h'(x) = 0$, and $h(x) = 0$. A solution of the differential equation is $xye^x + y^2e^x = c$.

**33.** We note that $(N_x - M_y)/M = 2/y$, so an integrating factor is $e^{\int 2\,dy/y} = y^2$. Let $M = 6xy^3$ and $N = 4y^3 + 9x^2y^2$ so that $M_y = 18xy^2 = N_x$. From $f_x = 6xy^3$ we obtain $f = 3x^2y^3 + h(y)$, $h'(y) = 4y^3$, and $h(y) = y^4$. A solution of the differential equation is $3x^2y^3 + y^4 = c$.

**34.** We note that $(M_y - N_x)/N = -\cot x$, so an integrating factor is $e^{-\int \cot x\,dx} = \csc x$. Let $M = \cos x\csc x = \cot x$ and $N = (1 + 2/y)\sin x\csc x = 1 + 2/y$, so that $M_y = 0 = N_x$. From $f_x = \cot x$ we obtain $f = \ln(\sin x) + h(y)$, $h'(y) = 1 + 2/y$, and $h(y) = y + \ln y^2$. A solution of the differential equation is $\ln(\sin x) + y + \ln y^2 = c$.

**35.** We note that $(M_y - N_x)/N = 3$, so an integrating factor is $e^{\int 3\,dx} = e^{3x}$. Let $M = (10 - 6y + e^{-3x})e^{3x} = 10e^{3x} - 6ye^{3x} + 1$ and $N = -2e^{3x}$, so that $M_y = -6e^{3x} = N_x$. From $f_x = 10e^{3x} - 6ye^{3x} + 1$ we obtain $f = \frac{10}{3}e^{3x} - 2ye^{3x} + x + h(y)$, $h'(y) = 0$, and $h(y) = 0$. A solution of the differential equation is $\frac{10}{3}e^{3x} - 2ye^{3x} + x = c$.

**36.** We note that $(N_x - M_y)/M = -3/y$, so an integrating factor is $e^{-3\int dy/y} = 1/y^3$. Let $M = (y^2 + xy^3)/y^3 = 1/y + x$ and $N = (5y^2 - xy + y^3\sin y)/y^3 = 5/y - x/y^2 + \sin y$, so that $M_y = -1/y^2 = N_x$. From $f_x = 1/y + x$ we obtain $f = x/y + \frac{1}{2}x^2 + h(y)$, $h'(y) = 5/y + \sin y$, and $h(y) = 5\ln|y| - \cos y$. A solution of the differential equation is $x/y + \frac{1}{2}x^2 + 5\ln|y| - \cos y = c$.

**37.** We note that $(M_y - N_x)/N = 2x/(4+x^2)$, so an integrating factor is $e^{-2\int x\,dx/(4+x^2)} = 1/(4+x^2)$. Let $M = x/(4+x^2)$ and $N = (x^2y+4y)/(4+x^2) = y$, so that $M_y = 0 = N_x$. From $f_x = x(4+x^2)$ we obtain $f = \frac{1}{2}\ln(4+x^2) + h(y)$, $h'(y) = y$, and $h(y) = \frac{1}{2}y^2$. A solution of the differential equation is $\frac{1}{2}\ln(4+x^2)+\frac{1}{2}y^2 = c$. Multiplying both sides by 2 the last equation can be written as $e^{y^2}\left(x^2+4\right) = c_1$. Using the initial condition $y(4) = 0$ we see that $c_1 = 20$. A solution of the initial-value problem is $e^{y^2}\left(x^2+4\right) = 20$.

**38.** We note that $(M_y - N_x)/N = -3/(1+x)$, so an integrating factor is $e^{-3\int dx/(1+x)} = 1/(1+x)^3$. Let $M = (x^2+y^2-5)/(1+x)^3$ and $N = -(y+xy)/(1+x)^3 = -y/(1+x)^2$, so that $M_y = 2y/(1+x)^3 = N_x$. From $f_y = -y/(1+x)^2$ we obtain $f = -\frac{1}{2}y^2/(1+x)^2 + h(x)$, $h'(x) = (x^2-5)/(1+x)^3$, and $h(x) = 2/(1+x)^2 + 2/(1+x) + \ln|1+x|$. A solution of the differential equation is

$$-\frac{y^2}{2(1+x)^2} + \frac{2}{(1+x)^2} + \frac{2}{(1+x)} + \ln|1+x| = c.$$

Using the initial condition $y(0) = 1$ we see that $c = 7/2$. A solution of the initial-value problem is

$$-\frac{y^2}{2\,(1+x)^2} + \frac{2}{(1+x)^2} + \frac{2}{1+x} + \ln|1+x| = \frac{7}{2}$$

**39.** **(a)** Implicitly differentiating $x^3 + 2x^2y + y^2 = c$ and solving for $dy/dx$ we obtain

$$3x^2 + 2x^2\frac{dy}{dx} + 4xy + 2y\frac{dy}{dx} = 0 \quad \text{and} \quad \frac{dy}{dx} = -\frac{3x^2+4xy}{2x^2+2y}.$$

By writing the last equation in differential form we get $(4xy+3x^2)dx + (2y+2x^2)dy = 0$.

**(b)** Setting $x = 0$ and $y = -2$ in $x^3 + 2x^2y + y^2 = c$ we find $c = 4$, and setting $x = y = 1$ we also find $c = 4$. Thus, both initial conditions determine the same implicit solution.

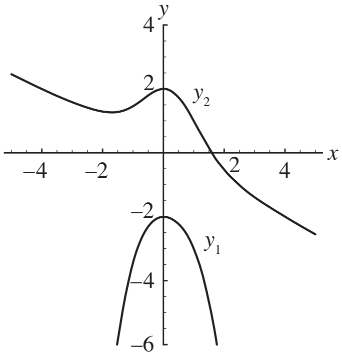
**(c)** Solving $x^3 + 2x^2y + y^2 = 4$ for $y$ we get

$$y_1(x) = -x^2 - \sqrt{4 - x^3 + x^4}$$

and

$$y_2(x) = -x^2 + \sqrt{4 - x^3 + x^4}\,.$$

Observe in the figure that $y_1(0) = -2$ and $y_2(1) = 1$.



**40.** To see that the equations are not equivalent consider $dx = -(x/y)\,dy$. An integrating factor is $\mu(x,y) = y$ resulting in $y\,dx + x\,dy = 0$. A solution of the latter equation is $y = 0$, but this is not a solution of the original equation.

**41.** The explicit solution is $y = \sqrt{(3 + \cos^2 x)/(1 - x^2)}$. Since $3 + \cos^2 x > 0$ for all $x$ we must have $1 - x^2 > 0$ or $-1 < x < 1$. Thus, the interval of definition is $(-1, 1)$.

**42.** **(a)** Since $f_y = N(x, y) = xe^{xy} + 2xy + 1/x$ we obtain $f = e^{xy} + xy^2 + \dfrac{y}{x} + h(x)$ so that $f_x = ye^{xy} + y^2 - \dfrac{y}{x^2} + h'(x)$. Let $M(x, y) = ye^{xy} + y^2 - \dfrac{y}{x^2}$.

**(b)** Since $f_x = M(x, y) = y^{1/2}x^{-1/2} + x\left(x^2 + y\right)^{-1}$ we obtain $f = 2y^{1/2}x^{1/2} + \dfrac{1}{2}\ln\left|x^2 + y\right| + g(y)$ so that $f_y = y^{-1/2}x^{1/2} + \dfrac{1}{2}\left(x^2 + y\right)^{-1} + g'(y)$. Let $N(x, y) = y^{-1/2}x^{1/2} + \dfrac{1}{2}\left(x^2 + y\right)^{-1}$.

**43.** First note that

$$d\left(\sqrt{x^2 + y^2}\right) = \frac{x}{\sqrt{x^2 + y^2}}\,dx + \frac{y}{\sqrt{x^2 + y^2}}\,dy.$$

Then $x\,dx + y\,dy = \sqrt{x^2 + y^2}\,dx$ becomes

$$\frac{x}{\sqrt{x^2 + y^2}}\,dx + \frac{y}{\sqrt{x^2 + y^2}}\,dy = d\left(\sqrt{x^2 + y^2}\right) = dx.$$

The left side is the total differential of $\sqrt{x^2 + y^2}$ and the right side is the total differential of $x + c$. Thus $\sqrt{x^2 + y^2} = x + c$ is a solution of the differential equation.

**44.** To see that the statement is true, write the separable equation as $-g(x)\,dx + dy/h(y) = 0$. Identifying $M = -g(x)$ and $N = 1/h(y)$, we see that $M_y = 0 = N_x$, so the differential equation is exact.

**45.** **(a)** In differential form

$$\left(v^2 - 32x\right)\,dx + xv\,dv = 0$$

This is not an exact equation, but $\mu(x) = x$ is an integrating factor. The new equation $\left(xv^2 - 32x^2\right)\,dx + x^2v\,dv = 0$ is exact and solving yields $\frac{1}{2}x^2v^2 - \frac{32}{3}x^3 = c$. When $x = 3$, $v = 0$ and so $c = -288$. Solving $\frac{1}{2}x^2v^2 - \frac{32}{3}x^3 = -288$ for $v$ yields the explicit solution

$$v(x) = 8\sqrt{\frac{x}{3} - \frac{9}{x^2}}\,.$$

**(b)** The chain leaves the platform when $x = 8$, and so

$$v(8) = 8\sqrt{\frac{8}{3} - \frac{9}{64}} \approx 12.7 \text{ ft/s}$$

**46.** **(a)** Letting

$$M(x, y) = \frac{2xy}{(x^2 + y^2)^2} \qquad \text{and} \qquad N(x, y) = 1 + \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

we compute

$$M_y = \frac{2x^3 - 8xy^2}{(x^2 + y^2)^3} = N_x,$$

so the differential equation is exact. Then we have

$$\begin{aligned}\frac{\partial f}{\partial x} &= M(x,y) = \frac{2xy}{(x^2+y^2)^2} = 2xy(x^2+y^2)^{-2} \\ f(x,y) &= -y(x^2+y^2)^{-1} + g(y) = -\frac{y}{x^2+y^2} + g(y) \\ \frac{\partial f}{\partial y} &= \frac{y^2-x^2}{(x^2+y^2)^2} + g'(y) = N(x,y) = 1 + \frac{y^2-x^2}{(x^2+y^2)^2}\,.\end{aligned}$$

Thus, $g'(y) = 1$ and $g(y) = y$. The solution is $y - \dfrac{y}{x^2+y^2} = c$. When $c = 0$ the solution is $x^2 + y^2 = 1$.

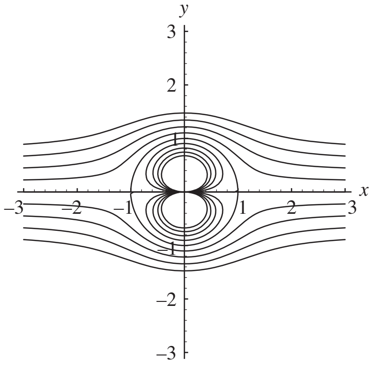
**(b)** The first graph below is obtained in *Mathematica* using $f(x,y) = y - y/(x^2+y^2)$ and

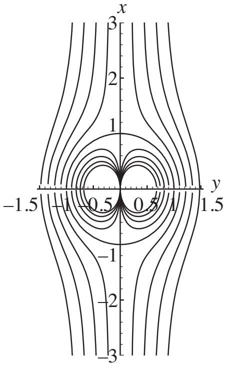
```
ContourPlot[f[x, y], {x, -3, 3}, {y, -3, 3},
    Axes->True, AxesOrigin->{0, 0}, AxesLabel->{x, y},
    Frame->False, PlotPoints->100, ContourShading->False,
    Contours->{0, -0.2, 0.2, -0.4, 0.4, -0.6, 0.6, -0.8, 0.8}]
```

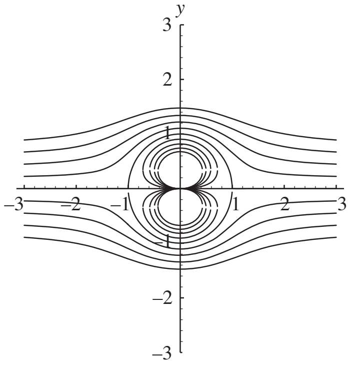
The second graph uses

$$x = -\sqrt{\frac{y^3 - cy^2 - y}{c - y}} \qquad \text{and} \qquad x = \sqrt{\frac{y^3 - cy^2 - y}{c - y}}\,.$$

In this case the $x$-axis is vertical and the $y$-axis is horizontal. To obtain the third graph, we solve $y - y/(x^2+y^2) = c$ for $y$ in a CAS. This appears to give one real and two complex solutions. When graphed in *Mathematica* however, all three solutions contribute to the graph. This is because the solutions involve the square root of expressions containing $c$. For some values of $c$ the expression is negative, causing an apparent complex solution to actually be real.







## 2.5 Solutions by Substitutions

**1.** Letting $y = ux$ we have

$$\begin{aligned} (x - ux)\,dx + x(u\,dx + x\,du) &= 0 \\ dx + x\,du &= 0 \\ \frac{dx}{x} + du &= 0 \\ \ln|x| + u &= c \\ x\ln|x| + y &= cx. \end{aligned}$$

**2.** Letting $y = ux$ we have

$$\begin{aligned} (x + ux)\,dx + x(u\,dx + x\,du) &= 0 \\ (1 + 2u)\,dx + x\,du &= 0 \\ \frac{dx}{x} + \frac{du}{1 + 2u} &= 0 \\ \ln|x| + \frac{1}{2}\ln|1 + 2u| &= c \\ x^2\left(1 + 2\frac{y}{x}\right) &= c_1 \\ x^2 + 2xy &= c_1. \end{aligned}$$

**3.** Letting $x = vy$ we have

$$\begin{aligned} vy(v\,dy + y\,dv) + (y - 2vy)\,dy &= 0 \\ vy^2\,dv + y\left(v^2 - 2v + 1\right)dy &= 0 \\ \frac{v\,dv}{(v-1)^2} + \frac{dy}{y} &= 0 \\ \ln|v - 1| - \frac{1}{v - 1} + \ln|y| &= c \\ \ln\left|\frac{x}{y} - 1\right| - \frac{1}{x/y - 1} + \ln|y| &= c \\ (x - y)\ln|x - y| - y &= c(x - y). \end{aligned}$$

**4.** Letting $x = vy$ we have

$$\begin{aligned}
y(v\,dy + y\,dv) - 2(vy + y)\,dy &= 0\\
y\,dv - (v+2)\,dy &= 0\\
\frac{dv}{v+2} - \frac{dy}{y} &= 0\\
\ln|v+2| - \ln|y| &= c\\
\ln\left|\frac{x}{y} + 2\right| - \ln|y| &= c\\
x + 2y &= c_1 y^2.
\end{aligned}$$

**5.** Letting $y = ux$ we have

$$\begin{aligned}
\left(u^2x^2 + ux^2\right)dx - x^2(u\,dx + x\,du) &= 0\\
u^2\,dx - x\,du &= 0\\
\frac{dx}{x} - \frac{du}{u^2} &= 0\\
\ln|x| + \frac{1}{u} &= c\\
\ln|x| + \frac{x}{y} &= c\\
y\ln|x| + x &= cy.
\end{aligned}$$

**6.** Letting $y = ux$ and using partial fractions, we have

$$\begin{aligned}
\left(u^2x^2 + ux^2\right)dx + x^2(u\,dx + x\,du) &= 0\\
x^2\left(u^2 + 2u\right)dx + x^3\,du &= 0\\
\frac{dx}{x} + \frac{du}{u(u+2)} &= 0\\
\ln|x| + \frac{1}{2}\ln|u| - \frac{1}{2}\ln|u+2| &= c\\
\frac{x^2u}{u+2} &= c_1\\
x^2\frac{y}{x} &= c_1\left(\frac{y}{x} + 2\right)\\
x^2y &= c_1(y + 2x).
\end{aligned}$$

**7.** Letting $y = ux$ we have

$$\begin{aligned}
(ux - x)\,dx - (ux + x)(u\,dx + x\,du) &= 0\\
\left(u^2 + 1\right)dx + x(u + 1)\,du &= 0\\
\frac{dx}{x} + \frac{u + 1}{u^2 + 1}\,du &= 0\\
\ln|x| + \frac{1}{2}\ln\left(u^2 + 1\right) + \tan^{-1}u &= c\\
\ln x^2\left(\frac{y^2}{x^2} + 1\right) + 2\tan^{-1}\frac{y}{x} &= c_1\\
\ln\left(x^2 + y^2\right) + 2\tan^{-1}\frac{y}{x} &= c_1
\end{aligned}$$

**8.** Letting $y = ux$ we have

$$\begin{aligned}
(x + 3ux)\,dx - (3x + ux)(u\,dx + x\,du) &= 0\\
\left(u^2 - 1\right)dx + x(u + 3)\,du &= 0\\
\frac{dx}{x} + \frac{u + 3}{(u - 1)(u + 1)}\,du &= 0\\
\ln|x| + 2\ln|u - 1| - \ln|u + 1| &= c\\
\frac{x(u - 1)^2}{u + 1} &= c_1\\
x\left(\frac{y}{x} - 1\right)^2 &= c_1\left(\frac{y}{x} + 1\right)\\
(y - x)^2 &= c_1(y + x).
\end{aligned}$$

**9.** Letting $y = ux$ we have

$$\begin{aligned}
-ux\,dx + (x + \sqrt{u}\,x)(u\,dx + x\,du) &= 0\\
(x^2 + x^2\sqrt{u}\,)\,du + xu^{3/2}\,dx &= 0\\
\left(u^{-3/2} + \frac{1}{u}\right)du + \frac{dx}{x} &= 0\\
-2u^{-1/2} + \ln|u| + \ln|x| &= c\\
\ln|y/x| + \ln|x| &= 2\sqrt{x/y} + c\\
y(\ln|y| - c)^2 &= 4x.
\end{aligned}$$

**10.** Letting $y = ux$ we have

$$\begin{aligned}
\sqrt{x^2 - u^2x^2}\,dx - x^2\,du &= 0\\
x\sqrt{1-u^2}\,dx - x^2\,du &= 0, \quad (x > 0)\\
\frac{dx}{x} - \frac{du}{\sqrt{1-u^2}} &= 0\\
\ln x - \sin^{-1} u &= c\\
\sin^{-1} u &= \ln x + c_1\\
\sin^{-1}\frac{y}{x} &= \ln x + c_2\\
\frac{y}{x} &= \sin(\ln x + c_2)\\
y &= x\sin(\ln x + c_2).
\end{aligned}$$

See Problem 33 in this section for an analysis of the solution.

**11.** Letting $y = ux$ we have

$$\begin{aligned}
\left(x^3 - u^3x^3\right)dx + u^2x^3(u\,dx + x\,du) &= 0\\
dx + u^2x\,du &= 0\\
\frac{dx}{x} + u^2\,du &= 0\\
\ln|x| + \frac{1}{3}u^3 &= c\\
3x^3\ln|x| + y^3 &= c_1x^3.
\end{aligned}$$

Using $y(1) = 2$ we find $c_1 = 8$. The solution of the initial-value problem is $3x^3\ln|x| + y^3 = 8x^3$.

**12.** Letting $y = ux$ we have

$$\begin{aligned}
(x^2 + 2u^2x^2)dx - ux^2(u\,dx + x\,du) &= 0\\
x^2(1+u^2)dx - ux^3\,du &= 0\\
\frac{dx}{x} - \frac{u\,du}{1+u^2} &= 0\\
\ln|x| - \frac{1}{2}\ln(1+u^2) &= c\\
\frac{x^2}{1+u^2} &= c_1\\
x^4 &= c_1(x^2+y^2).
\end{aligned}$$

Using $y(-1) = 1$ we find $c_1 = 1/2$. The solution of the initial-value problem is $2x^4 = y^2 + x^2$.

**13.** Letting $y = ux$ we have

$$\begin{aligned}(x + uxe^u)\,dx - xe^u(u\,dx + x\,du) &= 0\\ dx - xe^u\,du &= 0\\ \frac{dx}{x} - e^u\,du &= 0\\ \ln|x| - e^u &= c\\ \ln|x| - e^{y/x} &= c.\end{aligned}$$

Using $y(1) = 0$ we find $c = -1$. The solution of the initial-value problem is $\ln|x| = e^{y/x} - 1$.

**14.** Letting $x = vy$ we have

$$\begin{aligned}y(v\,dy + y\,dv) + vy(\ln vy - \ln y - 1)\,dy &= 0\\ y\,dv + v\ln v\,dy &= 0\\ \frac{dv}{v\ln v} + \frac{dy}{y} &= 0\\ \ln|\ln|v|| + \ln|y| &= c\\ y\ln\left|\frac{x}{y}\right| &= c_1.\end{aligned}$$

Using $y(1) = e$ we find $c_1 = -e$. The solution of the initial-value problem is $y\ln\left|\dfrac{x}{y}\right| = -e$.

**15.** From $y' + \dfrac{1}{x}y = \dfrac{1}{x}y^{-2}$ and $w = y^3$ we obtain $\dfrac{dw}{dx} + \dfrac{3}{x}w = \dfrac{3}{x}$. An integrating factor is $x^3$ so that $x^3w = x^3 + c$ or $y^3 = 1 + cx^{-3}$.

**16.** From $y' - y = e^xy^2$ and $w = y^{-1}$ we obtain $\dfrac{dw}{dx} + w = -e^x$. An integrating factor is $e^x$ so that $e^xw = -\frac{1}{2}e^{2x} + c$ or $y^{-1} = -\frac{1}{2}e^x + ce^{-x}$.

**17.** From $y' + y = xy^4$ and $w = y^{-3}$ we obtain $\dfrac{dw}{dx} - 3w = -3x$. An integrating factor is $e^{-3x}$ so that $e^{-3x}w = xe^{-3x} + \frac{1}{3}e^{-3x} + c$ or $y^{-3} = x + \frac{1}{3} + ce^{3x}$.

**18.** From $y' - \left(1 + \dfrac{1}{x}\right)y = y^2$ and $w = y^{-1}$ we obtain $\dfrac{dw}{dx} + \left(1 + \dfrac{1}{x}\right)w = -1$. An integrating factor is $xe^x$ so that $xe^xw = -xe^x + e^x + c$ or $y^{-1} = -1 + \dfrac{1}{x} + \dfrac{c}{x}e^{-x}$.

**19.** From $y' - \dfrac{1}{t}y = -\dfrac{1}{t^2}y^2$ and $w = y^{-1}$ we obtain $\dfrac{dw}{dt} + \dfrac{1}{t}w = \dfrac{1}{t^2}$. An integrating factor is $t$ so that $tw = \ln t + c$ or $y^{-1} = \dfrac{1}{t}\ln t + \dfrac{c}{t}$. Writing this in the form $\dfrac{t}{y} = \ln t + c$, we see that the solution can also be expressed in the form $e^{t/y} = c_1t$.

**20.** From $y' + \dfrac{2}{3\,(1+t^2)}y = \dfrac{2t}{3\,(1+t^2)}y^4$ and $w = y^{-3}$ we obtain $\dfrac{dw}{dt} - \dfrac{2t}{1+t^2}w = \dfrac{-2t}{1+t^2}$. An integrating factor is $\dfrac{1}{1+t^2}$ so that $\dfrac{w}{1+t^2} = \dfrac{1}{1+t^2} + c$ or $y^{-3} = 1 + c\left(1+t^2\right)$.

**21.** From $y' - \dfrac{2}{x}y = \dfrac{3}{x^2}y^4$ and $w = y^{-3}$ we obtain $\dfrac{dw}{dx} + \dfrac{6}{x}w = -\dfrac{9}{x^2}$. An integrating factor is $x^6$ so that $x^6 w = -\frac{9}{5}x^5 + c$ or $y^{-3} = -\frac{9}{5}x^{-1} + cx^{-6}$. If $y(1) = \frac{1}{2}$ then $c = \frac{49}{5}$ and $y^{-3} = -\frac{9}{5}x^{-1} + \frac{49}{5}x^{-6}$.

**22.** From $y' + y = y^{-1/2}$ and $w = y^{3/2}$ we obtain $\dfrac{dw}{dx} + \dfrac{3}{2}w = \dfrac{3}{2}$. An integrating factor is $e^{3x/2}$ so that $e^{3x/2}w = e^{3x/2} + c$ or $y^{3/2} = 1 + ce^{-3x/2}$. If $y(0) = 4$ then $c = 7$ and $y^{3/2} = 1 + 7e^{-3x/2}$.

**23.** Let $u = x + y + 1$ so that $du/dx = 1 + dy/dx$. Then $\dfrac{du}{dx} - 1 = u^2$ or $\dfrac{1}{1+u^2}\,du = dx$. Thus $\tan^{-1} u = x + c$ or $u = \tan\,(x + c)$, and $x + y + 1 = \tan\,(x + c)$ or $y = \tan\,(x + c) - x - 1$.

**24.** Let $u = x + y$ so that $du/dx = 1 + dy/dx$. Then $\dfrac{du}{dx} - 1 = \dfrac{1-u}{u}$ or $u\,du = dx$. Thus $\frac{1}{2}u^2 = x + c$ or $u^2 = 2x + c_1$, and $(x + y)^2 = 2x + c_1$.

**25.** Let $u = x + y$ so that $du/dx = 1 + dy/dx$. Then $\dfrac{du}{dx} - 1 = \tan^2 u$ or $\cos^2 u\,du = dx$. Thus $\frac{1}{2}u + \frac{1}{4}\sin 2u = x + c$ or $2u + \sin 2u = 4x + c_1$, and $2(x + y) + \sin 2(x + y) = 4x + c_1$ or $2y + \sin 2(x + y) = 2x + c_1$.

**26.** Let $u = x + y$ so that $du/dx = 1 + dy/dx$. Then $\dfrac{du}{dx} - 1 = \sin u$ or $\dfrac{1}{1+\sin u}\,du = dx$. Multiplying by $(1 - \sin u)/(1 - \sin u)$ we have $\dfrac{1-\sin u}{\cos^2 u}\,du = dx$ or $(\sec^2 u - \sec u \tan u)du = dx$. Thus $\tan u - \sec u = x + c$ or $\tan\,(x + y) - \sec\,(x + y) = x + c$.

**27.** Let $u = y - 2x + 3$ so that $du/dx = dy/dx - 2$. Then $\dfrac{du}{dx} + 2 = 2 + \sqrt{u}$ or $\dfrac{1}{\sqrt{u}}\,du = dx$. Thus $2\sqrt{u} = x + c$ and $2\sqrt{y - 2x + 3} = x + c$.

**28.** Let $u = y - x + 5$ so that $du/dx = dy/dx - 1$. Then $\dfrac{du}{dx} + 1 = 1 + e^u$ or $e^{-u}du = dx$. Thus $-e^{-u} = x + c$ and $-e^{-y+x-5} = x + c$.

**29.** Let $u = x + y$ so that $du/dx = 1 + dy/dx$. Then $\dfrac{du}{dx} - 1 = \cos u$ and $\dfrac{1}{1+\cos u}\,du = dx$. Now

$$\frac{1}{1+\cos u} = \frac{1-\cos u}{1-\cos^2 u} = \frac{1-\cos u}{\sin^2 u} = \csc^2 u - \csc u \cot u$$

so we have $\int (\csc^2 u - \csc u \cot u)\,du = \int dx$ and $-\cot u + \csc u = x + c$. Thus $-\cot\,(x + y) + \csc\,(x + y) = x + c$. Setting $x = 0$ and $y = \pi/4$ we obtain $c = \sqrt{2} - 1$. The solution is

$$\csc\,(x + y) - \cot\,(x + y) = x + \sqrt{2} - 1.$$

**30.** Let $u = 3x + 2y$ so that $du/dx = 3 + 2\,dy/dx$. Then $\dfrac{du}{dx} = 3 + \dfrac{2u}{u+2} = \dfrac{5u+6}{u+2}$ and $\dfrac{u+2}{5u+6}\,du = dx$. Now by long division

$$\frac{u+2}{5u+6} = \frac{1}{5} + \frac{4}{25u+30}$$

so we have

$$\int \left(\frac{1}{5} + \frac{4}{25u+30}\right) du = dx$$

and $\frac{1}{5}u + \frac{4}{25}\ln|25u + 30| = x + c$. Thus

$$\frac{1}{5}(3x+2y) + \frac{4}{25}\ln|75x + 50y + 30| = x + c.$$

Setting $x = -1$ and $y = -1$ we obtain $c = \frac{4}{25}\ln 95$. The solution is

$$\frac{1}{5}(3x+2y) + \frac{4}{25}\ln|75x + 50y + 30| = x + \frac{4}{25}\ln 95$$

or

$$5y - 5x + 2\ln|75x + 50y + 30| = 2\ln 95$$

**31.** We write the differential equation $M(x,y)dx + N(x,y)dy = 0$ as $dy/dx = f(x,y)$ where

$$f(x,y) = -\frac{M(x,y)}{N(x,y)}\,.$$

The function $f(x,y)$ must necessarily be homogeneous of degree 0 when $M$ and $N$ are homogeneous of degree $\alpha$. Since $M$ is homogeneous of degree $\alpha$, $M(tx,ty) = t^\alpha M(x,y)$, and letting $t = 1/x$ we have

$$M(1, y/x) = \frac{1}{x^\alpha}\,M(x,y) \quad \text{or} \quad M(x,y) = x^\alpha M(1, y/x).$$

Thus

$$\frac{dy}{dx} = f(x,y) = -\frac{x^\alpha M(1,y/x)}{x^\alpha N(1,y/x)} = -\frac{M(1,y/x)}{N(1,y/x)} = F\left(\frac{y}{x}\right).$$

**32.** Rewrite $(5x^2 - 2y^2)dx - xy\,dy = 0$ as

$$xy\,\frac{dy}{dx} = 5x^2 - 2y^2$$

and divide by $xy$, so that

$$\frac{dy}{dx} = 5\,\frac{x}{y} - 2\,\frac{y}{x}\,.$$

We then identify

$$F\left(\frac{y}{x}\right) = 5\left(\frac{y}{x}\right)^{-1} - 2\left(\frac{y}{x}\right).$$

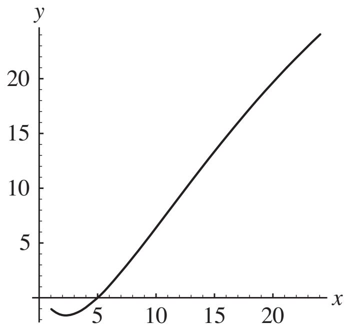
**33.** **(a)** By inspection $y = x$ and $y = -x$ are solutions of the differential equation and not members of the family $y = x\sin(\ln x + c_2)$.

**(b)** Letting $x = 5$ and $y = 0$ in $\sin^{-1}(y/x) = \ln x + c_2$ we get $\sin^{-1} 0 = \ln 5 + c_2$ or $c_2 = -\ln 5$. Then $\sin^{-1}(y/x) = \ln x - \ln 5 = \ln(x/5)$. Because the range of the arcsine function is $[-\pi/2, \pi/2]$ we must have

$$-\frac{\pi}{2} \le \ln\frac{x}{5} \le \frac{\pi}{2}$$

$$e^{-\pi/2} \le \frac{x}{5} \le e^{\pi/2}$$

$$5e^{-\pi/2} \le x \le 5e^{\pi/2}$$



The interval of definition of the solution is approximately $[1.04, 24.05]$.

**34.** As $x \to -\infty$, $e^{6x} \to 0$ and $y \to 2x+3$. Now write $(1+ce^{6x})/(1-ce^{6x})$ as $(e^{-6x}+c)/(e^{-6x}-c)$. Then, as $x \to \infty$, $e^{-6x} \to 0$ and $y \to 2x-3$.

**35.** **(a)** The substitutions $y = y_1 + u$ and

$$\frac{dy}{dx} = \frac{dy_1}{dx} + \frac{du}{dx}$$

lead to

$$\begin{aligned}\frac{dy_1}{dx} + \frac{du}{dx} &= P + Q(y_1+u) + R(y_1+u)^2 \\ &= P + Qy_1 + Ry_1^2 + Qu + 2y_1Ru + Ru^2\end{aligned}$$

or

$$\frac{du}{dx} - (Q + 2y_1R)u = Ru^2.$$

This is a Bernoulli equation with $n = 2$ which can be reduced to the linear equation

$$\frac{dw}{dx} + (Q + 2y_1R)w = -R$$

by the substitution $w = u^{-1}$.

**(b)** Identify $P(x) = -4/x^2$, $Q(x) = -1/x$, and $R(x) = 1$. Then $\dfrac{dw}{dx} + \left(-\dfrac{1}{x} + \dfrac{4}{x}\right)w = -1$. An integrating factor is $x^3$ so that $x^3w = -\frac{1}{4}x^4 + c$ or $u = \left[-\frac{1}{4}x + cx^{-3}\right]^{-1}$. Thus,

$$y = \frac{2}{x} + u \quad \text{or} \quad y = \frac{2}{x} + \left(-\frac{1}{4}x + cx^{-3}\right)^{-1}$$

**36.** Write the differential equation in the form $x(y'/y) = \ln x + \ln y$ and let $u = \ln y$. Then $du/dx = y'/y$ and the differential equation becomes $x(du/dx) = \ln x + u$ or $du/dx - u/x =$

$(\ln x)/x$, which is first-order and linear. An integrating factor is $e^{-\int dx/x} = 1/x$, so that (using integration by parts)

$$\frac{d}{dx}\left[\frac{1}{x}\,u\right] = \frac{\ln x}{x^2} \quad \text{and} \quad \frac{u}{x} = -\frac{1}{x} - \frac{\ln x}{x} + c.$$

The solution is

$$\ln y = -1 - \ln x + cx \quad \text{or} \quad y = \frac{e^{cx-1}}{x}\,.$$

**37.** Write the differential equation as

$$\frac{dv}{dx} + \frac{1}{x}\,v = 32v^{-1},$$

and let $u = v^2$ or $v = u^{1/2}$. Then

$$\frac{dv}{dx} = \frac{1}{2}\,u^{-1/2}\frac{du}{dx},$$

and substituting into the differential equation, we have

$$\frac{1}{2}\,u^{-1/2}\frac{du}{dx} + \frac{1}{x}u^{1/2} = 32u^{-1/2} \quad \text{or} \quad \frac{du}{dx} + \frac{2}{x}\,u = 64.$$

The latter differential equation is linear with integrating factor $e^{\int (2/x)\,dx} = x^2$, so

$$\frac{d}{dx}\left[x^2u\right] = 64x^2$$

and

$$x^2u = \frac{64}{3}\,x^3 + c \quad \text{or} \quad v^2 = \frac{64}{3}\,x + \frac{c}{x^2}\,.$$

**38.** Write the differential equation as $dP/dt - aP = -bP^2$ and let $u = P^{-1}$ or $P = u^{-1}$. Then

$$\frac{dp}{dt} = -u^{-2}\,\frac{du}{dt}\,,$$

and substituting into the differential equation, we have

$$-u^{-2}\,\frac{du}{dt} - au^{-1} = -bu^{-2} \qquad \text{or} \qquad \frac{du}{dt} + au = b.$$

The latter differential equation is linear with integrating factor $e^{\int a\,dt} = e^{at}$, so

$$\frac{d}{dt}\left[e^{at}u\right] = be^{at}$$

and

$$\begin{aligned} e^{at}u &= \frac{b}{a}\,e^{at} + c \\ e^{at}P^{-1} &= \frac{b}{a}\,e^{at} + c \\ P^{-1} &= \frac{b}{a} + ce^{-at} \\ P &= \frac{1}{b/a + ce^{-at}} = \frac{a}{b + c_1e^{-at}}\,. \end{aligned}$$

## 2.6 A Numerical Method

**1.** We identify $f(x,y)=2x-3y+1$. Then, for $h=0.1$,

$$y_{n+1}=y_n+0.1(2x_n-3y_n+1)=0.2x_n+0.7y_n+0.1,$$

and

$$\begin{aligned}y(1.1)&\approx y_1=0.2(1)+0.7(5)+0.1=3.8\\ y(1.2)&\approx y_2=0.2(1.1)+0.7(3.8)+0.1=2.98\end{aligned}$$

For $h=0.05$,

$$y_{n+1}=y_n+0.05(2x_n-3y_n+1)=0.1x_n+0.85y_n+0.1,$$

and

$$\begin{aligned}y(1.05)&\approx y_1=0.1(1)+0.85(5)+0.1=4.4\\ y(1.1)&\approx y_2=0.1(1.05)+0.85(4.4)+0.1=3.895\\ y(1.15)&\approx y_3=0.1(1.1)+0.85(3.895)+0.1=3.47075\\ y(1.2)&\approx y_4=0.1(1.15)+0.85(3.47075)+0.1=3.11514\end{aligned}$$

**2.** We identify $f(x,y)=x+y^2$. Then, for $h=0.1$,

$$y_{n+1}=y_n+0.1(x_n+y_n^2)=0.1x_n+y_n+0.1y_n^2,$$

and

$$\begin{aligned}y(0.1)&\approx y_1=0.1(0)+0+0.1(0)^2=0\\ y(0.2)&\approx y_2=0.1(0.1)+0+0.1(0)^2=0.01\end{aligned}$$

For $h=0.05$,

$$y_{n+1}=y_n+0.05(x_n+y_n^2)=0.05x_n+y_n+0.05y_n^2,$$

and

$$\begin{aligned}y(0.05)&\approx y_1=0.05(0)+0+0.05(0)^2=0\\ y(0.1)&\approx y_2=0.05(0.05)+0+0.05(0)^2=0.0025\\ y(0.15)&\approx y_3=0.05(0.1)+0.0025+0.05(0.0025)^2=0.0075\\ y(0.2)&\approx y_4=0.05(0.15)+0.0075+0.05(0.0075)^2=0.0150\end{aligned}$$

**3.** Separating variables and integrating, we have

$$\frac{dy}{y}=dx \quad\text{and}\quad \ln|y|=x+c.$$

Thus $y=c_1e^x$ and, using $y(0)=1$, we find $c=1$, so $y=e^x$ is the solution of the initial-value problem.

$h = 0.1$

| $x_n$ | $y_n$ | Actual Value | Abs. Error | %Rel. Error |
|---|---|---|---|---|
| 0.00 | 1.0000 | 1.0000 | 0.0000 | 0.00 |
| 1.10 | 1.1000 | 1.1052 | 0.0052 | 0.47 |
| 0.20 | 1.2100 | 1.2214 | 0.0114 | 0.93 |
| 0.30 | 1.3310 | 1.3499 | 0.0189 | 1.40 |
| 0.40 | 1.4641 | 1.4918 | 0.0277 | 1.86 |
| 0.50 | 1.6105 | 1.6487 | 0.0382 | 2.32 |
| 0.60 | 1.7716 | 1.8221 | 0.0506 | 2.77 |
| 0.70 | 1.9487 | 2.0138 | 0.0650 | 3.23 |
| 0.80 | 2.1436 | 2.2255 | 0.0820 | 3.68 |
| 0.90 | 2.3579 | 2.4596 | 0.1017 | 4.13 |
| 1.00 | 2.5937 | 2.7183 | 0.1245 | 4.58 |

$h = 0.05$

| $x_n$ | $y_n$ | Actual Value | Abs. Error | %Rel. Error |
|---|---|---|---|---|
| 0.00 | 1.0000 | 1.0000 | 0.0000 | 0.00 |
| 0.05 | 1.0500 | 1.0513 | 0.0013 | 0.12 |
| 0.10 | 1.1025 | 1.1052 | 0.0027 | 0.24 |
| 0.15 | 1.1576 | 1.1618 | 0.0042 | 0.36 |
| 0.20 | 1.2155 | 1.2214 | 0.0059 | 0.48 |
| 0.25 | 1.2763 | 1.2840 | 0.0077 | 0.60 |
| 0.30 | 1.3401 | 1.3499 | 0.0098 | 0.72 |
| 0.35 | 1.4071 | 1.4191 | 0.0120 | 0.84 |
| 0.40 | 1.4775 | 1.4918 | 0.0144 | 0.96 |
| 0.45 | 1.5513 | 1.5683 | 0.0170 | 1.08 |
| 0.50 | 1.6289 | 1.6487 | 0.0198 | 1.20 |
| 0.55 | 1.7103 | 1.7333 | 0.0229 | 1.32 |
| 0.60 | 1.7959 | 1.8221 | 0.0263 | 1.44 |
| 0.65 | 1.8856 | 1.9155 | 0.0299 | 1.56 |
| 0.70 | 1.9799 | 2.0138 | 0.0338 | 1.68 |
| 0.75 | 2.0789 | 2.1170 | 0.0381 | 1.80 |
| 0.80 | 2.1829 | 2.2255 | 0.0427 | 1.92 |
| 0.85 | 2.2920 | 2.3396 | 0.0476 | 2.04 |
| 0.90 | 2.4066 | 2.4596 | 0.0530 | 2.15 |
| 0.95 | 2.5270 | 2.5857 | 0.0588 | 2.27 |
| 1.00 | 2.6533 | 2.7183 | 0.0650 | 2.39 |

**4.** Separating variables and integrating, we have

$$\frac{dy}{y} = 2x\,dx \quad \text{and} \quad \ln|y| = x^2 + c.$$

Thus $y = c_1 e^{x^2}$ and, using $y(1) = 1$, we find $c = e^{-1}$, so $y = e^{x^2-1}$ is the solution of the initial-value problem.

$h = 0.1$

| $x_n$ | $y_n$ | Actual Value | Abs. Error | %Rel. Error |
|---|---|---|---|---|
| 1.00 | 1.0000 | 1.0000 | 0.0000 | 0.00 |
| 1.10 | 1.2000 | 1.2337 | 0.0337 | 2.73 |
| 1.20 | 1.4640 | 1.5527 | 0.0887 | 5.71 |
| 1.30 | 1.8154 | 1.9937 | 0.1784 | 8.95 |
| 1.40 | 2.2874 | 2.6117 | 0.3243 | 12.42 |
| 1.50 | 2.9278 | 3.4903 | 0.5625 | 16.12 |

$h = 0.05$

| $x_n$ | $y_n$ | Actual Value | Abs. Error | %Rel. Error |
|---|---|---|---|---|
| 1.00 | 1.0000 | 1.0000 | 0.0000 | 0.00 |
| 1.05 | 1.1000 | 1.1079 | 0.0079 | 0.72 |
| 1.10 | 1.2155 | 1.2337 | 0.0182 | 1.47 |
| 1.15 | 1.3492 | 1.3806 | 0.0314 | 2.27 |
| 1.20 | 1.5044 | 1.5527 | 0.0483 | 3.11 |
| 1.25 | 1.6849 | 1.7551 | 0.0702 | 4.00 |
| 1.30 | 1.8955 | 1.9937 | 0.0982 | 4.93 |
| 1.35 | 2.1419 | 2.2762 | 0.1343 | 5.90 |
| 1.40 | 2.4311 | 2.6117 | 0.1806 | 6.92 |
| 1.45 | 2.7714 | 3.0117 | 0.2403 | 7.98 |
| 1.50 | 3.1733 | 3.4903 | 0.3171 | 9.08 |

**5.** $h = 0.1$

| $x_n$ | $y_n$ |
|---|---|
| 0.00 | 0.0000 |
| 0.10 | 0.1000 |
| 0.20 | 0.1905 |
| 0.30 | 0.2731 |
| 0.40 | 0.3492 |
| 0.50 | 0.4198 |

$h = 0.05$

| $x_n$ | $y_n$ |
|---|---|
| 0.00 | 0.0000 |
| 0.05 | 0.0500 |
| 0.10 | 0.0976 |
| 0.15 | 0.1429 |
| 0.20 | 0.1863 |
| 0.25 | 0.2278 |
| 0.30 | 0.2676 |
| 0.35 | 0.3058 |
| 0.40 | 0.3427 |
| 0.45 | 0.3782 |
| 0.50 | 0.4124 |

**6.** $h = 0.1$

| $x_n$ | $y_n$ |
|---|---|
| 0.00 | 1.0000 |
| 0.10 | 1.1000 |
| 0.20 | 1.2220 |
| 0.30 | 1.3753 |
| 0.40 | 1.5735 |
| 0.50 | 1.8371 |

$h = 0.05$

| $x_n$ | $y_n$ |
|---|---|
| 0.00 | 1.0000 |
| 0.05 | 1.0500 |
| 0.10 | 1.1053 |
| 0.15 | 1.1668 |
| 0.20 | 1.2360 |
| 0.25 | 1.3144 |
| 0.30 | 1.4039 |
| 0.35 | 1.5070 |
| 0.40 | 1.6267 |
| 0.45 | 1.7670 |
| 0.50 | 1.9332 |

**7.** $h = 0.1$

| $x_n$ | $y_n$ |
|---|---|
| 0.00 | 0.5000 |
| 0.10 | 0.5250 |
| 0.20 | 0.5431 |
| 0.30 | 0.5548 |
| 0.40 | 0.5613 |
| 0.50 | 0.5639 |

$h = 0.05$

| $x_n$ | $y_n$ |
|---|---|
| 0.00 | 0.5000 |
| 0.05 | 0.5125 |
| 0.10 | 0.5232 |
| 0.15 | 0.5322 |
| 0.20 | 0.5395 |
| 0.25 | 0.5452 |
| 0.30 | 0.5496 |
| 0.35 | 0.5527 |
| 0.40 | 0.5547 |
| 0.45 | 0.5559 |
| 0.50 | 0.5565 |

**8.** $h = 0.1$

| $x_n$ | $y_n$ |
|---|---|
| 0.00 | 1.0000 |
| 0.10 | 1.1000 |
| 0.20 | 1.2159 |
| 0.30 | 1.3505 |
| 0.40 | 1.5072 |
| 0.50 | 1.6902 |

$h = 0.05$

| $x_n$ | $y_n$ |
|---|---|
| 0.00 | 1.0000 |
| 0.05 | 1.0500 |
| 0.10 | 1.1039 |
| 0.15 | 1.1619 |
| 0.20 | 1.2245 |
| 0.25 | 1.2921 |
| 0.30 | 1.3651 |
| 0.35 | 1.4440 |
| 0.40 | 1.5293 |
| 0.45 | 1.6217 |
| 0.50 | 1.7219 |

**9.** $h = 0.1$

| $x_n$ | $y_n$ |
|---|---|
| 1.00 | 1.0000 |
| 1.10 | 1.0000 |
| 1.20 | 1.0191 |
| 1.30 | 1.0588 |
| 1.40 | 1.1231 |
| 1.50 | 1.2194 |

$h = 0.05$

| $x_n$ | $y_n$ |
|---|---|
| 1.00 | 1.0000 |
| 1.05 | 1.0000 |
| 1.10 | 1.0049 |
| 1.15 | 1.0147 |
| 1.20 | 1.0298 |
| 1.25 | 1.0506 |
| 1.30 | 1.0775 |
| 1.35 | 1.1115 |
| 1.40 | 1.1538 |
| 1.45 | 1.2057 |
| 1.50 | 1.2696 |

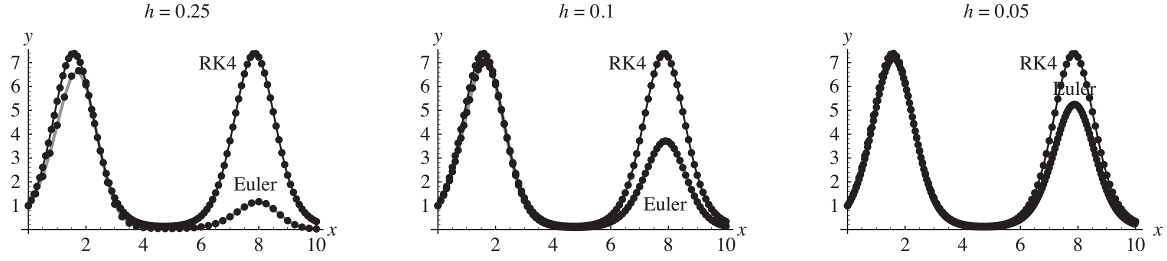
**10.** $h = 0.1$

| $x_n$ | $y_n$ |
|---|---|
| 0.00 | 0.5000 |
| 0.10 | 0.5250 |
| 0.20 | 0.5499 |
| 0.30 | 0.5747 |
| 0.40 | 0.5991 |
| 0.50 | 0.6231 |

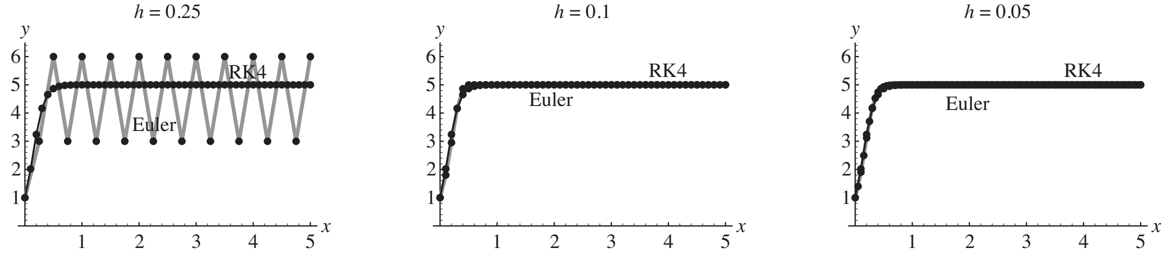
$h = 0.05$

| $x_n$ | $y_n$ |
|---|---|
| 0.00 | 0.5000 |
| 0.05 | 0.5125 |
| 0.10 | 0.5250 |
| 0.15 | 0.5375 |
| 0.20 | 0.5499 |
| 0.25 | 0.5623 |
| 0.30 | 0.5746 |
| 0.35 | 0.5868 |
| 0.40 | 0.5989 |
| 0.45 | 0.6109 |
| 0.50 | 0.6228 |

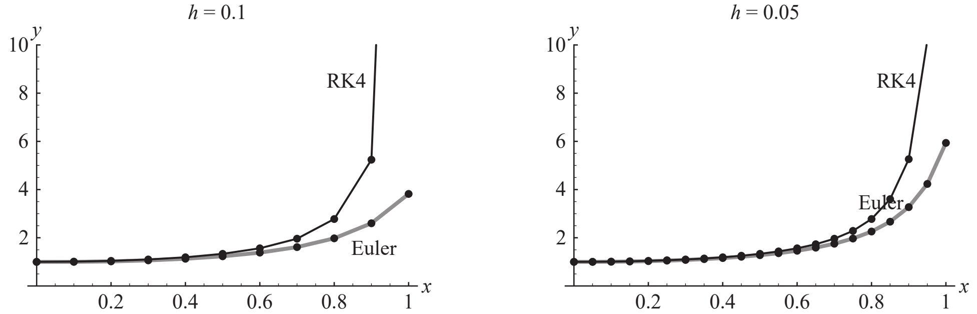
**11.** Tables of values were computed using the Euler and RK4 methods. The resulting points were plotted and joined using **ListPlot** in *Mathematica*.



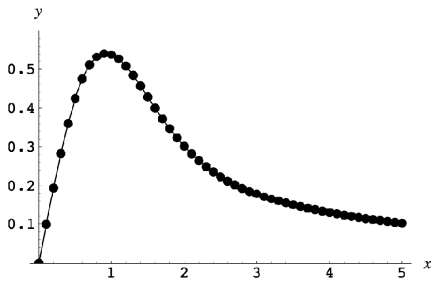
**12.** See the comments in Problem 11 above.



**13.** Using separation of variables we find that the solution of the differential equation is $y = 1/(1 - x^2)$, which is undefined at $x = 1$, where the graph has a vertical asymptote. Because the actual solution of the differential equation becomes unbounded at $x$ approaches 1, very small changes in the inputs $x$ will result in large changes in the corresponding outputs $y$. This can be expected to have a serious effect on numerical procedures. The graphs below were obtained as described in Problem 11.



**14. (a)** The graph to the right was obtained using RK4 and **ListPlot** in *Mathematica*with $h = 0.1$.



**(b)** Writing the differential equation in the form $y' + 2xy = 1$ we see that an integrating factor is $e^{\int 2x dx} = e^{x^2}$, so

$$\frac{d}{dx}[e^{x^2}y] = e^{x^2}$$

and

$$y = e^{-x^2}\int_0^x e^{t^2}\,dt + ce^{-x^2}.$$

This solution can also be expressed in terms of the inverse error function as

$$y = \frac{\sqrt{\pi}}{2}\,e^{-x^2}\operatorname{erfi}(x) + ce^{-x^2}.$$

Letting $x = 0$ and $y(0) = 0$ we find $c = 0$, so the solution of the initial-value problem is

$$y = e^{-x^2}\int_0^x e^{t^2}\,dt = \frac{\sqrt{\pi}}{2}\,e^{-x^2}\operatorname{erfi}(x).$$

**(c)** Using **FindRoot** in *Mathematica* we see that $y'(x) = 0$ when $x = 0.924139$. Since $y(0.924139) = 0.541044$, we see from the graph in part (a) that $(0.924139, 0.541044)$ is a relative maximum. Now, using the substitution $u = -t$ in the integral below, we have

$$y(-x) = e^{-(-x)^2}\int_0^{-x} e^{t^2}\,dt = e^{-x^2}\int_0^x e^{(-u)^2}\,(-du) = -e^{-x^2}\int_0^x e^{u^2}\,du = -y(x).$$

Thus, $y(x)$ is an odd function and $(-0.924139, -0.541044)$ is a relative minimum.

## Chapter 2 in Review

**1.** Writing the differential equation in the form $y' = k(y + A/k)$ we see that the critical point $-A/k$ is a repeller for $k > 0$ and an attractor for $k < 0$.

**2.** Separating variables and integrating we have

$$\frac{dy}{y} = \frac{4}{x}\,dx$$

$$\ln y = 4\ln x + c = \ln x^4 + c$$

$$y = c_1x^4.$$

We see that when $x = 0$, $y = 0$, so the initial-value problem has an infinite number of solutions for $k = 0$ and no solutions for $k \neq 0$.

**3.** True; $y = k_2/k_1$ is always a solution for $k_1 \neq 0$.

**4.** True; writing the differential equation as $a_1(x)\,dy + a_2(x)y\,dx = 0$ and separating variables yields

$$\frac{dy}{y} = -\frac{a_2(x)}{a_1(x)}\,dx.$$

**5.** $\dfrac{d^3y}{dx^3} = x\sin y$ (There are many answers.)

**6.** False: $\dfrac{dr}{d\theta} = r\theta + r + \theta + 1 = (r+1)(\theta+1)$.

**7.** True

**8.** Since the differential equation in the form $y' = 2 - |y|$ is seen to be autonomous, $2 - |y| = 0$ has critical points 2 and $-2$ so $y_1 = 2$ and $y_2 = -2$ are constant (equilibrium) solutions.

**9.** $\dfrac{dy}{y} = e^x\,dx$

$$\ln y = e^x + c$$
$$y = e^{e^x + c} = e^c e^{e^x} \quad \text{or} \quad y = c_1 e^{e^x}$$

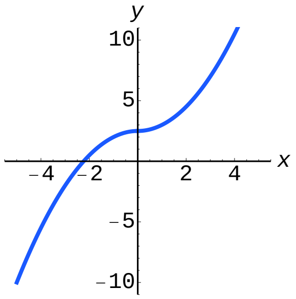
**10.** $y' = |x|, \quad y(-1) = 2$

$$\frac{dy}{dx} = \begin{cases} -x, & x < 0 \\ x, & x \ge 0 \end{cases}$$

$$y = \begin{cases} -\dfrac{1}{2}x^2 + c_1, & x < 0 \\ \dfrac{1}{2}x^2 + c_2, & x \ge 0 \end{cases}$$

The initial condition $y(-1) = 2$ implies $2 = -\dfrac{1}{2} + c_1$ and thus $c_1 = \dfrac{5}{2}$. Now $y(x)$ is supposed to be differentiable and so continuous. At $x = 0$ the two parts of the functions must agree and so $c_2 = c_1 = \dfrac{5}{2}$. So,

$$y = \begin{cases} \dfrac{1}{2}\left(5 - x^2\right), & x < 0 \\ \dfrac{1}{2}\left(x^2 + 5\right), & x \ge 0 \end{cases}$$



**11.** $y = e^{\cos x}\displaystyle\int_0^x te^{-\cos t}\,dt$

$$\frac{dy}{dx} = e^{\cos x} x e^{-\cos x} + (-\sin x)\,e^{\cos x}\int_0^x te^{-\cos t}\,dt$$

$$\frac{dy}{dx} = x - (\sin x)\,y \quad \text{or} \quad \frac{dy}{dx} + (\sin x)\,y = x.$$

**12.** $\dfrac{dy}{dx} = y + 3, \qquad \dfrac{dy}{dx} = (y+3)^2$

**13.** $\dfrac{dy}{dx} = (y-1)^2(y-3)^2$

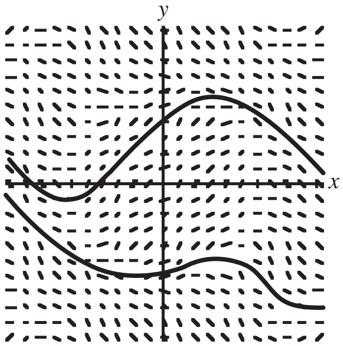
**14.** $\dfrac{dy}{dx} = y(y-2)^2(y-4)$

**15.** When $n$ is odd, $x^n < 0$ for $x < 0$ and $x^n > 0$ for $x > 0$. In this case 0 is unstable. When $n$ is even, $x^n > 0$ for $x < 0$ and for $x > 0$. In this case 0 is semi-stable.

When $n$ is odd, $-x^n > 0$ for $x < 0$ and $-x^n < 0$ for $x > 0$. In this case 0 is asymptotically stable. When $n$ is even, $-x^n < 0$ for $x < 0$ and for $x > 0$. In this case 0 is semi-stable.

**16.** Using a CAS we find that the zero of $f$ occurs at approximately $P = 1.3214$. From the graph we observe that $dP/dt > 0$ for $P < 1.3214$ and $dP/dt < 0$ for $P > 1.3214$, so $P = 1.3214$ is an asymptotically stable critical point. Thus, $\lim_{t\to\infty} P(t) = 1.3214$.

**17.**



**18.**

(**a**) linear in $y$, homogeneous, exact

(**b**) linear in $x$

(**c**) separable, exact, linear in $x$ and $y$

(**d**) Bernoulli in $x$

(**e**) separable

(**f**) separable, linear in $x$, Bernoulli

(**g**) linear in $x$

(**h**) homogeneous

(**i**) Bernoulli

(**j**) homogeneous, exact, Bernoulli

(**k**) linear in $x$ and $y$, exact, separable, homogeneous

(**l**) exact, linear in $y$

(**m**) homogeneous

(**n**) separable

**19.** Separating variables and using the identity $\cos^2 x = \frac{1}{2}(1+\cos 2x)$, we have

$$\cos^2 x\,dx = \frac{y}{y^2+1}\,dy,$$

$$\frac{1}{2}x + \frac{1}{4}\sin 2x = \frac{1}{2}\ln\left(y^2+1\right) + c,$$

and

$$2x + \sin 2x = 2\ln\left(y^2+1\right) + c.$$

**20.** Write the differential equation in the form

$$y\ln\frac{x}{y}\,dx = \left(x\ln\frac{x}{y} - y\right)\,dy.$$

This is a homogeneous equation, so let $x = uy$. Then $dx = u\,dy + y\,du$ and the differential equation becomes

$$y\ln u(u\,dy + y\,du) = (uy\ln u - y)\,dy \quad \text{or} \quad y\ln u\,du = -dy.$$

Separating variables, we obtain

$$\ln u\,du = -\frac{dy}{y}$$

$$u\ln|u| - u = -\ln|y| + c$$

$$\frac{x}{y}\ln\left|\frac{x}{y}\right| - \frac{x}{y} = -\ln|y| + c$$

$$x(\ln x - \ln y) - x = -y\ln|y| + cy.$$

**21.** The differential equation

$$\frac{dy}{dx} + \frac{2}{6x+1}y = -\frac{3x^2}{6x+1}y^{-2}$$

is Bernoulli. Using $w = y^3$, we obtain the linear equation

$$\frac{dw}{dx} + \frac{6}{6x+1}w = -\frac{9x^2}{6x+1}\,.$$

An integrating factor is $6x+1$, so

$$\frac{d}{dx}\left[(6x+1)w\right] = -9x^2,$$

$$w = -\frac{3x^3}{6x+1} + \frac{c}{6x+1},$$

and

$$(6x+1)y^3 = -3x^3 + c.$$

(Note: The differential equation is also exact.)

**22.** Write the differential equation in the form $(3y^2+2x)dx + (4y^2+6xy)dy = 0$. Letting $M = 3y^2 + 2x$ and $N = 4y^2 + 6xy$ we see that $M_y = 6y = N_x$, so the differential equation is exact. From $f_x = 3y^2 + 2x$ we obtain $f = 3xy^2 + x^2 + h(y)$. Then $f_y = 6xy + h'(y) = 4y^2 + 6xy$ and $h'(y) = 4y^2$ so $h(y) = \frac{4}{3}y^3$. A one-parameter family of solutions is

$$3xy^2 + x^2 + \frac{4}{3}y^3 = c.$$

**23.** Write the equation in the form

$$\frac{dQ}{dt} + \frac{1}{t}Q = t^3 \ln t.$$

An integrating factor is $e^{\ln t} = t$, so

$$\frac{d}{dt}[tQ] = t^4 \ln t$$

$$tQ = -\frac{1}{25}t^5 + \frac{1}{5}t^5 \ln t + c$$

and

$$Q = -\frac{1}{25}t^4 + \frac{1}{5}t^4 \ln t + \frac{c}{t}.$$

**24.** Letting $u = 2x + y + 1$ we have

$$\frac{du}{dx} = 2 + \frac{dy}{dx},$$

and so the given differential equation is transformed into

$$u\left(\frac{du}{dx} - 2\right) = 1 \quad \text{or} \quad \frac{du}{dx} = \frac{2u+1}{u}.$$

Separating variables and integrating we get

$$\frac{u}{2u+1}\,du = dx$$

$$\left(\frac{1}{2} - \frac{1}{2}\frac{1}{2u+1}\right)du = dx$$

$$\frac{1}{2}u - \frac{1}{4}\ln|2u+1| = x + c$$

$$2u - \ln|2u+1| = 4x + c_1.$$

Resubstituting for $u$ gives the solution

$$4x + 2y + 2 - \ln|4x + 2y + 3| = 4x + c_1$$

or

$$2y + 2 - \ln|4x + 2y + 3| = c_1.$$

**25.** Write the equation in the form

$$\frac{dy}{dx} + \frac{8x}{x^2+4}y = \frac{2x}{x^2+4}.$$

An integrating factor is $\left(x^2+4\right)^4$, so

$$\frac{d}{dx}\left[\left(x^2+4\right)^4 y\right] = 2x\left(x^2+4\right)^3$$

$$\left(x^2+4\right)^4 y = \frac{1}{4}\left(x^2+4\right)^4 + c$$

and

$$y = \frac{1}{4} + c\left(x^2 + 4\right)^{-4}.$$

**26.** Letting $M = 2r^2 \cos\theta \sin\theta + r\cos\theta$ and $N = 4r + \sin\theta - 2r\cos^2\theta$ we see that $M_r = 4r\cos\theta\sin\theta + \cos\theta = N_\theta$, so the differential equation is exact. From $f_\theta = 2r^2\cos\theta\sin\theta + r\cos\theta$ we obtain $f = -r^2\cos^2\theta + r\sin\theta + h(r)$. Then $f_r = -2r\cos^2\theta + \sin\theta + h'(r) = 4r + \sin\theta - 2r\cos^2\theta$ and $h'(r) = 4r$ so $h(r) = 2r^2$. The solution is

$$-r^2\cos^2\theta + r\sin\theta + 2r^2 = c.$$

**27.** We put the equation $\dfrac{dy}{dx} + 4\,(\cos x)\,y = x$ in the standard form $\dfrac{dy}{dx} + 2\,(\cos x)\,y = \dfrac{1}{2}x$ then the integrating factor is $e^{\int 2\cos x\,dx} = e^{2\sin x}$. Therefore

$$\begin{aligned}
\frac{d}{dx}\left[e^{2\sin x}y\right] &= \frac{1}{2}xe^{2\sin x}\\
\int_0^x \frac{d}{dt}\left[e^{2\sin t}y(t)\right]\,dt &= \frac{1}{2}\int_0^x te^{2\sin t}\,dt\\
e^{2\sin x}y(x) - e^0\,\overbrace{y(0)}^{1} &= \frac{1}{2}\int_0^x te^{2\sin t}\,dt\\
e^{2\sin x}y(x) - 1 &= \frac{1}{2}\int_0^x te^{2\sin t}\,dy\\
y(x) &= e^{-2\sin x} + \frac{1}{2}e^{-2\sin x}\int_0^x te^{2\sin t}\,dt
\end{aligned}$$

**28.** The equation $\dfrac{dy}{dx} - 4xy = \sin x^2$ is already in standard form so the integrating factor is $e^{-\int 4x\,dx} = e^{-2x^2}$. Therefore $\dfrac{d}{dx}\left[e^{-2x^2}y\right] = e^{-2x^2}\sin x^2$. Because of the initial condition $y(0) = 7$ we write

$$\begin{aligned}
\int_0^x \frac{d}{dt}\left[e^{-2t^2}y(t)\right]\,dt &= \int_0^x e^{-2t^2}\sin t^2\,dt\\
e^{-2x^2}y(x) - e^0\,\overbrace{y(0)}^{7} &= \int_0^x e^{-2t^2}\sin t^2\,dt\\
y(x) &= 7e^{2x^2} + e^{2x^2}\int_0^x e^{-2t^2}\sin t^2\,dt
\end{aligned}$$

**29.** We put the equation $x\frac{dy}{dx}+2y = xe^{x^2}$ into standard form $\dfrac{dy}{dx}+\dfrac{2}{x}y = e^{x^2}$. Then the integrating factor is $e^{\int \frac{2}{x}\,dx} = e^{\ln x^2} = x^2$. Therefore

$$\begin{aligned}
x^2\frac{dy}{dx} + 2xy &= x^2e^{x^2} \\
\frac{d}{dx}\left[x^2y\right] &= x^2e^{x^2} \\
\int_1^x \frac{d}{dt}\left[t^2y(t)\right]\,dt &= \int_1^x t^2e^{t^2}\,dt \\
x^2y(x) - \overbrace{y(1)}^{3} &= \int_1^x t^2e^{t^2}\,dt \\
y(x) &= \frac{3}{x^2} + \frac{1}{x^2}\int_1^x t^2e^{t^2}\,dt
\end{aligned}$$

**30.**

$$\begin{aligned}
x\frac{dy}{dx} + (\sin x)\,y &= 0 \\
\frac{dy}{dx} + \frac{\sin x}{x}y &= 0
\end{aligned}$$

The integrating factor is $e^{\int_0^x \frac{\sin t}{t}\,dy}$. Therefore,

$$\begin{aligned}
\frac{d}{dx}\left[e^{\int_0^x \frac{\sin t}{t}\,dt}y\right] &= 0 \\
\int_0^x \frac{d}{dt}\left[e^{\int_0^t \frac{\sin u}{u}\,du}y(t)\right]\,dt &= \int_0^x 0\,dt = 0 \\
e^{\int_0^x \frac{\sin t}{t}\,dt}y(x) - e^0\,\overbrace{y(0)}^{10} &= 0 \\
y(x) &= 10e^{-\int_0^x \frac{\sin t}{t}\,dt}
\end{aligned}$$

**31.**

$$\frac{dy}{dx} + y = f(x), \quad y(0) = 5, \qquad \text{where} \quad f(x) = \begin{cases} e^{-x}, & 0 \le x < 1 \\ 0, & x \ge 1 \end{cases}$$

For $0 \le x < 1$,

$$\begin{aligned}
\frac{d}{dx}\left[e^xy\right] &= 1 \\
e^xy &= x + c_1 \\
y &= xe^{-x} + c_1e^{-x}
\end{aligned}$$

Using $y(0) = 5$, we have $c_1 = 5$. Therefore $y = xe^{-x} + 5e^{-x}$. Then for $x \geq 1$,

$$\begin{aligned} \frac{d}{dx}\left[e^x y\right] &= 0 \\ e^x y &= c_2 \\ y &= c_2 e^{-x} \end{aligned}$$

Requiring that $y(x)$ be continuous at $x = 1$ yields

$$\begin{aligned} c_2 e^{-1} &= e^{-1} + 5e^{-1} \\ c_2 &= 6 \end{aligned}$$

Therefore

$$y(x) = \begin{cases} xe^{-x} + 5e^{-x}, & 0 \leq x < 1 \\ 6e^{-x}, & x \geq 1 \end{cases}$$

**32.**

$$\frac{dy}{dx} + P(x)y = e^x, \quad y(0) = -1, \qquad \text{where} \quad P(x) = \begin{cases} 1, & 0 \leq x < 1 \\ -1, & x \geq 1 \end{cases}$$

For $0 \leq x < 1$,

$$\begin{aligned} \frac{d}{dx}\left[e^x y\right] &= e^{2x} \\ e^x y &= \frac{1}{2}e^{2x} + c_1 \\ y &= \frac{1}{2}e^x + c_1 e^{-x} \end{aligned}$$

Using $y(0) = -1$, we have $c_1 = -\frac{3}{2}$. Therefore $y = \frac{1}{2}e^x - \frac{3}{2}e^{-x}$. Then for $x \geq 1$,

$$\begin{aligned} \frac{d}{dx}\left[e^{-x} y\right] &= 1 \\ e^{-x} y &= x + c_2 \\ y &= xe^x + c_2 e^x \end{aligned}$$

Requiring that $y(x)$ be continuous at $x = 1$ yields

$$\begin{aligned} e + c_2 e &= \frac{1}{2}e - \frac{3}{2}e^{-1} \\ c_2 &= -\frac{1}{2} - \frac{3}{2}e^{-2} \end{aligned}$$

Therefore

$$y(x) = \begin{cases} \dfrac{1}{2}e^x - \dfrac{3}{2}e^{-x}, & 0 \leq x < 1 \\ xe^x - \dfrac{1}{2}e^x - \dfrac{3}{2}e^{x-2}, & x \geq 1 \end{cases}$$

**33.** The differential equation has the form $(d/dx)\left[(\sin x)y\right] = 0$. Integrating, we have $(\sin x)y = c$ or $y = c/\sin x$. The initial condition implies $c = -2\sin(7\pi/6) = 1$. Thus, $y = 1/\sin x$, where the interval $(\pi, 2\pi)$ is chosen to include $x = 7\pi/6$.

**34.** Separating variables and integrating we have

$$\begin{aligned} \frac{dy}{y^2} &= -2(t+1)\,dt \\ -\frac{1}{y} &= -(t+1)^2 + c \\ y &= \frac{1}{(t+1)^2 + c_1}\,, \qquad \text{where } -c = c_1 \end{aligned}$$

The initial condition $y(0) = -\frac{1}{8}$ implies $c_1 = -9$, so a solution of the initial-value problem is

$$y = \frac{1}{(t+1)^2 - 9} \qquad \text{or} \qquad y = \frac{1}{t^2 + 2t - 8},$$

where $-4 < t < 2$.

**35.** **(a)** For $y < 0$, $\sqrt{y}$ is not a real number.

**(b)** Separating variables and integrating we have

$$\frac{dy}{\sqrt{y}} = dx \quad \text{and} \quad 2\sqrt{y} = x + c.$$

Letting $y(x_0) = y_0$ we get $c = 2\sqrt{y_0} - x_0$, so that

$$2\sqrt{y} = x + 2\sqrt{y_0} - x_0 \quad \text{and} \quad y = \frac{1}{4}(x + 2\sqrt{y_0} - x_0)^2.$$

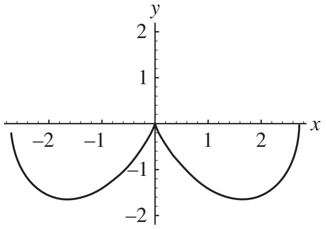
Since $\sqrt{y} > 0$ for $y \neq 0$, we see that $dy/dx = \frac{1}{2}(x + 2\sqrt{y_0} - x_0)$ must be positive. Thus, the interval on which the solution is defined is $(x_0 - 2\sqrt{y_0}, \infty)$.

**36.** **(a)** The differential equation is homogeneous and we let $y = ux$. Then

$$\begin{aligned} (x^2 - y^2)\,dx + xy\,dy &= 0 \\ (x^2 - u^2x^2)\,dx + ux^2(u\,dx + x\,du) &= 0 \\ dx + ux\,du &= 0 \\ u\,du &= -\frac{dx}{x} \\ \frac{1}{2}u^2 &= -\ln|x| + c \\ \frac{y^2}{x^2} &= -2\ln|x| + c_1. \end{aligned}$$

The initial condition gives $c_1 = 2$, so an implicit solution is $y^2 = x^2(2 - 2\ln|x|)$.

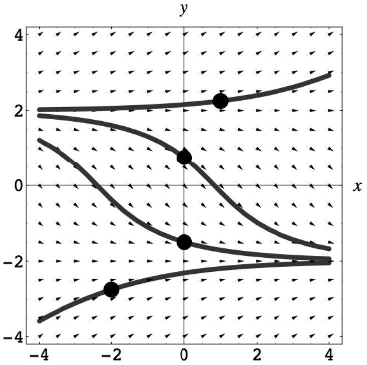
**(b)** Solving for $y$ in part (a) and being sure that the initial condition is still satisfied, we have $y = -\sqrt{2}\,|x|(1 - \ln|x|)^{1/2}$, where $-e \le x \le e$ so that $1 - \ln|x| \ge 0$. The graph of this function indicates that the derivative is not defined at $x = 0$ and $x = e$. Thus, the solution of the initial-value problem is $y = -\sqrt{2}\,x(1 - \ln x)^{1/2}$, for $0 < x < e$.



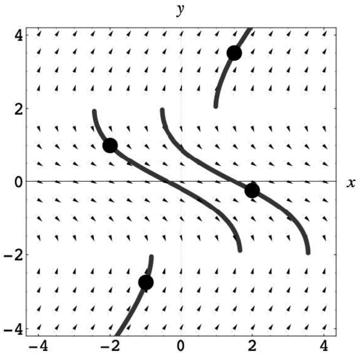
**37.** The graph of $y_1(x)$ is the portion of the closed blue curve lying in the fourth quadrant. Its interval of definition is approximately $(0.7, 4.3)$. The graph of $y_2(x)$ is the portion of the left-hand blue curve lying in the third quadrant. Its interval of definition is $(-\infty, 0)$.

**38.** The first step of Euler's method gives $y(1.1) \approx 9 + 0.1(1 + 3) = 9.4$. Applying Euler's method one more time gives $y(1.2) \approx 9.4 + 0.1(1 + 1.1\sqrt{9.4}\,) \approx 9.8373$.

**39.** Since the differential equation is autonomous, all lineal elements on a given horizontal line have the same slope. The direction field is then as shown in the figure at the right. It appears from the figure that the differential equation has critical points at $-2$ (an attractor) and at 2 (a repeller). Thus, $-2$ is an aymptotically stable critical point and 2 is an unstable critical point.



**40.** Since the differential equation is autonomous, all lineal elements on a given horizontal line have the same slope. The direction field is then as shown in the figure at the right. It appears from the figure that the differential equation has no critical points.



# Chapter 3

## Modeling with First-Order Differential Equations

### 3.1 Linear Models

**1.** Let $P = P(t)$ be the population at time $t$, and $P_0$ the initial population. From $dP/dt = kP$ we obtain $P = P_0e^{kt}$. Using $P(5) = 2P_0$ we find $k = \frac{1}{5}\ln 2$ and $P = P_0e^{(\ln 2)t/5}$. Setting $P(t) = 3P_0$ we have $3 = e^{(\ln 2)t/5}$, so

$$\ln 3 = \frac{(\ln 2)t}{5} \qquad \text{and} \qquad t = \frac{5\ln 3}{\ln 2} \approx 7.9 \text{ years.}$$

Setting $P(t) = 4P_0$ we have $4 = e^{(\ln 2)t/5}$, so

$$\ln 4 = \frac{(\ln 2)t}{5} \qquad \text{and} \qquad t = 10 \text{ years.}$$

**2.** From Problem 1 the growth constant is $k = \frac{1}{5}\ln 2$. Then $P = P_0e^{(1/5)(\ln 2)t}$ and $10{,}000 = P_0e^{(3/5)\ln 2}$. Solving for $P_0$ we get $P_0 = 10{,}000e^{-(3/5)\ln 2} = 6{,}597.5$. Now

$$P(10) = P_0e^{(1/5)(\ln 2)(10)} = 6{,}597.5e^{2\ln 2} = 4P_0 = 26{,}390.$$

The rate at which the population is growing is

$$P'(10) = kP(10) = \frac{1}{5}(\ln 2)26{,}390 = 3658 \text{ persons/year.}$$

**3.** Let $P = P(t)$ be the population at time $t$. Then $dP/dt = kP$ and $P = ce^{kt}$. From $P(0) = c = 500$ we see that $P = 500e^{kt}$. Since 15% of 500 is 75, we have
$P(10) = 500e^{10k} = 575$. Solving for $k$, we get $k = \frac{1}{10}\ln\frac{575}{500} = \frac{1}{10}\ln 1.15$. When $t = 30$,

$$P(30) = 500e^{(1/10)(\ln 1.15)30} = 500e^{3\ln 1.15} \approx 760 \text{ years}$$

and

$$P'(30) = kP(30) \approx \frac{1}{10}(\ln 1.15)760 \approx 10.62 \text{ persons/year.}$$

**4.** Let $P = P(t)$ be bacteria population at time $t$ and $P_0$ the initial number. From $dP/dt = kP$ we obtain $P = P_0e^{kt}$. Using $P(3) = 400$ and $P(10) = 2000$ we find $400 = P_0e^{3k}$ or $e^k = (400/P_0)^{1/3}$. From $P(10) = 2000$ we then have $2000 = P_0e^{10k} = P_0(400/P_0)^{10/3}$, so

$$\frac{2000}{400^{10/3}} = P_0^{-7/3} \qquad \text{and} \qquad P_0 = \left(\frac{2000}{400^{10/3}}\right)^{-3/7} \approx 201.$$

**5.** Let $A = A(t)$ be the amount of lead present at time $t$. From $dA/dt = kA$ and $A(0) = 1$ we obtain $A = e^{kt}$. Using $A(3.3) = 1/2$ we find $k = \frac{1}{3.3}\ln(1/2)$. When 90% of the lead has decayed, 0.1 grams will remain. Setting $A(t) = 0.1$ we have $e^{t(1/3.3)\ln(1/2)} = 0.1$, so

$$\frac{t}{3.3}\ln\frac{1}{2} = \ln 0.1 \qquad \text{and} \qquad t = \frac{3.3\ln 0.1}{\ln(1/2)} \approx 10.96 \text{ hours}.$$

**6.** Let $A = A(t)$ be the amount present at time $t$. From $dA/dt = kA$ and $A(0) = 100$ we obtain $A = 100e^{kt}$. Using $A(6) = 97$ we find $k = \frac{1}{6}\ln 0.97$. Then $A(24) = 100e^{(1/6)(\ln 0.97)24} = 100(0.97)^4 \approx 88.5$ mg.

**7.** Setting $A(t) = 50$ in Problem 6 we obtain $50 = 100e^{kt}$, so

$$kt = \ln\frac{1}{2} \qquad \text{and} \qquad t = \frac{\ln(1/2)}{(1/6)\ln 0.97} \approx 136.5 \text{ hours}.$$

**8.** **(a)** The solution of $dA/dt = kA$ is $A(t) = A_0e^{kt}$. Letting $A = \frac{1}{2}A_0$ and solving for $t$ we obtain the half-life $T = -(\ln 2)/k$.

**(b)** Since $k = -(\ln 2)/T$ we have

$$A(t) = A_0e^{-(\ln 2)t/T} = A_0 2^{-t/T}.$$

**(c)** Writing $\frac{1}{8}A_0 = A_0 2^{-t/T}$ as $2^{-3} = 2^{-t/T}$ and solving for $t$ we get $t = 3T$. Thus, an initial amount $A_0$ will decay to $\frac{1}{8}A_0$ in three half-lives.

**9.** Let $I = I(t)$ be the intensity, $t$ the thickness, and $I(0) = I_0$. If $dI/dt = kI$ and $I(3) = 0.25I_0$, then $I = I_0e^{kt}$, $k = \frac{1}{3}\ln 0.25$, and $I(15) = 0.00098I_0$.

**10.** From $dS/dt = rS$ we obtain $S = S_0e^{rt}$ where $S(0) = S_0$.

**(a)** If $S_0 = \$5000$ and $r = 5.75\%$ then $S(5) = \$6665.45$.

**(b)** If $S(t) = \$10{,}000$ then $t = 12$ years.

**(c)** $S \approx \$6651.82$

**11.** Using 5730 years as the half-life of C-14 we have from Example 3 in the text $A(t) = A_0e^{-0.00012097t}$. Since 85.5% of the C-14 has decayed, $1 - 0.855 = 0.145$ times the original amount is now present, so

$$0.145A_0 = A_0e^{-0.00012097t}, \quad e^{-0.00012097t} = 0.145, \quad \text{and} \quad t = -\frac{\ln 0.145}{0.00012097} \approx 15,968 \text{ years}$$

is the approximate age.

**12.** From Example 3 in the text, the amount of carbon present at time $t$ is $A(t) = A_0e^{-0.00012097t}$. Letting $t = 660$ and solving for $A_0$ we have $A(660) = A_0e^{-0.00012097(660)} = 0.923264A_0$. Thus, approximately 92% of the original amount of C-14 remained in the cloth as of 1988.

**13.** Assume that $dT/dt = k(T - 10)$ so that $T = 10 + ce^{kt}$. If $T(0) = 70^\circ$ and $T(1/2) = 50^\circ$ then $c = 60$ and $k = 2\ln(2/3)$ so that $T(1) = 36.67^\circ$. If $T(t) = 15^\circ$ then $t = 3.06$ minutes.

**14.** Assume that $dT/dt = k(T - 5)$ so that $T = 5 + ce^{kt}$. If $T(1) = 55^\circ$ and $T(5) = 30^\circ$ then $k = -\frac{1}{4}\ln 2$ and $c = 59.4611$ so that $T(0) = 64.4611^\circ$.

**15.** Assume that $dT/dt = k(T - 100)$ so that $T = 100 + ce^{kt}$. If $T(0) = 20^\circ$ and $T(1) = 22^\circ$, then $c = -80$ and $k = \ln(39/40)$ so that $T(t) = 90^\circ$, which implies $t = 82.1$ seconds. If $T(t) = 98^\circ$ then $t = 145.7$ seconds.

**16.** The differential equation for the first container is $dT_1/dt = k_1(T_1 - 0) = k_1T_1$, whose solution is $T_1(t) = c_1e^{k_1t}$. Since $T_1(0) = 100$ (the initial temperature of the metal bar), we have $100 = c_1$ and $T_1(t) = 100e^{k_1t}$. After 1 minute, $T_1(1) = 100e^{k_1} = 90^\circ$C, so $k_1 = \ln 0.9$ and $T_1(t) = 100e^{t\ln 0.9}$. After 2 minutes, $T_1(2) = 100e^{2\ln 0.9} = 100(0.9)^2 = 81^\circ$C.

The differential equation for the second container is $dT_2/dt = k_2(T_2 - 100)$, whose solution is $T_2(t) = 100 + c_2e^{k_2t}$. When the metal bar is immersed in the second container, its initial temperature is $T_2(0) = 81$, so

$$T_2(0) = 100 + c_2e^{k_2(0)} = 100 + c_2 = 81$$

and $c_2 = -19$. Thus, $T_2(t) = 100 - 19e^{k_2t}$. After 1 minute in the second tank, the temperature of the metal bar is 91°C, so

$$\begin{aligned} T_2(1) &= 100 - 19e^{k_2} = 91 \\ e^{k_2} &= \frac{9}{19} \\ k_2 &= \ln\frac{9}{19} \end{aligned}$$

and $T_2(t) = 100 - 19e^{t\ln(9/19)}$. Setting $T_2(t) = 99.9$ we have

$$\begin{aligned}100 - 19e^{t\ln(9/19)} &= 99.9\\ e^{t\ln(9/19)} &= \frac{0.1}{19}\\ t &= \frac{\ln(0.1/19)}{\ln(9/19)} \approx 7.02.\end{aligned}$$

Thus, from the start of the "double dipping" process, the total time until the bar reaches 99.9°C in the second container is approximately 9.02 minutes.

**17.** Using separation of variables to solve $dT/dt = k(T - T_m)$ we get $T(t) = T_m + ce^{kt}$. Using $T(0) = 70$ we find $c = 70 - T_m$, so $T(t) = T_m + (70 - T_m)e^{kt}$. Using the given observations, we obtain

$$\begin{aligned}T\left(\frac{1}{2}\right) &= T_m + (70 - T_m)e^{k/2} = 110\\ T(1) &= T_m + (70 - T_m)e^{k} = 145.\end{aligned}$$

Then, from the first equation, $e^{k/2} = (110 - T_m)/(70 - T_m)$ and

$$\begin{aligned}e^k = (e^{k/2})^2 = \left(\frac{110 - T_m}{70 - T_m}\right)^2 &= \frac{145 - T_m}{70 - T_m}\frac{(110 - T_m)^2}{70 - T_m} \qquad = 145 - T_m\\ 12100 - 220T_m + T_m^2 &= 10150 - 250T_m + T_m^2\\ T_m &= 390.\end{aligned}$$

The temperature in the oven is 390°.

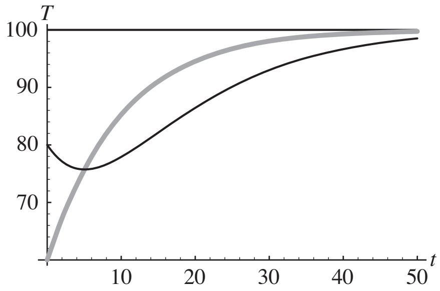
**18.** **(a)** The initial temperature of the bath is $T_m(0) = 60°$, so in the short term the temperature of the chemical, which starts at 80°, should decrease or cool. Over time, the temperature of the bath will increase toward 100° since $e^{-0.1t}$ decreases from 1 toward 0 as $t$ increases from 0. Thus, in the long term, the temperature of the chemical should increase or warm toward 100°.

**(b)** Adapting the model for Newton's law of cooling, we have

$$\frac{dT}{dt} = -0.1(T - 100 + 40e^{-0.1t}), \quad T(0) = 80.$$

Writing the differential equation in the form

$$\frac{dT}{dt} + 0.1T = 10 - 4e^{-0.1t}$$



we see that it is linear with integrating factor $e^{\int 0.1\,dt} = e^{0.1t}$.

Thus

$$\begin{aligned}\frac{d}{dt}[e^{0.1t}T] &= 10e^{0.1t} - 4\\ e^{0.1t}T &= 100e^{0.1t} - 4t + c\end{aligned}$$

and

$$T(t) = 100 - 4te^{-0.1t} + ce^{-0.1t}.$$

Now $T(0) = 80$ so $100 + c = 80$, $c = -20$ and

$$T(t) = 100 - 4te^{-0.1t} - 20e^{-0.1t} = 100 - (4t + 20)e^{-0.1t}.$$

The thinner curve verifies the prediction of cooling followed by warming toward 100°. The wider curve shows the temperature $T_m$ of the liquid bath.

**19.** According to Newton's Law of Cooling

$$\frac{dT}{dt} = k(T - T_m).$$

Separating variables we have

$$\frac{dT}{T - T_m} = k\,dt \qquad \text{so} \qquad \ln|T - T_m| = kt + c \qquad \text{and} \qquad T = T_m + c_1e^{kt}.$$

Setting $T(0) = T_0$ we find $c_1 = T_0 - T_m$. Thus

$$T(t) = T_m + (T_0 - T_m)e^{kt}.$$

In this problem we use $T_0 = 98.6$ and $T_m = 70$. Now, let $n$ denote the number of hours elapsed before the body was found. Then $T(n) = 85$ and $T(n + 1) = 80$. Using this information, we have

$$70 + (98.6 - 70)e^{kn} = 85 \qquad \text{and} \qquad 70 + (98.6 - 70)e^{k(n+1)} = 80$$

or

$$28.6e^{kn} = 15 \qquad \text{and} \qquad 28.6e^{kn+k} = 28.6e^{kn}e^k = 10.$$

The last equation is the same as $15e^k = 10$. Solving for $k$, we have $k = \ln\frac{2}{3} \approx -0.4055$. Finally, solving $e^{-0.4055n} = 15/28.6$ for $n$, we have

$$\begin{aligned}-0.4055n &= \ln\left(\frac{15}{28.6}\right)\\ n &= \frac{1}{-0.4055}\ln\left(\frac{15}{28.6}\right) \approx 1.6.\end{aligned}$$

Thus, about 1.6 hours elapsed before the body was found.

**20.** Solving the differential equation $dT/dt = kS(T - T_m)$ subject to $T(0) = T_0$ gives

$$T(t) = T_m + (T_0 - T_m)e^{kSt}.$$

The temperatures of the coffee in cups $A$ and $B$ are, respectively,

$$T_A(t) = 70 + 80e^{kSt} \quad \text{and} \quad T_B(t) = 70 + 80e^{2kSt}.$$

Then $T_A(30) = 70 + 80e^{30kS} = 100$, which implies $e^{30kS} = \frac{3}{8}$. Hence

$$\begin{aligned} T_B(30) &= 70 + 80e^{60kS} = 70 + 80\left(e^{30kS}\right)^2 \\ &= 70 + 80\left(\frac{3}{8}\right)^2 = 70 + 80\left(\frac{9}{64}\right) = 81.25°\text{F}. \end{aligned}$$

**21.** From $dA/dt = 4 - A/50$ we obtain $A = 200 + ce^{-t/50}$. If $A(0) = 30$ then $c = -170$ and $A = 200 - 170e^{-t/50}$.

**22.** From $dA/dt = 0 - A/50$ we obtain $A = ce^{-t/50}$. If $A(0) = 30$ then $c = 30$ and $A = 30e^{-t/50}$.

**23.** From $dA/dt = 10 - A/100$ we obtain $A = 1000 + ce^{-t/100}$. If $A(0) = 0$ then $c = -1000$ and $A(t) = 1000 - 1000e^{-t/100}$.

**24.** From Problem 23 the number of pounds of salt in the tank at time $t$ is $A(t) = 1000 - 1000e^{-t/100}$. The concentration at time $t$ is $c(t) = A(t)/500 = 2 - 2e^{-t/100}$. Therefore $c(5) = 2 - 2e^{-1/20} = 0.0975\,\text{lb/gal}$ and $\lim_{t\to\infty} c(t) = 2$. Solving $c(t) = 1 = 2 - 2e^{-t/100}$ for $t$ we obtain $t = 100\ln 2 \approx 69.3\,\text{min}$.

**25.** From

$$\frac{dA}{dt} = 10 - \frac{10A}{500 - (10 - 5)t} = 10 - \frac{2A}{100 - t}$$

we obtain $A = 1000 - 10t + c(100 - t)^2$. If $A(0) = 0$ then $c = -\frac{1}{10}$. The tank is empty in 100 minutes.

**26.** With $c_{in}(t) = 2 + \sin(t/4)\,\text{lb/gal}$, the initial-value problem is

$$\frac{dA}{dt} + \frac{1}{100}A = 6 + 3\sin\frac{t}{4}, \quad A(0) = 50.$$

The differential equation is linear with integrating factor $e^{\int dt/100} = e^{t/100}$, so

$$\begin{aligned} \frac{d}{dt}[e^{t/100}A(t)] &= \left(6 + 3\sin\frac{t}{4}\right)e^{t/100} \\ e^{t/100}A(t) &= 600e^{t/100} + \frac{150}{313}e^{t/100}\sin\frac{t}{4} - \frac{3750}{313}e^{t/100}\cos\frac{t}{4} + c, \end{aligned}$$

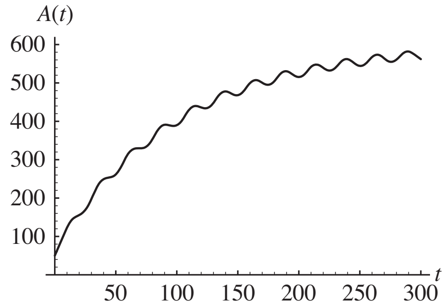
and

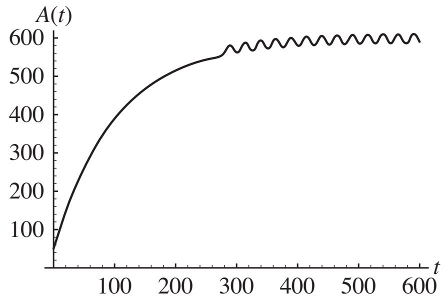
$$A(t) = 600 + \frac{150}{313}\sin\frac{t}{4} - \frac{3750}{313}\cos\frac{t}{4} + ce^{-t/100}.$$

Letting $t = 0$ and $A = 50$ we have $600 - 3750/313 + c = 50$ and $c = -168400/313$. Then

$$A(t) = 600 + \frac{150}{313}\sin\frac{t}{4} - \frac{3750}{313}\cos\frac{t}{4} - \frac{168400}{313}e^{-t/100}.$$

The graphs on $[0, 300]$ and $[0, 600]$ below show the effect of the sine function in the input when compared with the graph in Figure 2.7.4(a) in the text.





**27.** From

$$\frac{dA}{dt} = 3 - \frac{4A}{100 + (6-4)t} = 3 - \frac{2A}{50 + t}$$

we obtain $A = 50 + t + c(50 + t)^{-2}$. If $A(0) = 10$ then $c = -100{,}000$ and $A(30) = 64.38$ pounds.

**28.** **(a)** Initially the tank contains 300 gallons of solution. Since brine is pumped in at a rate of 3 gal/min and the mixture is pumped out at a rate of 2 gal/min, the net change is an increase of 1 gal/min. Thus, in 100 minutes the tank will contain its capacity of 400 gallons.

**(b)** The differential equation describing the amount of salt in the tank is $A'(t) = 6 - 2A/(300 + t)$ with solution

$$A(t) = 600 + 2t - (4.95 \times 10^7)(300 + t)^{-2}, \qquad 0 \le t \le 100,$$

as noted in the discussion following Example 5 in the text. Thus, the amount of salt in the tank when it overflows is

$$A(100) = 800 - (4.95 \times 10^7)(400)^{-2} = 490.625 \text{ lbs.}$$

**(c)** When the tank is overflowing the amount of salt in the tank is governed by the differential equation

$$\begin{aligned}\frac{dA}{dt} &= (3 \text{ gal/min})(2 \text{ lb/gal}) - \left(\frac{A}{400}\text{ lb/gal}\right)(3 \text{ gal/min})\\ &= 6 - \frac{3A}{400}, \qquad A(100) = 490.625.\end{aligned}$$

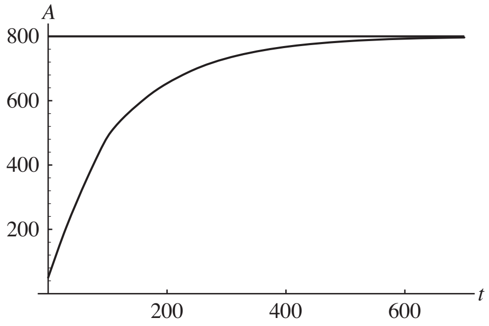
Solving the equation, we obtain $A(t) = 800 + ce^{-3t/400}$. The initial condition yields $c = -654.947$, so that

$$A(t) = 800 - 654.947e^{-3t/400}.$$

When $t = 150$, $A(150) = 587.37$ lbs.

**(d)** As $t \to \infty$, the amount of salt is 800 lbs, which is to be expected since (400 gal)(2 lb/gal)= 800 lbs.

**(e)**



**29.** Assume $L\,di/dt + Ri = E(t)$, $L = 0.1$, $R = 50$, and $E(t) = 50$ so that $i = \frac{3}{5} + ce^{-500t}$. If $i(0) = 0$ then $c = -3/5$ and $\lim\limits_{t\to\infty} i(t) = 3/5$.

**30.** Assume $L\,di/dt + Ri = E(t)$, $E(t) = E_0 \sin\omega t$, and $i(0) = i_0$ so that

$$i = \frac{E_0 R}{L^2\omega^2 + R^2}\sin\omega t - \frac{E_0 L\omega}{L^2\omega^2 + R^2}\cos\omega t + ce^{-Rt/L}.$$

Since $i(0) = i_0$ we obtain $c = i_0 + \dfrac{E_0 L\omega}{L^2\omega^2 + R^2}$.

**31.** Assume $R\,dq/dt + (1/C)q = E(t)$, $R = 200$, $C = 10^{-4}$, and $E(t) = 100$ so that $q = 1/100 + ce^{-50t}$. If $q(0) = 0$ then $c = -1/100$ and $i = \frac{1}{2}e^{-50t}$.

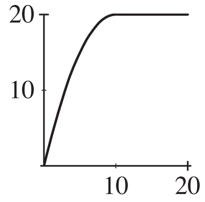
**32.** Assume $R\,dq/dt + (1/C)q = E(t)$, $R = 1000$, $C = 5 \times 10^{-6}$, and $E(t) = 200$. Then $q = \frac{1}{1000} + ce^{-200t}$ and $i = -200ce^{-200t}$. If $i(0) = 0.4$ then $c = -\frac{1}{500}$, $q(0.005) = 0.0003$ coulombs, and $i(0.005) = 0.1472$ amps. We have $q \to \frac{1}{1000}$ as $t \to \infty$.

**33.** For $0 \le t \le 20$ the differential equation is $20\,di/dt + 2i = 120$. An integrating factor is $e^{t/10}$, so $(d/dt)[e^{t/10}i] = 6e^{t/10}$ and $i = 60 + c_1e^{-t/10}$. If $i(0) = 0$ then $c_1 = -60$ and $i = 60 - 60e^{-t/10}$. For $t > 20$ the differential equation is $20\,di/dt + 2i = 0$ and $i = c_2e^{-t/10}$. At $t = 20$ we want $c_2e^{-2} = 60 - 60e^{-2}$ so that $c_2 = 60\left(e^2 - 1\right)$. Thus

$$i(t) = \begin{cases} 60 - 60e^{-t/10}, & 0 \le t \le 20 \\ 60\left(e^2 - 1\right)e^{-t/10}, & t > 20 \end{cases}$$

**34.** We first solve $(1-t/10)di/dt+0.2i=4$. Separating variables we obtain $di/(40-2i)=dt/(10-t)$. Then

$$-\frac{1}{2}\ln|40-2i|=-\ln|10-t|+c \quad \text{or} \quad \sqrt{40-2i}=c_1(10-t).$$



Since $i(0)=0$ we must have $c_1=2/\sqrt{10}$. Solving for $i$ we get $i(t)=4t-\frac{1}{5}t^2$, $0\le t<10$. For $t\ge 10$ the equation for the current becomes $0.2i=4$ or $i=20$. Thus

$$i(t)=\begin{cases} 4t-\dfrac{1}{5}t^2, & 0\le t<10 \\ 20, & t\ge 10. \end{cases}$$

The graph of $i(t)$ is given in the figure.

**35.** **(a)** From $m\,dv/dt=mg-kv$ we obtain $v=mg/k+ce^{-kt/m}$. If $v(0)=v_0$ then $c=v_0-mg/k$ and the solution of the initial-value problem is

$$v(t)=\frac{mg}{k}+\left(v_0-\frac{mg}{k}\right)e^{-kt/m}.$$

**(b)** As $t\to\infty$ the limiting velocity is $mg/k$.

**(c)** From $ds/dt=v$ and $s(0)=0$ we obtain

$$s(t)=\frac{mg}{k}t-\frac{m}{k}\left(v_0-\frac{mg}{k}\right)e^{-kt/m}+\frac{m}{k}\left(v_0-\frac{mg}{k}\right).$$

**36.** **(a)** Integrating $d^2s/dt^2=-g$ we get $v(t)=ds/dt=-gt+c$. From $v(0)=300$ we find $c=300$, and we are given $g=32$, so the velocity is $v(t)=-32t+300$.

**(b)** Integrating again and using $s(0)=0$ we get $s(t)=-16t^2+300t$. The maximum height is attained when $v=0$, that is, at $t_a=9.375$. The maximum height will be $s(9.375)=1406.25$ ft.

**37.** When air resistance is proportional to velocity, the model for the velocity is $m\,dv/dt=-mg-kv$ (using the fact that the positive direction is upward.) Solving the differential equation using separation of variables we obtain $v(t)=-mg/k+ce^{-kt/m}$. From $v(0)=300$ we get

$$v(t)=-\frac{mg}{k}+\left(300+\frac{mg}{k}\right)e^{-kt/m}.$$

Integrating and using $s(0)=0$ we find

$$s(t)=-\frac{mg}{k}t+\frac{m}{k}\left(300+\frac{mg}{k}\right)(1-e^{-kt/m}).$$

Setting $k=0.0025$, $m=16/32=0.5$, and $g=32$ we have

$$s(t)=1{,}340{,}000-6{,}400t-1{,}340{,}000e^{-0.005t}$$

and

$$v(t) = -6{,}400 + 6{,}700e^{-0.005t}.$$

The maximum height is attained when $v = 0$, that is, at $t_a = 9.162$. The maximum height will be $s(9.162) = 1363.79\,\text{ft}$, which is less than the maximum height in Problem 36.

**38.** Assuming that the air resistance is proportional to velocity and the positive direction is downward with $s(0) = 0$, the model for the velocity is $m\,dv/dt = mg - kv$. Using separation of variables to solve this differential equation, we obtain $v(t) = mg/k + ce^{-kt/m}$. Then, using $v(0) = 0$, we get $v(t) = (mg/k)(1 - e^{-kt/m})$. Letting $k = 0.5$, $m = (125 + 35)/32 = 5$, and $g = 32$, we have $v(t) = 320(1 - e^{-0.1t})$. Integrating, we find $s(t) = 320t + 3200e^{-0.1t} + c_1$. Solving $s(0) = 0$ for $c_1$ we find $c_1 = -3200$, therefore $s(t) = 320t + 3200e^{-0.1t} - 3200$. At $t = 15$, when the parachute opens, $v(15) = 248.598$ and $s(15) = 2314.02$. At this time the value of $k$ changes to $k = 10$ and the new initial velocity is $v_0 = 248.598$. With the parachute open, the skydiver's velocity is $v_p(t) = mg/k + c_2e^{-kt/m}$, where $t$ is reset to 0 when the parachute opens. Letting $m = 5$, $g = 32$, and $k = 10$, this gives $v_p(t) = 16 + c_2e^{-2t}$. From $v(0) = 248.598$ we find $c_2 = 232.598$, so $v_p(t) = 16 + 232.598e^{-2t}$. Integrating, we get $s_p(t) = 16t - 116.299e^{-2t} + c_3$. Solving $s_p(0) = 0$ for $c_3$, we find $c_3 = 116.299$, so $s_p(t) = 16t - 116.299e^{-2t} + 116.299$. Twenty seconds after leaving the plane is five seconds after the parachute opens. The skydiver's velocity at this time is $v_p(5) = 16.0106$ ft/s and she has fallen a total of $s(15) + s_p(5) = 2314.02 + 196.294 = 2510.31$ ft. Her terminal velocity is $\lim_{t\to\infty} v_p(t) = 16$, so she has very nearly reached her terminal velocity five seconds after the parachute opens. When the parachute opens, the distance to the ground is $15{,}000 - s(15) = 15{,}000 - 2{,}314 = 12{,}686$ ft. Solving $s_p(t) = 12{,}686$ we get $t = 785.6$ s $= 13.1$ min. Thus, it will take her approximately 13.1 minutes to reach the ground after her parachute has opened and a total of $(785.6 + 15)/60 = 13.34$ minutes after she exits the plane.

**39.** **(a)** With the values given in the text the initial-value problem becomes

$$\frac{dv}{dt} + \frac{2}{200-t}v = -9.8 + \frac{2000}{200-t}, \quad v(0) = 0.$$

This is a linear differential equationwith integrating factor

$$e^{\int [2/(200-t)]\,dt} = e^{-2\ln|200-t|} = (200-t)^{-2}.$$

Then

$$\begin{aligned}
\frac{d}{dt}\left[(200-t)^{-2}v\right] &= -9.8(200-t)^{-2} + 2000(200-t)^{-3} \quad \longleftarrow \text{integrate}\\
(200-t)^{-2}v &= -9.8(200-t)^{-1} + 1000(200-t)^{-2} + c \quad \longleftarrow \text{multiply by}(200-t)^2\\
v &= -9.8(200-t) + 1000 + c(200-t)^2\\
&= -960 + 9.8t + c(200-t)^2.
\end{aligned}$$

Using the initial condition we have

$$0 = v(0) = -960 + 40{,}000c \qquad \text{so} \qquad c = \frac{960}{40{,}000} = 0.024.$$

Thus

$$\begin{aligned} v(t) &= -960 + 9.8t + 0.024(200 - t)^2 \\ &= -960 + 9.8t + 0.024(40,000 - 400t + t^2) \\ &= -960 + 9.8t + 960 - 9.6t + 0.024t^2 \\ &= 0.024t^2 + 0.2t. \end{aligned}$$

**(b)** Integrating both sides of

$$\frac{ds}{dt} = v(t) = 0.024t^2 + 0.2t$$

we find

$$s(t) = 0.008t^3 + 0.1t^2 + c_1.$$

We assume that the height of the rocket is measured from $s = 0$, so that $s(0) = 0$ and $c_1 = 0$. Then the height of the rocket at time $t$ is $s(t) = 0.008t^3 + 0.1t^2$.

**40. (a)** From Problem 22 in Exercises 1.3, $t_b = m_f(0)/\lambda$. In this case $m_f(0) = 50$ and $\lambda = 1$, so the time of burnout is 50 s.

**(b)** The velocity at burnout time is

$$v(50) = 0.024(50)^2 + 0.2(50) = 70 \text{ m/s}.$$

**(c)** The height of the rocket at burnout is

$$s(50) = 0.008(50)^3 + 0.1(50)^2 = 1250 \text{ m}.$$

**(d)** At burnout the rocket will have upward momentum which will carry it higher.

**(e)** After burnout the total mass of the rocket is a constant $200 - 50 = 150$ kg. By Problem 22 in Exercises 1.3 the velocity for a rocket with variable mass due to fuel consumption is

$$m\frac{dv}{dt} + v\frac{dm}{dt} + kv = -mg + R.$$

Here $m$ is the total mass of the rocket, and in this case $m$ is constant after burnout, so $dm/dt = 0$ and the velocity of the rocket satisfies

$$m\frac{dv}{dt} + kv = -mg + R.$$

We identify $m = 150, k = 3, g = 9.8$ and $R = 0$, since the thrust is 0 after burnout. Then

$$150\frac{dv}{dt} + 3v = -150(9.8) = -1470 \quad \text{or} \quad \frac{dv}{dt} + \frac{1}{50}v = -9.8.$$

**41.** **(a)** The differential equation is first-order and linear. Letting $b = k/\rho$, the integrating factor is $e^{\int 3b\,dt/(bt+r_0)} = (r_0 + bt)^3$. Then

$$\frac{d}{dt}[(r_0 + bt)^3 v] = g(r_0 + bt)^3 \quad \text{and} \quad (r_0 + bt)^3 v = \frac{g}{4b}(r_0 + bt)^4 + c.$$

The solution of the differential equation is $v(t) = (g/4b)(r_0 + bt) + c(r_0 + bt)^{-3}$. Using $v(0) = 0$ we find $c = -gr_0^4/4b$, so that

$$v(t) = \frac{g}{4b}(r_0 + bt) - \frac{gr_0^4}{4b(r_0 + bt)^3} = \frac{g\rho}{4k}\left(r_0 + \frac{k}{\rho}t\right) - \frac{g\rho r_0^4}{4k(r_0 + kt/\rho)^3}.$$

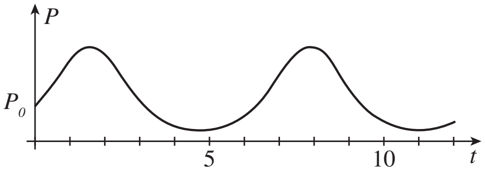
**(b)** Integrating $dr/dt = k/\rho$ we get $r = kt/\rho + c$. Using $r(0) = r_0$ we have $c = r_0$, so $r(t) = kt/\rho + r_0$.

**(c)** If $r = 0.007$ ft when $t = 10$ s, then solving $r(10) = 0.007$ for $k/\rho$, we obtain $k/\rho = -0.0003$ and $r(t) = 0.01 - 0.0003t$. Solving $r(t) = 0$ we get $t = 33.3$, so the raindrop will have evaporated completely at 33.3 seconds.

**42.** Separating variables, we obtain $dP/P = k\cos t\,dt$, so

$$\ln|P| = k\sin t + c \qquad \text{and} \qquad P = c_1 e^{k\sin t}.$$

If $P(0) = P_0$, then $c_1 = P_0$ and $P = P_0 e^{k\sin t}$.



**43.** **(a)** From $dP/dt = (k_1 - k_2)P$ we obtain $P = P_0 e^{(k_1 - k_2)t}$ where $P_0 = P(0)$.

**(b)** If $k_1 > k_2$ then $P \to \infty$ as $t \to \infty$. If $k_1 = k_2$ then $P = P_0$ for every $t$. If $k_1 < k_2$ then $P \to 0$ as $t \to \infty$.

**44.** **(a)** The solution of the differential equation is $P(t) = c_1 e^{kt} + h/k$. If we let the initial population of fish be $P_0$ then $P(0) = P_0$ which implies that

$$c_1 = P_0 - \frac{h}{k} \quad \text{and} \quad P(t) = \left(P_0 - \frac{h}{k}\right)e^{kt} + \frac{h}{k}.$$

**(b)** For $P_0 > h/k$ all terms in the solution are positive. In this case $P(t)$ increases as time $t$ increases. That is, $P(t) \to \infty$ as $t \to \infty$. For $P_0 = h/k$ the population remains constant for all time $t$:

$$P(t) = \left(\frac{h}{k} - \frac{h}{k}\right)e^{kt} + \frac{h}{k} = \frac{h}{k}.$$

For $0 < P_0 < h/k$ the coefficient of the exponential function is negative and so the function decreases as time $t$ increases.

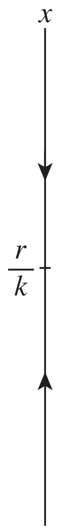
**(c)** For $0 < P_0 < h/k$ the function decreases and is concave down, therefore the graph of $P(t)$ crosses the $t$-axis. That is, there exists a time $T > 0$ such that $P(T) = 0$. Solving

$$\left(P_0 - \frac{h}{k}\right) e^{kT} + \frac{h}{k} = 0$$

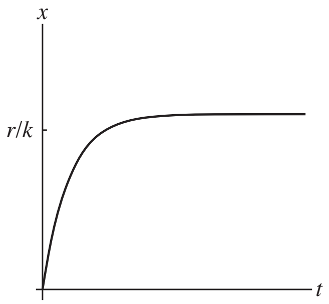
for $T$ shows that the time of extinction is

$$T = \frac{1}{k} \ln \left(\frac{h}{h - kP_0}\right).$$

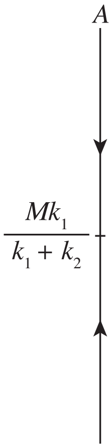
**45. (a)** Solving $r - kx = 0$ for $x$ we find the equilibrium solution $x = r/k$. When $x < r/k$, $dx/dt > 0$ and when $x > r/k$, $dx/dt < 0$. From the phase portrait we see that $\lim_{t\to\infty} x(t) = r/k$.



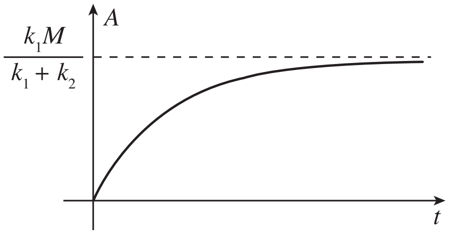
**(b)** From $dx/dt = r - kx$ and $x(0) = 0$ we obtain $x = r/k - (r/k)e^{-kt}$ so that $x \to r/k$ as $t \to \infty$. If $x(T) = r/2k$ then $T = (\ln 2)/k$.



**46. (a)** Solving $k_1(M - A) - k_2A = 0$ for $A$ we find the equilibrium solution $A = k_1M/(k_1 + k_2)$. From the phase portrait we see that $\lim_{t\to\infty} A(t) = k_1M/(k_1 + k_2)$. Since $k_2 > 0$, the material will never be completely memorized and the larger $k_2$ is, the less the amount of material will be memorized over time.



**(b)** Write the differential equation in the form $dA/dt + (k_1 + k_2)A = k_1M$. Then an integrating factor is $e^{(k_1+k_2)t}$, and



$$\frac{d}{dt}\left[e^{(k_1+k_2)t}A\right] = k_1Me^{(k_1+k_2)t}$$

$$e^{(k_1+k_2)t}A = \frac{k_1M}{k_1+k_2}e^{(k_1+k_2)t} + c$$

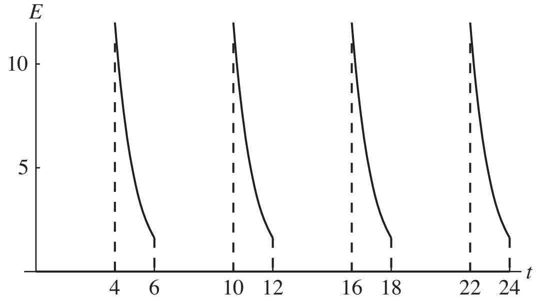
$$A = \frac{k_1M}{k_1+k_2} + ce^{-(k_1+k_2)t}.$$

Using $A(0) = 0$ we find $c = -\dfrac{k_1M}{k_1+k_2}$ and $A = \dfrac{k_1M}{k_1+k_2}\left(1 - e^{-(k_1+k_2)t}\right)$. As $t \to \infty$, $A \to \dfrac{k_1M}{k_1+k_2}$.

**47. (a)** For $0 \le t < 4$, $6 \le t < 10$ and $12 \le t < 16$, no voltage is applied to the heart and $E(t) = 0$. At the other times, the differential equation is $dE/dt = -E/RC$. Separating variables, integrating, and solving for $e$, we get $E = ke^{-t/RC}$, subject to $E(4) = E(10) = E(16) = 12$. These intitial conditions yield, respectively, $k = 12e^{4/RC}$, $k = 12e^{10/RC}$, $k = 12e^{16/RC}$, and $k = 12e^{22/RC}$. Thus

$$E(t) = \begin{cases} 0, & 0 \le t < 4,\ 6 \le t < 10,\ 12 \le t < 16 \\ 12e^{(4-t)/RC}, & 4 \le t < 6 \\ 12e^{(10-t)/RC}, & 10 \le t < 12 \\ 12e^{(16-t)/RC}, & 16 \le t < 18 \\ 12e^{(22-t)/RC}, & 22 \le t < 24 \end{cases}$$

**(b)**



**48. (a)** ($i$) Using Newton's second law of motion, $F = ma = m\,dv/dt$, the differential equation for the velocity $v$ is

$$m\frac{dv}{dt} = mg\sin\theta \qquad \text{or} \qquad \frac{dv}{dt} = g\sin\theta,$$

where $mg\sin\theta$, $0 < \theta < \pi/2$, is the component of the weight along the plane in the direction of motion.

(*ii*) The model now becomes

$$m\,\frac{dv}{dt} = mg\sin\theta - \mu mg\cos\theta,$$

where $\mu mg\cos\theta$ is the component of the force of sliding friction (which acts perpendicular to the plane) along the plane. The negative sign indicates that this component of force is a retarding force which acts in the direction opposite to that of motion.

(*iii*) If air resistance is taken to be proportional to the instantaneous velocity of the body, the model becomes

$$m\,\frac{dv}{dt} = mg\sin\theta - \mu mg\cos\theta - kv,$$

where $k$ is a constant of proportionality.

**(b)** (*i*) With $m = 3$ slugs, the differential equation is

$$3\,\frac{dv}{dt} = (96)\cdot\frac{1}{2} \qquad \text{or} \qquad \frac{dv}{dt} = 16.$$

Integrating the last equation gives $v(t) = 16t + c_1$. Since $v(0) = 0$, we have $c_1 = 0$ and so $v(t) = 16t$.

(*ii*) With $m = 3$ slugs, the differential equation is

$$3\,\frac{dv}{dt} = (96)\cdot\frac{1}{2} - \frac{\sqrt{3}}{4}\cdot(96)\cdot\frac{\sqrt{3}}{2} \qquad \text{or} \qquad \frac{dv}{dt} = 4.$$

In this case $v(t) = 4t$.

(*iii*) When the retarding force due to air resistance is taken into account, the differential equation for velocity $v$ becomes

$$3\,\frac{dv}{dt} = (96)\cdot\frac{1}{2} - \frac{\sqrt{3}}{4}\cdot(96)\cdot\frac{\sqrt{3}}{2} - \frac{1}{4}v \qquad \text{or} \qquad 3\,\frac{dv}{dt} = 12 - \frac{1}{4}v.$$

The last differential equation is linear and has solution $v(t) = 48 + c_1e^{-t/12}$. Since $v(0) = 0$, we find $c_1 = -48$, so $v(t) = 48 - 48e^{-t/12}$.

**49. (a)** (*i*) If $s(t)$ is distance measured down the plane from the highest point, then $ds/dt = v$. Integrating $ds/dt = 16t$ gives $s(t) = 8t^2 + c_2$. Using $s(0) = 0$ then gives $c_2 = 0$. Now the length $L$ of the plane is $L = 50/\sin 30° = 100$ ft. The time it takes the box to slide completely down the plane is the solution of $s(t) = 100$ or $t^2 = 25/2$, so $t \approx 3.54$ s.

(*ii*) Integrating $ds/dt = 4t$ gives $s(t) = 2t^2 + c_2$. Using $s(0) = 0$ gives $c_2 = 0$, so $s(t) = 2t^2$ and the solution of $s(t) = 100$ is now $t \approx 7.07$ s.

(*iii*) Integrating $ds/dt = 48 - 48e^{-t/12}$ and using $s(0) = 0$ to determine the constant of integration, we obtain $s(t) = 48t + 576e^{-t/12} - 576$. With the aid of a CAS we find that the solution of $s(t) = 100$, or

$$100 = 48t + 576e^{-t/12} - 576 \qquad \text{or} \qquad 0 = 48t + 576e^{-t/12} - 676,$$

is now $t \approx 7.84$ s.

**(b)** The differential equation $m\,dv/dt = mg\sin\theta - \mu mg\cos\theta$ can be written

$$m\frac{dv}{dt} = mg\cos\theta(\tan\theta - \mu).$$

If $\tan\theta = \mu$, $dv/dt = 0$ and $v(0) = 0$ implies that $v(t) = 0$. If $\tan\theta < \mu$ and $v(0) = 0$, then integration implies $v(t) = g\cos\theta(\tan\theta - \mu)t < 0$ for all time $t$.

**(c)** Since $\tan 23° = 0.4245$ and $\mu = \sqrt{3}/4 = 0.4330$, we see that $\tan 23° < 0.4330$. The differential equation is $dv/dt = 32\cos 23°(\tan 23° - \sqrt{3}/4) = -0.251493$. Integration and the use of the initial condition gives $v(t) = -0.251493t + 1$. When the box stops, $v(t) = 0$ or $0 = -0.251493t + 1$ or $t = 3.976254\,\text{s}$. From $s(t) = -0.125747t^2 + t$ we find $s(3.976254) = 1.988119\,\text{ft}$.

**(d)** With $v_0 > 0$, $v(t) = -0.251493t + v_0$ and $s(t) = -0.125747t^2 + v_0t$. Because two real positive solutions of the equation $s(t) = 100$, or $0 = -0.125747t^2 + v_0t - 100$, would be physically meaningless, we use the quadratic formula and require that $b^2 - 4ac = 0$ or $v_0^2 - 50.2987 = 0$. From this last equality we find $v_0 \approx 7.092164\,\text{ft/s}$. For the time it takes the box to traverse the entire inclined plane, we must have $0 = -0.125747t^2 + 7.092164t - 100$. *Mathematica* gives complex roots for the last equation: $t = 28.2001 \pm 0.0124458i$. But, for

$$0 = -0.125747t^2 + 7.092164691t - 100,$$

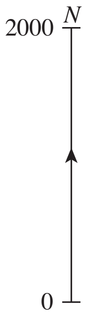
the roots are $t = 28.1999\,\text{s}$ and $t = 28.2004\,\text{s}$. So if $v_0 > 7.092164$, we are guaranteed that the box will slide completely down the plane.

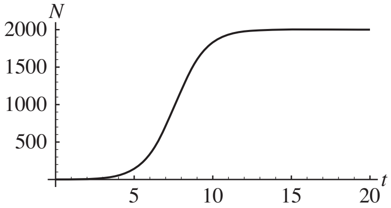
**50. (a)** We saw in part (a) of Problem 36 that the ascent time is $t_a = 9.375$. To find when the cannonball hits the ground we solve $s(t) = -16t^2 + 300t = 0$, getting a total time in flight of $t = 18.75\,\text{s}$. Thus, the time of descent is $t_d = 18.75 - 9.375 = 9.375$. The impact velocity is $v_i = v(18.75) = -300$, which has the same magnitude as the initial velocity.

**(b)** We saw in Problem 37 that the ascent time in the case of air resistance is $t_a = 9.162$. Solving $s(t) = 1{,}340{,}000 - 6{,}400t - 1{,}340{,}000e^{-0.005t} = 0$ we see that the total time of flight is $18.466\,\text{s}$. Thus, the descent time is $t_d = 18.466 - 9.162 = 9.304$. The impact velocity is $v_i = v(18.466) = -290.91$, compared to an initial velocity of $v_0 = 300$.

## 3.2 Nonlinear Models

**1. (a)** Solving $N(1 - 0.0005N) = 0$ for $N$ we find the equilibrium solutions $N = 0$ and $N = 2000$. When $0 < N < 2000$, $dN/dt > 0$. From the phase portrait we see that $\lim_{t\to\infty} N(t) = 2000$. A graph of the solution is shown in part (b).





**(b)** Separating variables and integrating we have

$$\frac{dN}{N(1-0.0005N)} = \left(\frac{1}{N} - \frac{1}{N-2000}\right) dN = dt$$

and

$$\ln N - \ln|N-2000| = t + c.$$

Solving for $N$ we get $N(t) = 2000e^{c+t}/(1+e^{c+t}) = 2000e^c e^t/(1+e^c e^t)$. Using $N(0)=1$ and solving for $e^c$ we find $e^c = 1/1999$ and so $N(t) = 2000e^t/(1999+e^t)$. Then $N(10) = 1833.59$, so 1834 companies are expected to adopt the new technology when $t = 10$.

**2.** From $dN/dt = N(a-bN)$ and $N(0) = 500$ we obtain

$$N = \frac{500a}{500b + (a-500b)e^{-at}}.$$

Since $\lim_{t\to\infty} N = a/b = 50,000$ and $N(1) = 1000$ we have $a = 0.7033$, $b = 0.00014$, and $N = 50{,}000/(1+99e^{-0.7033t})$.

**3.** From $dP/dt = P\left(10^{-1} - 10^{-7}P\right)$ and $P(0) = 5000$ we obtain $P = 500/(0.0005+0.0995e^{-0.1t})$ so that $P \to 1{,}000{,}000$ as $t \to \infty$. If $P(t) = 500{,}000$ then $t = 52.9$ months.

**4.** **(a)** We have $dP/dt = P(a-bP)$ with $P(0) = 3.929$ million. Using separation of variables we obtain

$$P(t) = \frac{3.929a}{3.929b + (a-3.929b)e^{-at}} = \frac{a/b}{1+(a/3.929b-1)e^{-at}}$$
$$= \frac{c}{1+(c/3.929-1)e^{-at}},$$

where $c = a/b$. At $t = 60(1850)$ the population is 23.192 million, so

$$23.192 = \frac{c}{1+(c/3.929-1)e^{-60a}}$$

or $c = 23.192 + 23.192(c/3.929-1)e^{-60a}$. At $t = 120(1910)$,

$$91.972 = \frac{c}{1+(c/3.929-1)e^{-120a}}$$

or $c = 91.972 + 91.972(c/3.929-1)(e^{-60a})^2$. Combining the two equations for $c$ we get

$$\left(\frac{(c-23.192)/23.192}{c/3.929-1}\right)^2 \left(\frac{c}{3.929} - 1\right) = \frac{c-91.972}{91.972}$$

or

$$91.972(3.929)(c-23.192)^2 = (23.192)^2(c-91.972)(c-3.929).$$

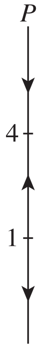
The solution of this quadratic equation is $c = 197.274$. This in turn gives $a = 0.0313$. Therefore,

$$P(t) = \frac{197.274}{1+49.21e^{-0.0313t}}.$$

**(b)**

| Year | Census Population | Predicted Population | Error | % Error |
|---|---|---|---|---|
| 1790 | 3.929 | 3.929 | 0.000 | 0.00 |
| 1800 | 5.308 | 5.334 | –0.026 | –0.49 |
| 1810 | 7.240 | 7.222 | 0.018 | 0.24 |
| 1820 | 9.638 | 9.746 | –0.108 | –1.12 |
| 1830 | 12.866 | 13.090 | –0.224 | –1.74 |
| 1840 | 17.069 | 17.475 | –0.406 | –2.38 |
| 1850 | 23.192 | 23.143 | 0.049 | 0.21 |
| 1860 | 31.433 | 30.341 | 1.092 | 3.47 |
| 1870 | 38.558 | 39.272 | –0.714 | –1.85 |
| 1880 | 50.156 | 50.044 | 0.112 | 0.22 |
| 1890 | 62.948 | 62.600 | 0.348 | 0.55 |
| 1900 | 75.996 | 76.666 | –0.670 | –0.88 |
| 1910 | 91.972 | 91.739 | 0.233 | 0.25 |
| 1920 | 105.711 | 107.143 | –1.432 | –1.35 |
| 1930 | 122.775 | 122.140 | 0.635 | 0.52 |
| 1940 | 131.669 | 136.068 | –4.399 | –3.34 |
| 1950 | 150.697 | 148.445 | 2.252 | 1.49 |

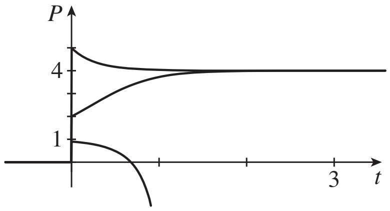
**5. (a)** The differential equation is $dP/dt = P(5-P)-4$. Solving $P(5-P)-4=0$ for $P$ we obtain equilibrium solutions $P=1$ and $P=4$. The phase portrait is shown on the right and solution curves are shown in part (b). We see that for $P_0 > 4$ and $1 < P_0 < 4$ the population approaches 4 as $t$ increases. For $0 < P < 1$ the population decreases to 0 in finite time.



**(b)** The differential equation is

$$\frac{dP}{dt} = P(5-P)-4 = -(P^2-5P+4) = -(P-4)(P-1).$$

Separating variables and integrating, we obtain



$$\frac{dP}{(P-4)(P-1)} = -dt$$

$$\left(\frac{1/3}{P-4} - \frac{1/3}{P-1}\right) dP = -dt$$

$$\frac{1}{3}\ln\left|\frac{P-4}{P-1}\right| = -t + c$$

$$\frac{P-4}{P-1} = c_1 e^{-3t}.$$

Setting $t=0$ and $P=P_0$ we find $c_1 = (P_0-4)/(P_0-1)$. Solving for $P$ we obtain

$$P(t) = \frac{4(P_0-1)-(P_0-4)e^{-3t}}{(P_0-1)-(P_0-4)e^{-3t}}.$$

**(c)** To find when the population becomes extinct in the case $0 < P_0 < 1$ we set $P = 0$ in

$$\frac{P-4}{P-1} = \frac{P_0-4}{P_0-1}\,e^{-3t}$$

from part (a) and solve for $t$. This gives the time of extinction

$$t = -\frac{1}{3}\ln\frac{4(P_0-1)}{P_0-4}\,.$$

**6.** Solving $P(5-P) - \frac{25}{4} = 0$ for $P$ we obtain the equilibrium solution $P = \frac{5}{2}$. For $P \neq \frac{5}{2}$, $dP/dt < 0$. Thus, if $P_0 < \frac{5}{2}$, the population becomes extinct (otherwise there would be another equilibrium solution.) Using separation of variables to solve the initial-value problem, we get

$$P(t) = [4P_0 + (10P_0 - 25)t]/[4 + (4P_0 - 10)t].$$

To find when the population becomes extinct for $P_0 < \frac{5}{2}$ we solve $P(t) = 0$ for $t$. We see that the time of extinction is $t = 4P_0/5(5 - 2P_0)$.

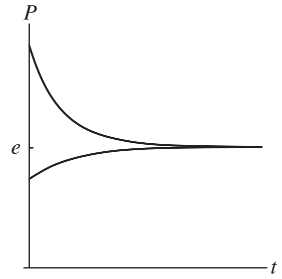
**7.** Solving $P(5-P) - 7 = 0$ for $P$ we obtain complex roots, so there are no equilibrium solutions. Since $dP/dt < 0$ for all values of $P$, the population becomes extinct for any initial condition. Using separation of variables to solve the initial-value problem, we get

$$P(t) = \frac{5}{2} + \frac{\sqrt{3}}{2}\tan\left[\tan^{-1}\left(\frac{2P_0-5}{\sqrt{3}}\right) - \frac{\sqrt{3}}{2}t\right].$$

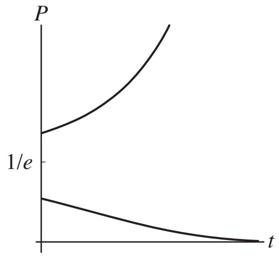
Solving $P(t) = 0$ for $t$ we see that the time of extinction is

$$t = \frac{2}{3}\left(\sqrt{3}\tan^{-1}(5/\sqrt{3}) + \sqrt{3}\tan^{-1}\left[(2P_0-5)/\sqrt{3}\right]\right).$$

**8.** **(a)** The differential equation is $dP/dt = P(1 - \ln P)$, which has the equilibrium solution $P = e$. When $P_0 > e$, $dP/dt < 0$, and when $P_0 < e$, $dP/dt > 0$.



**(b)** The differential equation is $dP/dt = P(1 + \ln P)$, which has the equilibrium solution $P = 1/e$. When $P_0 > 1/e$, $dP/dt > 0$, and when $P_0 < 1/e$, $dP/dt < 0$.



**9.** Let $X = X(t)$ be the amount of $C$ at time $t$ and $dX/dt = k(120 - 2X)(150 - X)$. If $X(0) = 0$ and $X(5) = 10$, then

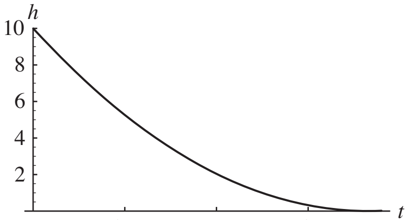
$$X(t) = \frac{150 - 150e^{180kt}}{1 - 2.5e^{180kt}},$$

where $k = .0001259$ and $X(20) = 29.3$ grams. Now by L'Hôpital's rule, $X \to 60$ as $t \to \infty$, so that the amount of $A \to 0$ and the amount of $B \to 30$ as $t \to \infty$.

**10.** From $dX/dt = k(150 - X)^2$, $X(0) = 0$, and $X(5) = 10$ we obtain $X = 150 - 150/(150kt + 1)$ where $k = .000095238$. Then $X(20) = 33.3$ grams and $X \to 150$ as $t \to \infty$ so that the amount of $A \to 0$ and the amount of $B \to 0$ as $t \to \infty$. If $X(t) = 75$ then $t = 70$ minutes.

**11.** **(a)** The initial-value problem is $dh/dt = -8A_h\sqrt{h}/A_w$, $h(0) = H$. Separating variables and integrating we have

$$\frac{dh}{\sqrt{h}} = -\frac{8A_h}{A_w}\,dt \quad \text{and} \quad 2\sqrt{h} = -\frac{8A_h}{A_w}t + c.$$



Using $h(0) = H$ we find $c = 2\sqrt{H}$, so the solution of the initial-value problem is $\sqrt{h(t)} = (A_w\sqrt{H} - 4A_h t)/A_w$, where $A_w\sqrt{H} - 4A_h t \geq 0$. Thus,

$$h(t) = (A_w\sqrt{H} - 4A_h t)^2/A_w^2 \quad \text{for} \quad 0 \leq t \leq A_w\sqrt{H}/4A_h.$$

**(b)** Identifying $H = 10$, $A_w = 4\pi$, and $A_h = \pi/576$ we have $h(t) = t^2/331{,}776 - (\sqrt{5/2}/144)t + 10$. Solving $h(t) = 0$ we see that the tank empties in $576\sqrt{10}$ seconds or 30.36 minutes.

**12.** To obtain the solution of this differential equation we use $h(t)$ from Problem 13 in Exercises 1.3. Then $h(t) = (A_w\sqrt{H} - 4cA_h t)^2/A_w^2$. Solving $h(t) = 0$ with $c = 0.6$ and the values from Problem 13 we see that the tank empties in 3035.79 seconds or 50.6 minutes.

**13.** **(a)** Separating variables and integrating gives

$$6h^{3/2}\,dh = -5t \quad \text{and} \quad \frac{12}{5}h^{5/2} = -5t + c.$$

Using $h(0) = 20$ we find $c = 1920\sqrt{5}$, so the solution of the initial-value problem is $h(t) = \left(800\sqrt{5} - \frac{25}{12}t\right)^{2/5}$. Solving $h(t) = 0$ we see that the tank empties in $384\sqrt{5}$ seconds or 14.31 minutes.

**(b)** When the height of the water is $h$, the radius of the top of the water is $r = h\tan 30° = h/\sqrt{3}$ and $A_w = \pi h^2/3$. The differential equation is

$$\frac{dh}{dt} = -c\frac{A_h}{A_w}\sqrt{2gh} = -0.6\frac{\pi(2/12)^2}{\pi h^2/3}\sqrt{64h} = -\frac{2}{5h^{3/2}}.$$

Separating variables and integrating gives

$$5h^{3/2}\,dh = -2\,dt \quad \text{and} \quad 2h^{5/2} = -2t + c.$$

Using $h(0) = 9$ we find $c = 486$, so the solution of the initial-value problem is $h(t) = (243 - t)^{2/5}$. Solving $h(t) = 0$ we see that the tank empties in 243 seconds or 4.05 minutes.

**14.** When the height of the water is $h$, the radius of the top of the water is $\frac{2}{5}(20 - h)$ and $A_w = 4\pi(20 - h)^2/25$. The differential equation is

$$\frac{dh}{dt} = -c\frac{A_h}{A_w}\sqrt{2gh} = -0.6\frac{\pi(2/12)^2}{4\pi(20-h)^2/25}\sqrt{64h} = -\frac{5}{6}\frac{\sqrt{h}}{(20-h)^2}\,.$$

Separating variables and integrating we have

$$\frac{(20-h)^2}{\sqrt{h}}\,dh = -\frac{5}{6}\,dt \quad \text{and} \quad 800\sqrt{h} - \frac{80}{3}h^{3/2} + \frac{2}{5}h^{5/2} = -\frac{5}{6}t + c.$$

Using $h(0) = 20$ we find $c = 2560\sqrt{5}/3$, so an implicit solution of the initial-value problem is

$$800\sqrt{h} - \frac{80}{3}h^{3/2} + \frac{2}{5}h^{5/2} = -\frac{5}{6}t + \frac{2560\sqrt{5}}{3}\,.$$

To find the time it takes the tank to empty we set $h = 0$ and solve for $t$. The tank empties in $1024\sqrt{5}$ seconds or 38.16 minutes. Thus, the tank empties more slowly when the base of the cone is on the bottom.

**15.** **(a)** After separating variables we obtain

$$\begin{aligned}
\frac{m\,dv}{mg - kv^2} &= dt \\
\frac{1}{g}\,\frac{dv}{1 - (\sqrt{k}\,v/\sqrt{mg}\,)^2} &= dt \\
\frac{\sqrt{mg}}{\sqrt{k}\,g}\,\frac{\sqrt{k/mg}\,dv}{1 - (\sqrt{k}\,v/\sqrt{mg}\,)^2} &= dt \\
\sqrt{\frac{m}{kg}}\tanh^{-1}\frac{\sqrt{k}\,v}{\sqrt{mg}} &= t + c \\
\tanh^{-1}\frac{\sqrt{k}\,v}{\sqrt{mg}} &= \sqrt{\frac{kg}{m}}\,t + c_1.
\end{aligned}$$

Thus the velocity at time $t$ is

$$v(t) = \sqrt{\frac{mg}{k}}\tanh\left(\sqrt{\frac{kg}{m}}t + c_1\right).$$

Setting $t = 0$ and $v = v_0$ we find $c_1 = \tanh^{-1}(\sqrt{k}\,v_0/\sqrt{mg}\,)$.

**(b)** Since $\tanh t \to 1$ as $t \to \infty$, we have $v \to \sqrt{mg/k}$ as $t \to \infty$.

**(c)** Integrating the expression for $v(t)$ in part (a) we obtain an integral of the form $\int du/u$:

$$s(t) = \sqrt{\frac{mg}{k}} \int \tanh\left(\sqrt{\frac{kg}{m}}t + c_1\right) dt = \frac{m}{k} \ln\left[\cosh\left(\sqrt{\frac{kg}{m}}t + c_1\right)\right] + c_2.$$

Setting $t = 0$ and $s = 0$ we find $c_2 = -(m/k)\ln(\cosh c_1)$, where $c_1$ is given in part (a).

**16.** The differential equation is $m\,dv/dt = -mg - kv^2$. Separating variables and integrating, we have

$$\begin{aligned}
\frac{dv}{mg + kv^2} &= -\frac{dt}{m} \\
\frac{1}{\sqrt{mgk}} \tan^{-1}\left(\frac{\sqrt{k}\,v}{\sqrt{mg}}\right) &= -\frac{1}{m}t + c \\
\tan^{-1}\left(\frac{\sqrt{k}\,v}{\sqrt{mg}}\right) &= -\sqrt{\frac{gk}{m}}\,t + c_1 \\
v(t) &= \sqrt{\frac{mg}{k}} \tan\left(c_1 - \sqrt{\frac{gk}{m}}\,t\right)
\end{aligned}$$

Setting $v(0) = 300$, $m = \frac{16}{32} = \frac{1}{2}$, $g = 32$, and $k = 0.0003$, we find $v(t) = 230.94\tan(c_1 - 0.138564t)$ and $c_1 = 0.914743$. Integrating

$$v(t) = 230.94\tan(0.914743 - 0.138564t)$$

we get

$$s(t) = 1666.67\ln|\cos(0.914743 - 0.138564t)| + c_2.$$

Using $s(0) = 0$ we find $c_2 = 823.843$. Solving $v(t) = 0$ we see that the maximum height is attained when $t = 6.60159$. The maximum height is $s(6.60159) = 823.843$ ft.

**17. (a)** Let $\rho$ be the weight density of the water and $V$ the volume of the object. Archimedes' principle states that the upward buoyant force has magnitude equal to the weight of the water displaced. Taking the positive direction to be down, the differential equation is

$$m\frac{dv}{dt} = mg - kv^2 - \rho V.$$

**(b)** Using separation of variables we have

$$\begin{aligned}
\frac{m\,dv}{(mg - \rho V) - kv^2} &= dt \\
\frac{m}{\sqrt{k}} \frac{\sqrt{k}\,dv}{(\sqrt{mg - \rho V}\,)^2 - (\sqrt{k}\,v)^2} &= dt \\
\frac{m}{\sqrt{k}} \frac{1}{\sqrt{mg - \rho V}} \tanh^{-1} \frac{\sqrt{k}\,v}{\sqrt{mg - \rho V}} &= t + c.
\end{aligned}$$

Thus

$$v(t)=\sqrt{\frac{mg-\rho V}{k}}\tanh\left(\frac{\sqrt{kmg-k\rho V}}{m}t+c_1\right).$$

**(c)** Since $\tanh t\to 1$ as $t\to\infty$, the terminal velocity is $\sqrt{(mg-\rho V)/k}$.

**18. (a)** Writing the equation in the form $(x-\sqrt{x^2+y^2}\,)dx+y\,dy=0$ we identify $M=x-\sqrt{x^2+y^2}$ and $N=y$. Since $M$ and $N$ are both homogeneous functions of degree 1 we use the substitution $y=ux$. It follows that

$$\begin{aligned}\left(x-\sqrt{x^2+u^2x^2}\right)dx+ux(u\,dx+x\,du)&=0\\ x\left[1-\sqrt{1+u^2}+u^2\right]dx+x^2u\,du&=0\\ -\frac{u\,du}{1+u^2-\sqrt{1+u^2}}&=\frac{dx}{x}\\ \frac{u\,du}{\sqrt{1+u^2}\,(1-\sqrt{1+u^2}\,)}&=\frac{dx}{x}.\end{aligned}$$

Letting $w=1-\sqrt{1+u^2}$ we have $dw=-u\,du/\sqrt{1+u^2}$ so that

$$\begin{aligned}-\ln\left|1-\sqrt{1+u^2}\right|&=\ln|x|+c\\ \frac{1}{1-\sqrt{1+u^2}}&=c_1x\\ 1-\sqrt{1+u^2}&=-\frac{c_2}{x}\qquad\qquad(-c_2=1/c_1)\\ 1+\frac{c_2}{x}&=\sqrt{1+\frac{y^2}{x^2}}\\ 1+\frac{2c_2}{x}+\frac{c_2^2}{x^2}&=1+\frac{y^2}{x^2}.\end{aligned}$$

Solving for $y^2$ we have

$$y^2=2c_2x+c_2^2=4\left(\frac{c_2}{2}\right)\left(x+\frac{c_2}{2}\right)$$

which is a family of parabolas symmetric with respect to the $x$-axis with vertex at $(-c_2/2,0)$ and focus at the origin.

**(b)** Let $u=x^2+y^2$ so that

$$\frac{du}{dx}=2x+2y\frac{dy}{dx}.$$

Then

$$y\frac{dy}{dx}=\frac{1}{2}\frac{du}{dx}-x$$

and the differential equation can be written in the form

$$\frac{1}{2}\frac{du}{dx}-x=-x+\sqrt{u}\quad\text{or}\quad\frac{1}{2}\frac{du}{dx}=\sqrt{u}.$$

Separating variables and integrating gives

$$\begin{aligned} \frac{du}{2\sqrt{u}} &= dx \\ \sqrt{u} &= x + c \\ u &= x^2 + 2cx + c^2 \\ x^2 + y^2 &= x^2 + 2cx + c^2 \\ y^2 &= 2cx + c^2. \end{aligned}$$

**19.** **(a)** From $2W^2 - W^3 = W^2(2 - W) = 0$ we see that $W = 0$ and $W = 2$ are constant solutions.

**(b)** Separating variables and using a CAS to integrate we get

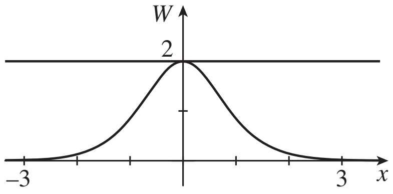
$$\frac{dW}{W\sqrt{4-2W}} = dx \quad \text{and} \quad -\tanh^{-1}\left(\frac{1}{2}\sqrt{4-2W}\right) = x + c.$$

Using the facts that the hyperbolic tangent is an odd function and $1 - \tanh^2 x = \operatorname{sech}^2 x$ we have

$$\begin{aligned} \frac{1}{2}\sqrt{4-2W} &= \tanh(-x-c) = -\tanh(x+c) \\ \frac{1}{4}(4-2W) &= \tanh^2(x+c) \\ 1 - \frac{1}{2}W &= \tanh^2(x+c) \\ \frac{1}{2}W &= 1 - \tanh^2(x+c) = \operatorname{sech}^2(x+c) \end{aligned}$$

Thus, $W(x) = 2\operatorname{sech}^2(x+c)$.

**(c)** Letting $x = 0$ and $W = 2$ we find that $\operatorname{sech}^2(c) = 1$ and $c = 0$.



**20.** **(a)** Solving $r^2 + (10-h)^2 = 10^2$ for $r^2$ we see that $r^2 = 20h - h^2$. Combining the rate of input of water, $\pi$, with the rate of output due to evaporation, $k\pi r^2 = k\pi(20h - h^2)$, we have $dV/dt = \pi - k\pi(20h - h^2)$. Using $V = 10\pi h^2 - \frac{1}{3}\pi h^3$, we see also that $dV/dt = (20\pi h - \pi h^2)dh/dt$. Thus,

$$(20\pi h - \pi h^2)\frac{dh}{dt} = \pi - k\pi(20h - h^2) \quad \text{and} \quad \frac{dh}{dt} = \frac{1 - 20kh + kh^2}{20h - h^2}.$$

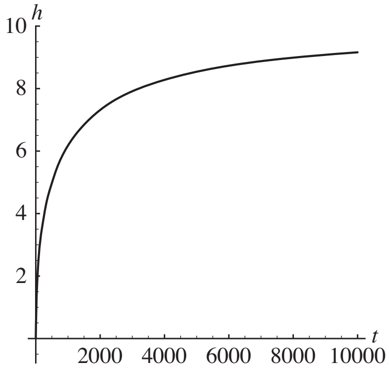
**(b)** Letting $k = 1/100$, separating variables and integrating (with the help of a CAS), we get

$$\frac{100h(h-20)}{(h-10)^2}\,dh = dt$$

and

$$\frac{100(h^2-10h+100)}{10-h} = t + c.$$

Using $h(0) = 0$ we find $c = 1000$, and solving for $h$ we get $h(t) = 0.005\left(\sqrt{t^2+4000t} - t\right)$, where the positive square root is chosen because $h \geq 0$.



**(c)** The volume of the tank is $V = \frac{2}{3}\pi(10)^3$ feet, so at a rate of $\pi$ cubic feet per minute, the tank will fill in $\frac{2}{3}(10)^3 \approx 666.67$ minutes $\approx 11.11$ hours.

**(d)** At 666.67 minutes, the depth of the water is $h(666.67) = 5.486$ feet. From the graph in (b) we suspect that $\lim_{t\to\infty} h(t) = 10$, in which case the tank will never completely fill. To prove this we compute the limit of $h(t)$:

$$\lim_{t\to\infty} h(t) = 0.005 \lim_{t\to\infty}\left(\sqrt{t^2+4000t} - t\right) = 0.005 \lim_{t\to\infty} \frac{t^2+4000t-t^2}{\sqrt{t^2+4000t}+t}$$

$$= 0.005 \lim_{t\to\infty} \frac{4000t}{t\sqrt{1+4000/t}+t} = 0.005\,\frac{4000}{1+1} = 0.005(2000) = 10.$$

**21. (a)** With $c = 0.01$ the differential equation is $dP/dt = kP^{1.01}$. Separating variables and integrating we obtain

$$\begin{aligned}
P^{-1.01}\,dP &= k\,dt \\
\frac{P^{-0.01}}{-0.01} &= kt + c_1 \\
P^{-0.01} &= -0.01kt + c_2 \\
P(t) &= (-0.01kt + c_2)^{-100} \\
P(0) &= c_2^{-100} = 10 \\
c_2 &= 10^{-0.01}.
\end{aligned}$$

Then

$$P(t) = \frac{1}{(-0.01kt + 10^{-0.01})^{100}}$$

and, since $P$ doubles in 5 months from 10 to 20,

$$P(5) = \frac{1}{(-0.01k(5) + 10^{-0.01})^{100}} = 20$$

so

$$\begin{aligned} \left(-0.01k(5)+10^{-0.01}\right)^{100} &= \frac{1}{20} \\ -0.01k &= \frac{\left[\left(\frac{1}{20}\right)^{1/100}-\left(\frac{1}{10}\right)^{1/100}\right]}{5} \\ &= -0.001350\,. \end{aligned}$$

Thus $P(t)=1/\left(-0.001350t+10^{-0.01}\right)^{100}$.

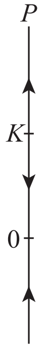
**(b)** Define $T=\left(\frac{1}{10}\right)^{1/100}/0.001350 \approx 724$ months $=60$ years. As $t \to 724$ (from the left), $P \to \infty$.

**(c)**

$$P(50)=1/\left[-0.001350\,(50)+10^{-0.01}\right]^{100} \approx 12,839 \qquad \text{and}$$

$$P(100)=1/\left[-0.001350\,(100)+10^{-0.01}\right]^{100} \approx 28,630,966$$

**22.** **(a)** From the phase portrait we see that $P=0$ is an attractor for $0<P_0<K=a/b$ and $P=K$ is a repeller for $P_0>K$.



**(b)** Letting $a=0.1$, $b=0.0005$ and using separation of variables gives

$$\left(-\frac{1}{P}+\frac{b}{bP-a}\right)dP=a\,dt$$

Integrating we have

$$\begin{aligned} -\ln P+\ln(bP-a) &= at+c_1 \\ \ln\left(\frac{bP-a}{P}\right) &= at+c_1 \\ \frac{bP-a}{P} &= c_2e^{at} \\ P &= \frac{a}{b-c_2e^{at}}. \end{aligned}$$

Since $P(0)=300$,

$$c_2=\frac{300b-a}{300} \qquad \text{and} \qquad P(t)=\frac{300a}{300b-(300b-a)\,e^{at}}.$$

Then, with $b = 0.0005$ and $a = 0.1$,

$$P(t) = \frac{300(0.1)}{00(0.0005) - [300(0.0005) - 0.1]\, e^{0.1t}} = \frac{30}{0.15 - 0.05e^{0.1t}} = \frac{600}{3 - e^{0.1t}}$$

and

$$3 - e^{0.1} = 0 \qquad \text{implies} \qquad 0.1t \ln 3 \qquad \text{so} \qquad t = 10 \ln 3.$$

This is doomsday in finite time, since $P(t) \to \infty$ as $t \to 10 \ln 3$ (from the left) $\approx 10.99$.

**(c)** For $P_0 = 100$

$$P(t) = \frac{100a}{100b - (100b - a)\, e^{at}} = \frac{100(0.1)}{100(0.0005) - [100(0.0005) - 0.1]\, e^{0.1t}}$$

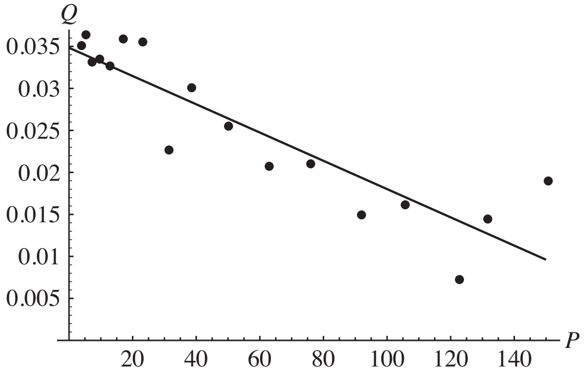
$$= \frac{10}{0.05 + 0.05e^{0.1t}} = \frac{200}{1 + e^{0.1t}},$$

and $P(t) \to 0$ as $t \to \infty$.

**23. (a)**

| t | P(t) | Q(t) |
|---|---|---|
| 0 | 3.929 | 0.035 |
| 10 | 5.308 | 0.036 |
| 20 | 7.240 | 0.033 |
| 30 | 9.638 | 0.033 |
| 40 | 12.866 | 0.033 |
| 50 | 17.069 | 0.036 |
| 60 | 23.192 | 0.036 |
| 70 | 31.433 | 0.023 |
| 80 | 38.558 | 0.030 |
| 90 | 50.156 | 0.026 |
| 100 | 62.948 | 0.021 |
| 110 | 75.996 | 0.021 |
| 120 | 91.972 | 0.015 |
| 130 | 105.711 | 0.016 |
| 140 | 122.775 | 0.007 |
| 150 | 131.669 | 0.014 |
| 160 | 150.697 | 0.019 |
| 170 | 179.300 | |

**(b)** The regression line is $Q = 0.0348391 - 0.000168222P$.



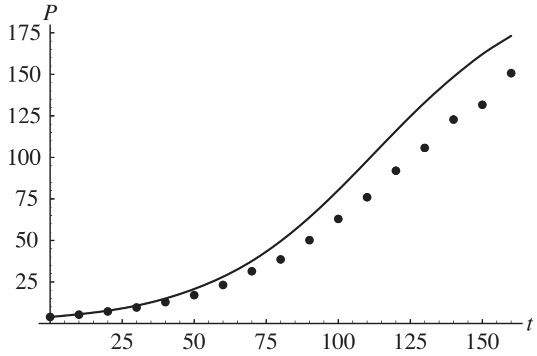
**(c)** The solution of the logistic equation is given in Equation (5) in the text. Identifying $a = 0.0348391$ and $b = 0.000168222$ we have

$$P(t) = \frac{aP_0}{bP_0 + (a - bP_0)e^{-at}}.$$

**(d)** With $P_0 = 3.929$ the solution becomes

$$P(t) = \frac{0.136883}{0.000660944 + 0.0341781e^{-0.0348391t}}.$$

**(e)**



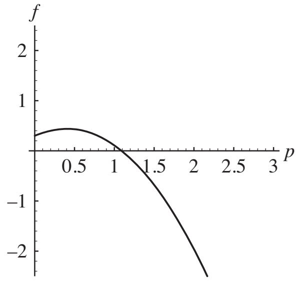
**(f)** We identify $t = 180$ with 1970, $t = 190$ with 1980, and $t = 200$ with 1990. The model predicts $P(180) = 188.661$, $P(190) = 193.735$, and $P(200) = 197.485$. The actual population figures for these years are 203.303, 226.542, and 248.765 millions. As $t \to \infty$, $P(t) \to a/b = 207.102$.

**24. (a)** Using a CAS to solve $P(1 - P) + 0.3e^{-P} = 0$ for $P$ we see that $P = 1.09216$ is an equilibrium solution.

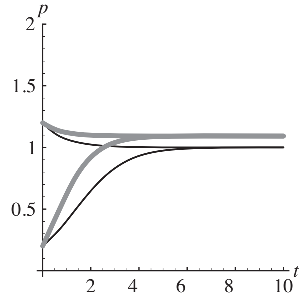
**(b)** Since $f(P) > 0$ for $0 < P < 1.09216$, the solution $P(t)$ of

$$dP/dt = P(1-P) + 0.3e^{-P}, \quad P(0) = P_0,$$

is increasing for $P_0 < 1.09216$. Since $f(P) < 0$ for $P > 1.09216$, the solution $P(t)$ is decreasing for $P_0 > 1.09216$. Thus $P = 1.09216$ is an attractor.



**(c)** The curves for the second initial-value problem are thicker. The equilibrium solution for the logic model is $P = 1$. Comparing 1.09216 and 1, we see that the percentage increase is 9.216



**25.** To find $t_d$ we solve

$$m\frac{dv}{dt} = mg - kv^2, \qquad v(0) = 0$$

using separation of variables. This gives

$$v(t) = \sqrt{\frac{mg}{k}}\tanh\sqrt{\frac{kg}{m}}\,t.$$

Integrating and using $s(0) = 0$ gives

$$s(t) = \frac{m}{k}\ln\left(\cosh\sqrt{\frac{kg}{m}}\,t\right).$$

To find the time of descent we solve $s(t) = 823.84$ and find $t_d = 7.77882$. The impact velocity is $v(t_d) = 182.998$, which is positive because the positive direction is downward.

**26.** **(a)** Solving $v_t = \sqrt{mg/k}$ for $k$ we obtain $k = mg/v_t^2$. The differential equation then becomes

$$m\frac{dv}{dt} = mg - \frac{mg}{v_t^2}v^2 \quad \text{or} \quad \frac{dv}{dt} = g\left(1 - \frac{1}{v_t^2}v^2\right).$$

Separating variables and integrating gives

$$v_t\tanh^{-1}\frac{v}{v_t} = gt + c_1.$$

The initial condition $v(0) = 0$ implies $c_1 = 0$, so

$$v(t) = v_t\tanh\frac{gt}{v_t}.$$

We find the distance by integrating:

$$s(t) = \int v_t \tanh \frac{gt}{v_t}\, dt = \frac{v_t^2}{g} \ln\left(\cosh \frac{gt}{v_t}\right) + c_2.$$

The initial condition $s(0) = 0$ implies $c_2 = 0$, so

$$s(t) = \frac{v_t^2}{g} \ln\left(\cosh \frac{gt}{v_t}\right).$$

In 25 seconds she has fallen 20,000 − 14,800 = 5,200 feet. Using a CAS to solve

$$5200 = (v_t^2/32) \ln\left(\cosh \frac{32(25)}{v_t}\right)$$

for $v_t$ gives $v_t \approx 271.711$ ft/s. Then

$$s(t) = \frac{v_t^2}{g} \ln\left(\cosh \frac{gt}{v_t}\right) = 2307.08 \ln(\cosh 0.117772t).$$

**(b)** At $t = 15$, $s(15) = 2{,}542.94$ ft and $v(15) = s'(15) = 256.287$ ft/sec.

**27.** While the object is in the air its velocity is modeled by the linear differential equation $m\,dv/dt = mg - kv$. Using $m = 160$, $k = \frac{1}{4}$, and $g = 32$, the differential equation becomes $dv/dt + (1/640)v = 32$. The integrating factor is $e^{\int dt/640} = e^{t/640}$ and the solution of the differential equation is $e^{t/640}v = \int 32e^{t/640}\,dt = 20{,}480e^{t/640} + c$. Using $v(0) = 0$ we see that $c = -20{,}480$ and $v(t) = 20{,}480 - 20{,}480e^{-t/640}$. Integrating we get $s(t) = 20{,}480t + 13{,}107{,}200e^{-t/640} + c$. Since $s(0) = 0$, $c = -13{,}107{,}200$ and $s(t) = -13{,}107{,}200 + 20{,}480t + 13{,}107{,}200e^{-t/640}$. To find when the object hits the liquid we solve $s(t) = 500 - 75 = 425$, obtaining $t_a = 5.16018$. The velocity at the time of impact with the liquid is $v_a = v(t_a) = 164.482$. When the object is in the liquid its velocity is modeled by the nonlinear differential equation $m\,dv/dt = mg - kv^2$. Using $m = 160$, $g = 32$, and $k = 0.1$ this becomes $dv/dt = (51{,}200 - v^2)/1600$. Separating variables and integrating we have

$$\frac{dv}{51{,}200 - v^2} = \frac{dt}{1600} \quad \text{and} \quad \frac{\sqrt{2}}{640} \ln\left|\frac{v - 160\sqrt{2}}{v + 160\sqrt{2}}\right| = \frac{1}{1600}t + c.$$

Solving $v(0) = v_a = 164.482$ we obtain $c = -0.00407537$. Then, for $v < 160\sqrt{2} = 226.274$,

$$\left|\frac{v - 160\sqrt{2}}{v + 160\sqrt{2}}\right| = e^{\sqrt{2}t/5 - 1.8443} \quad \text{or} \quad -\frac{v - 160\sqrt{2}}{v + 160\sqrt{2}} = e^{\sqrt{2}t/5 - 1.8443}.$$

Solving for $v$ we get

$$v(t) = \frac{13964.6 - 2208.29e^{\sqrt{2}t/5}}{61.7153 + 9.75937e^{\sqrt{2}t/5}}.$$

Integrating we find

$$s(t) = 226.275t - 1600 \ln(6.3237 + e^{\sqrt{2}t/5}) + c.$$

Solving $s(0) = 0$ we see that $c = 3185.78$, so

$$s(t) = 3185.78 + 226.275t - 1600 \ln\left(6.3237 + e^{\sqrt{2}t/5}\right).$$

To find when the object hits the bottom of the tank we solve $s(t) = 75$, obtaining $t_b = 0.466273$. The time from when the object is dropped from the helicopter to when it hits the bottom of the tank is $t_a + t_b = 5.62708$ seconds.

**28.** The velocity vector of the swimmer is

$$\mathbf{v} = \mathbf{v}_s + \mathbf{v}_r = (-v_s \cos\theta, -v_s \sin\theta) + (0, v_r) = (-v_s \cos\theta, -v_s \sin\theta + v_r) = \left(\frac{dx}{dt}, \frac{dy}{dt}\right).$$

Equating components gives

$$\frac{dx}{dt} = -v_s \cos\theta \qquad \text{and} \qquad \frac{dy}{dt} = v_s \sin\theta + v_r$$

so

$$\frac{dx}{dt} = -v_s \frac{x}{\sqrt{x^2 + y^2}} \qquad \text{and} \qquad \frac{dy}{dt} = -v_s \frac{y}{\sqrt{x^2 + y^2}} + v_r.$$

Thus,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-v_s y + v_r \sqrt{x^2 + y^2}}{-v_s x} = \frac{v_s y - v_r \sqrt{x^2 + y^2}}{v_s x}.$$

**29. (a)** With $k = v_r/v_s$,

$$\frac{dy}{dx} = \frac{y - k\sqrt{x^2 + y^2}}{x}$$

is a first-order homogeneous differential equation (see Section 2.5). Substituting $y = ux$ into the differential equation gives

$$u + x\frac{du}{dx} = u - k\sqrt{1 + u^2} \quad \text{or} \quad \frac{du}{dx} = -k\sqrt{1 + u^2}.$$

Separating variables and integrating we obtain

$$\int \frac{du}{\sqrt{1 + u^2}} = -\int k\, dx \quad \text{or} \quad \ln\left(u + \sqrt{1 + u^2}\right) = -k \ln x + \ln c.$$

This implies

$$\ln x^k \left(u + \sqrt{1 + u^2}\right) = \ln c \quad \text{or} \quad x^k \left(\frac{y}{x} + \frac{\sqrt{x^2 + y^2}}{x}\right) = c.$$

The condition $y(1) = 0$ gives $c = 1$ and so $y + \sqrt{x^2 + y^2} = x^{1-k}$. Solving for $y$ gives

$$y(x) = \frac{1}{2}\left(x^{1-k} - x^{1+k}\right).$$

**(b)** If $k = 1$, then $v_s = v_r$ and $y = \frac{1}{2}(1 - x^2)$. Since $y(0) = \frac{1}{2}$, the swimmer lands on the west beach at $(0, \frac{1}{2})$. That is, $\frac{1}{2}$ mile north of $(0, 0)$. If $k > 1$, then $v_r > v_s$ and $1 - k < 0$. This means $\lim_{x \to 0^+} y(x)$ becomes infinite, since $\lim_{x \to 0^+} x^{1-k}$ becomes infinite. The swimmer never makes it to the west beach and is swept northward with the current. If $0 < k < 1$, then $v_s > v_r$ and $1 - k > 0$. The value of $y(x)$ at $x = 0$ is $y(0) = 0$. The swimmer has made it to the point $(0, 0)$.

**30.** The velocity vector of the swimmer is

$$\mathbf{v} = \mathbf{v}_s + \mathbf{v}_r = (-v_s, 0) + (0, v_r) = \left(\frac{dx}{dt}, \frac{dy}{dt}\right).$$

Equating components gives

$$\frac{dx}{dt} = -v_s \quad \text{and} \quad \frac{dy}{dt} = v_r$$

so

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{v_r}{-v_s} = -\frac{v_r}{v_s}.$$

**31.** The differential equation

$$\frac{dy}{dx} = -\frac{30x(1-x)}{2}$$

separates into $dy = 15(-x + x^2)dx$. Integration gives $y(x) = -\frac{15}{2}x^2 + 5x^3 + c$. The condition $y(1) = 0$ gives $c = \frac{5}{2}$ and so $y(x) = \frac{1}{2}(-15x^2 + 10x^3 + 5)$. Since $y(0) = \frac{5}{2}$, the swimmer has to walk 2.5 miles back down the west beach to reach $(0, 0)$.

**32.** This problem has a great many components, so we will consider the case in which air resistance is assumed to be proportional to the velocity. By Problem 35 in Section 3.1 the differential equation is

$$m\frac{dv}{dt} = mg - kv,$$

and the solution is

$$v(t) = \frac{mg}{k} + \left(v_0 - \frac{mg}{k}\right)e^{-kt/m}.$$

If we take the initial velocity to be 0, then the velocity at time $t$ is

$$v(t) = \frac{mg}{k} - \frac{mg}{k}e^{-kt/m}.$$

The mass of the raindrop is about $m = 62 \times 0.000000155/32 \approx 0.0000003$ and $g = 32$, so the velocity at time $t$ is

$$v(t) = \frac{0.0000096}{k} - \frac{0.0000096}{k}e^{-3333333kt}$$

If we let $k = 0.0000007$, then $v(100) \approx 13.7$ ft/s. In this case 100 is the time in seconds. Since 7 mph $\approx 10.3$ ft/s, the assertion that the average velocity is 7 mph is not unreasonable. Of course, this assumes that the air resistance is proportional to the velocity, and, more importantly, that the constant of proportionality is 0.0000007. The assumption about the constant is particularly suspect.

**33.** **(a)** Letting $c = 0.6$, $A_h = \pi(\frac{1}{32}\cdot\frac{1}{12})^2$, $A_w = \pi\cdot 1^2 = \pi$, and $g = 32$, the differential equation in Problem 12 becomes $dh/dt = -0.00003255\sqrt{h}$. Separating variables and integrating, we get $2\sqrt{h} = -0.00003255t + c$, so $h = (c_1 - 0.00001628t)^2$. Setting $h(0) = 2$, we find $c = \sqrt{2}$, so $h(t) = (\sqrt{2} - 0.00001628t)^2$, where $h$ is measured in feet and $t$ in seconds.

**(b)** One hour is 3,600 seconds, so the hour mark should be placed at

$$h(3600) = [\sqrt{2} - 0.00001628(3600)]^2 \approx 1.838\,\text{ft} \approx 22.0525\,\text{in.}$$

up from the bottom of the tank. The remaining marks corresponding to the passage of 2, 3, 4, . . . , 12 hours are placed at the values shown in the table. The marks are not evenly spaced because the water is not draining out at a uniform rate; that is, $h(t)$ is not a linear function of time.

| **time (seconds )** | **height (inches )** |
|---|---|
| 0 | 24.0000 |
| 1 | 22.0520 |
| 2 | 20.1864 |
| 3 | 18.4033 |
| 4 | 16.7026 |
| 5 | 15.0844 |
| 6 | 13.5485 |
| 7 | 12.0952 |
| 8 | 10.7242 |
| 9 | 9.4357 |
| 10 | 8.2297 |
| 11 | 7.1060 |
| 12 | 6.0648 |

**34.** **(a)** In this case $A_w = \pi h^2/4$ and the differential equation is

$$\frac{dh}{dt} = -\frac{1}{7680}h^{-3/2}.$$

Separating variables and integrating, we have

$$h^{3/2}\,dh = -\frac{1}{7680}\,dt$$

$$\frac{2}{5}h^{5/2} = -\frac{1}{7680}t + c_1.$$

Setting $h(0) = 2$ we find $c_1 = 8\sqrt{2}/5$, so that

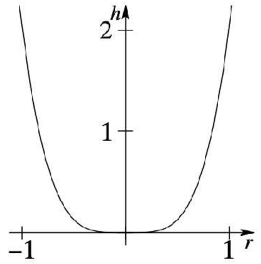
$$\frac{2}{5}h^{5/2} = -\frac{1}{7680}t + \frac{8\sqrt{2}}{5},$$

$$h^{5/2} = 4\sqrt{2} - \frac{1}{3072}t,$$

and

$$h = \left(4\sqrt{2} - \frac{1}{3072}t\right)^{2/5}.$$

**(b)** In this case $h(4\text{ hr}) = h(14{,}400\text{ s}) = 11.8515$ inches and $h(5\text{ hr}) = h(18{,}000\text{ s})$ is not a real number. Using a CAS to solve $h(t) = 0$, we see that the tank runs dry at $t \approx 17{,}378\text{ s} \approx 4.83$ hr. Thus, this particular conical water clock can only measure time intervals of less than 4.83 hours.



**35.** If we let $r_h$ denote the radius of the hole and $A_w = \pi[f(h)]^2$, then the differential equation $dh/dt = -k\sqrt{h}$, where $k = cA_h\sqrt{2g}/A_w$, becomes

$$\frac{dh}{dt} = -\frac{c\pi r_h^2\sqrt{2g}}{\pi[f(h)]^2}\sqrt{h} = -\frac{8cr_h^2\sqrt{h}}{[f(h)]^2}\,.$$

For the time marks to be equally spaced, the rate of change of the height must be a constant; that is, $dh/dt = -a$. (The constant is negative because the height is decreasing.) Thus

$$-a = -\frac{8cr_h^2\sqrt{h}}{[f(h)]^2}\,, \qquad [f(h)]^2 = \frac{8cr_h^2\sqrt{h}}{a}\,, \qquad \text{and} \qquad r = f(h) = 2r_h\sqrt{\frac{2c}{a}}\,h^{1/4}.$$

Solving for $h$, we have

$$h = \frac{a^2}{64c^2r_h^4}\,r^4.$$

The shape of the tank with $c = 0.6$, $a = 2$ ft/12 hr $= 1$ ft/21,600 s, and $r_h = 1/32(12) = 1/384$ is shown in the above figure.

## 3.3 Modeling with Systems of First-Order DEs

**1.** The linear equation $dx/dt = -\lambda_1 x$ can be solved by either separation of variables or by an integrating factor. Integrating both sides of $dx/x = -\lambda_1 dt$ we obtain $\ln|x| = -\lambda_1 t + c$ from which we get $x = c_1e^{-\lambda_1 t}$. Using $x(0) = x_0$ we find $c_1 = x_0$ so that $x = x_0e^{-\lambda_1 t}$. Substituting this result into the second differential equation we have

$$\frac{dy}{dt} + \lambda_2 y = \lambda_1 x_0 e^{-\lambda_1 t}$$

which is linear. An integrating factor is $e^{\lambda_2 t}$ so that

$$\frac{d}{dt}\left[e^{-\lambda_2 t}y\right] + \lambda_2 y = \lambda_1 x_0 e^{(\lambda_2-\lambda_1)t}$$

$$y = \frac{\lambda_1 x_0}{\lambda_2-\lambda_1}e^{(\lambda_2-\lambda_1)t}e^{-\lambda_2 t} + c_2e^{-\lambda_2 t} = \frac{\lambda_1 x_0}{\lambda_2-\lambda_1}e^{-\lambda_1 t} + c_2e^{-\lambda_2 t}.$$

Using $y(0) = 0$ we find $c_2 = -\lambda_1 x_0/(\lambda_2-\lambda_1)$. Thus

$$y = \frac{\lambda_1 x_0}{\lambda_2-\lambda_1}\left(e^{-\lambda_1 t} - e^{-\lambda_2 t}\right).$$

Substituting this result into the third differential equation we have

$$\frac{dz}{dt} = \frac{\lambda_1\lambda_2 x_0}{\lambda_2-\lambda_1}\left(e^{-\lambda_1 t} - e^{-\lambda_2 t}\right).$$

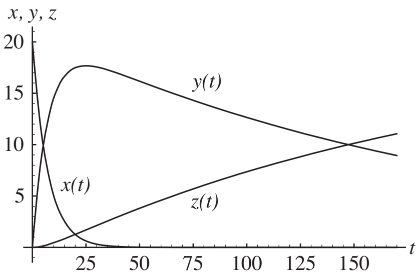
Integrating we find

$$z = -\frac{\lambda_2 x_0}{\lambda_2-\lambda_1}e^{-\lambda_1 t} + \frac{\lambda_1 x_0}{\lambda_2-\lambda_1}e^{-\lambda_2 t} + c_3.$$

Using $z(0) = 0$ we find $c_3 = x_0$. Thus

$$z = x_0 \left(1 - \frac{\lambda_2}{\lambda_2 - \lambda_1} e^{-\lambda_1 t} + \frac{\lambda_1}{\lambda_2 - \lambda_1} e^{-\lambda_2 t}\right).$$

**2.** We see from the graph that the half-life of $A$ is approximately 4.7 days. To determine the half-life of $B$ we use $t = 50$ as a base, since at this time the amount of substance $A$ is so small that it contributes very little to substance $B$. Now we see from the graph that $y(50) \approx 16.2$ and $y(191) \approx 8.1$. Thus, the half-life of $B$ is approximately 141 days.



**3.** The amounts $x$ and $y$ are the same at about $t = 5$ days. The amounts $x$ and $z$ are the same at about $t = 20$ days. The amounts $y$ and $z$ are the same at about $t = 147$ days. The time when $y$ and $z$ are the same makes sense because most of $A$ and half of $B$ are gone, so half of $C$ should have been formed.

**4.** Suppose that the series is described schematically by $W \Longrightarrow -\lambda_1 X \Longrightarrow -\lambda_2 Y \Longrightarrow -\lambda_3 Z$ where $-\lambda_1$, $-\lambda_2$, and $-\lambda_3$ are the decay constants for $W$, $X$ and $Y$, respectively, and $Z$ is a stable element. Let $w(t)$, $x(t)$, $y(t)$, and $z(t)$ denote the amounts of substances $W$, $X$, $Y$, and $Z$, respectively. A model for the radioactive series is

$$\frac{dw}{dt} = -\lambda_1 w$$

$$\frac{dx}{dt} = \lambda_1 w - \lambda_2 x$$

$$\frac{dy}{dt} = \lambda_2 x - \lambda_3 y$$

$$\frac{dz}{dt} = \lambda_3 y.$$

**5. (a)** Since the third equation in the system is linear and containts only the variable $K(t)$ we have

$$\frac{dK}{dt} = -\left(\lambda_1 + \lambda_2\right) P$$

$$K(t) = c_1 e^{-(\lambda_1 + \lambda_2)t}$$

Using $K(0) = K_0$ yields $K(t) = K_0 e^{-(\lambda_1 + \lambda_2)t}$.

We can now solve for $A(t)$ and $C(t)$:

$$\frac{dA}{dt} = \lambda_2 P(t) = \lambda_2 K_0 e^{-(\lambda_1 + \lambda_2)t}$$

$$A(t) = -\frac{\lambda_2}{\lambda_1 + \lambda_2} K_0 e^{-(\lambda_1 + \lambda_2)t} + c_2$$

Using $A(0)=0$ implies $c_2=\dfrac{\lambda_2}{\lambda_1+\lambda_2}K_0$. Therefore,

$$A(t)=\frac{\lambda_2}{\lambda_1+\lambda_2}K_0\left[1-e^{-(\lambda_1+\lambda_2)t}\right].$$

We use the same approach to solve for $C(t)$:

$$\frac{dC}{dt}=\lambda_1K(t)=\lambda_1K_0e^{-(\lambda_1+\lambda_2)t}$$
$$C(t)=-\frac{\lambda_1}{\lambda_1+\lambda_2}K_0e^{-(\lambda_1+\lambda_2)t}+c_3$$

Using $C(0)=0$ implies $c_3=\dfrac{\lambda_1}{\lambda_1+\lambda_2}K_0$. Therefore,

$$C(t)=\frac{\lambda_1}{\lambda_1+\lambda_2}K_0\left[1-e^{-(\lambda_1+\lambda_2)t}\right].$$

**(b)** It is known that $\lambda_1=4.7526\times10^{-10}$ and $\lambda_2=0.5874\times10^{-10}$ so

$$\begin{aligned}\lambda_1+\lambda_2&=5.34\times10^{-10}\\ K(t)&=K_0e^{-0.000000000534t}\\ K(t)&=\frac{1}{2}K_0\\ t&=\frac{\ln\frac{1}{2}}{-0.000000000534}\approx1.3\times10^9\text{ years}\end{aligned}$$

or the half-life of K-40 is about 1.3 billion years.

**(c)** Using the solutions $A(t)$, $C(t)$, and the values of $\lambda_1$ and $\lambda_2$ from part (b) we see that

$$\begin{aligned}\lim_{t\to\infty}A(t)&=\frac{\lambda_2}{\lambda_1+\lambda_2}\lim_{t\to\infty}\left[K_0\left(1-e^{-(\lambda_1+\lambda_2)t}\right)\right]\\ &=\frac{\lambda_2}{\lambda_1+\lambda_2}K_0=\frac{0.5874\times10^{-10}}{5.34\times10^{-10}}K_0=0.11K_0\text{ or }11\%\text{ of }K_0\\ \lim_{t\to\infty}C(t)&=\frac{\lambda_1}{\lambda_1+\lambda_2}\lim_{t\to\infty}\left[K_0\left(1-e^{-(\lambda_1+\lambda_2)t}\right)\right]\\ &=\frac{\lambda_1}{\lambda_1+\lambda_2}K_0=\frac{4.7526\times10^{-10}}{5.34\times10^{-10}}K_0=0.89K_0\text{ or }89\%\text{ of }K_0\end{aligned}$$

**6. (a)** From part (a) of Problem 5:

$$\frac{A(t)}{K(t)}=\frac{\frac{\lambda_2}{\lambda_1+\lambda_2}K_0\left(1-e^{-(\lambda_1+\lambda_2)t}\right)}{K_0e^{-(\lambda_1+\lambda_2)t}}=\frac{\lambda_2}{\lambda_1+\lambda_2}\left(e^{(\lambda_1+\lambda_2)t}-1\right)$$

**(b)** Solving for $t$ in part (a) we get

$$\begin{aligned}\left(\frac{\lambda_1+\lambda_2}{\lambda_2}\right)\frac{A(t)}{K(t)} &= e^{(\lambda_1+\lambda_2)t}-1\\ e^{(\lambda_1+\lambda_2)t} &= 1+\left(\frac{\lambda_1+\lambda_2}{\lambda_2}\right)\frac{A(t)}{K(t)}\\ t &= \frac{1}{\lambda_1+\lambda_2}\ln\left[1+\left(\frac{\lambda_1+\lambda_2}{\lambda_2}\right)\frac{A(t)}{K(t)}\right]\end{aligned}$$

**(c)** From part (b)

$$t=\frac{1}{5.34\times10^{-10}}\ln\left[1+\left(\frac{5.34\times10^{-10}}{0.5874\times10^{-10}}\right)\frac{8.5\times10^{-7}}{5.4\times10^{-6}}\right]\approx 1.66 \text{ billion years}$$

**7.** The system is

$$\begin{aligned}x_1' &= 2\cdot3+\frac{1}{50}x^2-\frac{1}{50}x_1\cdot4=-\frac{2}{25}x_1+\frac{1}{50}x_2+6\\ x_2' &= \frac{1}{50}x_1\cdot4-\frac{1}{50}x_2-\frac{1}{50}x_2\cdot3=\frac{2}{25}x_1-\frac{2}{25}x_2\,.\end{aligned}$$

**8.** Let $x_1$, $x_2$, and $x_3$ be the amounts of salt in tanks $A$, $B$, and $C$, respectively, so that

$$\begin{aligned}x_1' &= \frac{1}{100}x_2\cdot2-\frac{1}{100}x_1\cdot6=\frac{1}{50}x_2-\frac{3}{50}x_1\\ x_2' &= \frac{1}{100}x_1\cdot6+\frac{1}{100}x_3-\frac{1}{100}x_2\cdot2-\frac{1}{100}x_2\cdot5=\frac{3}{50}x_1-\frac{7}{100}x_2+\frac{1}{100}x_3\\ x_3' &= \frac{1}{100}x_2\cdot5-\frac{1}{100}x_3-\frac{1}{100}x_3\cdot4=\frac{1}{20}x_2-\frac{1}{20}x_3.\end{aligned}$$

**9. (a)** A model is

$$\begin{aligned}\frac{dx_1}{dt} &= 3\cdot\frac{x_2}{100-t}-2\cdot\frac{x_1}{100+t}, \qquad x_1(0)=100\\ \frac{dx_2}{dt} &= 2\cdot\frac{x_1}{100+t}-3\cdot\frac{x_2}{100-t}, \qquad x_2(0)=50.\end{aligned}$$

**(b)** Since the system is closed, no salt enters or leaves the system and $x_1(t)+x_2(t)=100+50=150$ for all time. Thus $x_1=150-x_2$ and the second equation in part (a) becomes

$$\frac{dx_2}{dt}=\frac{2(150-x_2)}{100+t}-\frac{3x_2}{100-t}=\frac{300}{100+t}-\frac{2x_2}{100+t}-\frac{3x_2}{100-t}$$

or

$$\frac{dx_2}{dt}+\left(\frac{2}{100+t}+\frac{3}{100-t}\right)x_2=\frac{300}{100+t}\,,$$

which is linear in $x_2$. An integrating factor is

$$e^{2\ln(100+t)-3\ln(100-t)}=(100+t)^2(100-t)^{-3}$$

so

$$\frac{d}{dt}[(100+t)^2(100-t)^{-3}x_2] = 300(100+t)(100-t)^{-3}.$$

Using integration by parts, we obtain

$$(100+t)^2(100-t)^{-3}x_2 = 300\left[\frac{1}{2}(100+t)(100-t)^{-2} - \frac{1}{2}(100-t)^{-1} + c\right].$$

Thus

$$\begin{aligned} x_2 &= \frac{300}{(100+t)^2}\left[c(100-t)^3 - \frac{1}{2}(100-t)^2 + \frac{1}{2}(100+t)(100-t)\right] \\ &= \frac{300}{(100+t)^2}[c(100-t)^3 + t(100-t)]. \end{aligned}$$

Using $x_2(0) = 50$ we find $c = 5/3000$. At $t = 30$, $x_2 = (300/130^2)(70^3c + 30 \cdot 70) \approx$ 47.4 lbs.

**10.** A model is

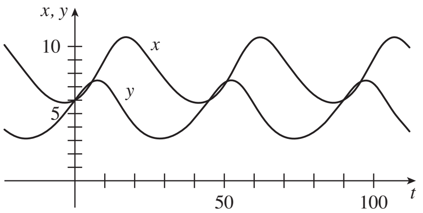
$$\begin{aligned} \frac{dx_1}{dt} &= (4 \text{ gal/min})(0 \text{ lb/gal}) - (4 \text{ gal/min})\left(\frac{1}{200}x_1 \text{ lb/gal}\right) \\ \frac{dx_2}{dt} &= (4 \text{ gal/min})\left(\frac{1}{200}x_1 \text{ lb/gal}\right) - (4 \text{ gal/min})\left(\frac{1}{150}x_2 \text{ lb/gal}\right) \\ \frac{dx_3}{dt} &= (4 \text{ gal/min})\left(\frac{1}{150}x_2 \text{ lb/gal}\right) - (4 \text{ gal/min})\left(\frac{1}{100}x_3 \text{ lb/gal}\right) \end{aligned}$$

or

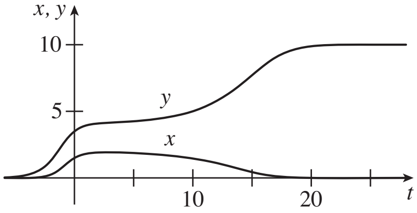
$$\begin{aligned} \frac{dx_1}{dt} &= -\frac{1}{50}x_1 \\ \frac{dx_2}{dt} &= \frac{1}{50}x_1 - \frac{2}{75}x_2 \\ \frac{dx_3}{dt} &= \frac{2}{75}x_2 - \frac{1}{25}x_3. \end{aligned}$$

Over a long period of time we would expect $x_1$, $x_2$, and $x_3$ to approach 0 because the entering pure water should flush the salt out of all three tanks.

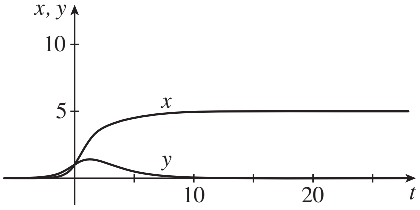
**11.** Zooming in on the graph it can be seen that the populations are first equal at about $t = 5.6$. The approximate periods of $x$ and $y$ are both 45.



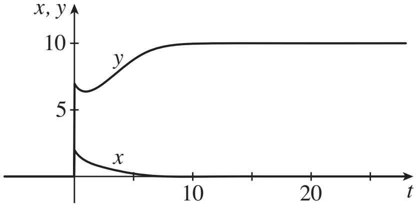
**12.** **(a)** The population $y(t)$ approaches 10,000, while the population $x(t)$ approaches extinction.



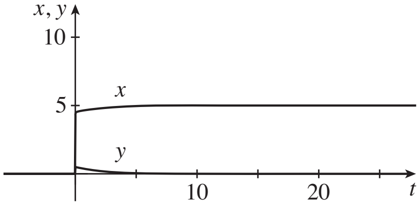
**(b)** The population $x(t)$ approaches 5,000, while the population $y(t)$ approaches extinction.



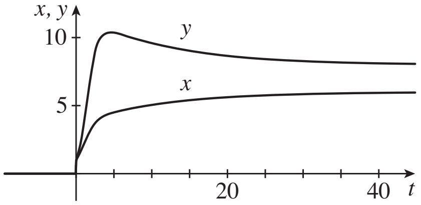
**(c)** The population $y(t)$ approaches 10,000, while the population $x(t)$ approaches extinction.



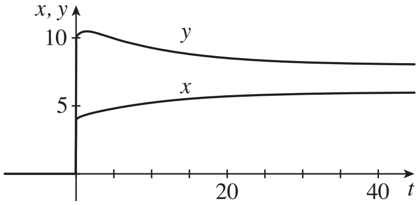
**(d)** The population $x(t)$ approaches 5,000, while the population $y(t)$ approaches extinction.



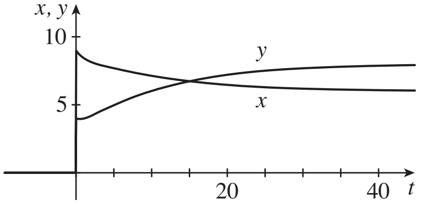
**13.** **(a)**



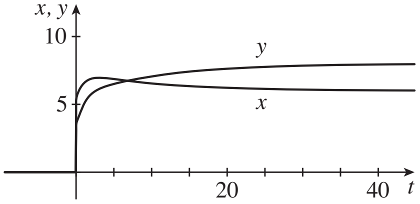
**(b)**



**(c)**



**(d)**



In each case the population $x(t)$ approaches 6,000, while the population $y(t)$ approaches 8,000.

**14.** By Kirchhoff's first law we have $i_1 = i_2 + i_3$. By Kirchhoff's second law, on each loop we have $E(t) = Li_1' + R_1 i_2$ and $E(t) = Li_1' + R_2 i_3 + q/C$ so that $q = CR_1 i_2 - CR_2 i_3$. Then

$i_3 = q' = CR_1 i_2' - CR_2 i_3$ so that the system is

$$\begin{aligned} Li_2' + Li_3' + R_1 i_2 &= E(t) \\ -R_1 i_2' + R_2 i_3' + \frac{1}{C} i_3 &= 0. \end{aligned}$$

**15.** By Kirchhoff's first law we have $i_1 = i_2 + i_3$. Applying Kirchhoff's second law to each loop we obtain

$$E(t) = i_1 R_1 + L_1 \frac{di_2}{dt} + i_2 R_2$$

and

$$E(t) = i_1 R_1 + L_2 \frac{di_3}{dt} + i_3 R_3.$$

Combining the three equations, we obtain the system

$$\begin{aligned} L_1 \frac{di_2}{dt} + (R_1 + R_2) i_2 + R_1 i_3 &= E \\ L_2 \frac{di_3}{dt} + R_1 i_2 + (R_1 + R_3) i_3 &= E. \end{aligned}$$

**16.** By Kirchhoff's first law we have $i_1 = i_2 + i_3$. By Kirchhoff's second law, on each loop we have $E(t) = Li_1' + Ri_2$ and $E(t) = Li_1' + q/C$ so that $q = CRi_2$. Then $i_3 = q' = CRi_2'$ so that system is

$$\begin{aligned} Li' + Ri_2 &= E(t) \\ CRi_2' + i_2 - i_1 &= 0. \end{aligned}$$

**17.** We first note that $s(t) + i(t) + r(t) = n$. Now the rate of change of the number of susceptible persons, $s(t)$, is proportional to the number of contacts between the number of people infected and the number who are susceptible; that is, $ds/dt = -k_1 si$. We use $-k_1 < 0$ because $s(t)$ is decreasing. Next, the rate of change of the number of persons who have recovered is proportional to the number infected; that is, $dr/dt = k_2 i$ where $k_2 > 0$ since $r$ is increasing. Finally, to obtain $di/dt$ we use

$$\frac{d}{dt}(s + i + r) = \frac{d}{dt} n = 0.$$

This gives

$$\frac{di}{dt} = -\frac{dr}{dt} - \frac{ds}{dt} = -k_2 i + k_1 si.$$

The system of differential equations is then

$$\begin{aligned} \frac{ds}{dt} &= -k_1 si \\ \frac{di}{dt} &= -k_2 i + k_1 si \\ \frac{dr}{dt} &= k_2 i. \end{aligned}$$

A reasonable set of initial conditions is $i(0) = i_0$, the number of infected people at time 0, $s(0) = n - i_0$, and $r(0) = 0$.

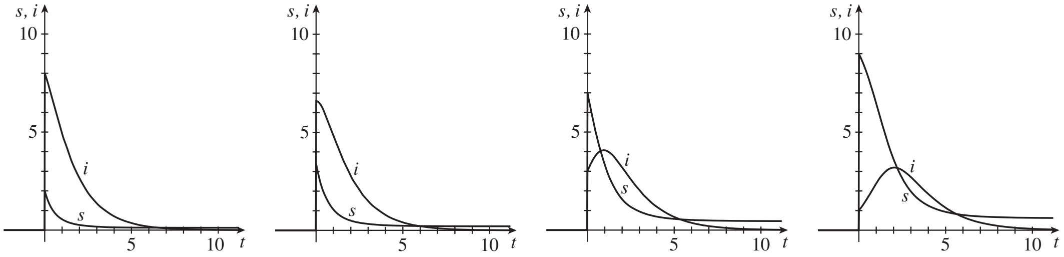
**18.** **(a)** If we know $s(t)$ and $i(t)$ then we can determine $r(t)$ from $s + i + r = n$.

**(b)** In this case the system is

$$\frac{ds}{dt} = -0.2si$$

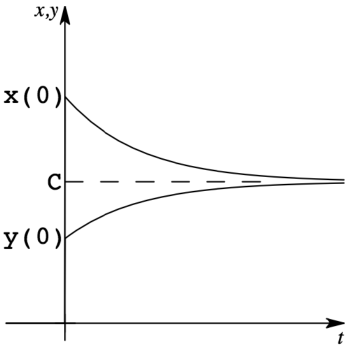
$$\frac{di}{dt} = -0.7i + 0.2si.$$

We also note that when $i(0) = i_0$, $s(0) = 10 - i_0$ since $r(0) = 0$ and $i(t) + s(t) + r(t) = 0$ for all values of $t$. Now $k_2/k_1 = 0.7/0.2 = 3.5$, so we cconsider initial conditions $s(0) = 2$, $i(0) = 8$; $s(0) = 3.4$, $i(0) = 6.6$; $s(0) = 7$, $i(0) = 3$; and $s(0) = 9$, $i(0) = 1$.



We see that an initial susceptible population greater than $k_2/k_1$ results in an epidemic in the sense that the number of infected persons increases to a maximum before decreasing to 0. On the other hand, when $s(0) < k_2/k_1$, the number of infected persons decreases from the start and there is no epidemic.

**19.** Since $x_0 > y_0 > 0$ we have $x(t) > y(t)$ and $y - x < 0$. Thus $dx/dt < 0$ and $dy/dt > 0$. We conclude that $x(t)$ is decreasing and $y(t)$ is increasing. As $t \to \infty$ we expect that $x(t) \to C$ and $y(t) \to C$, where $C$ is a constant common equilibrium concentration.



**20.** We write the system in the form

$$\frac{dx}{dt} = k_1(y - x)$$

$$\frac{dy}{dt} = k_2(x - y),$$

where $k_1 = \kappa/V_A$ and $k_2 = \kappa/V_B$. Letting $z(t) = x(t) - y(t)$ we have

$$\frac{dx}{dt} - \frac{dy}{dt} = k_1(y - x) - k_2(x - y)$$

$$\frac{dz}{dt} = k_1(-z) - k_2 z$$

$$\frac{dz}{dt} + (k_1 + k_2)z = 0.$$

This is a linear first-order differential equation with solution $z(t) = c_1 e^{-(k_1+k_2)t}$. Now

$$\frac{dx}{dt} = -k_1(y - x) = -k_1 z = -k_1 c_1 e^{-(k_1+k_2)t}$$

and

$$x(t) = c_1 \frac{k_1}{k_1 + k_2} e^{-(k_1+k_2)t} + c_2.$$

Since $y(t) = x(t) - z(t)$ we have

$$y(t) = -c_1 \frac{k_2}{k_1 + k_2} e^{-(k_1+k_2)t} + c_2.$$

The initial conditions $x(0) = x_0$ and $y(0) = y_0$ imply

$$c_1 = x_0 - y_0 \qquad \text{and} \qquad c_2 = \frac{x_0 k_2 + y_0 k_1}{k_1 + k_2}\,.$$

The solution of the system is

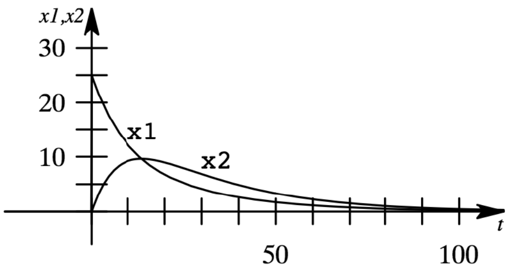
$$\begin{aligned} x(t) &= \frac{(x_0 - y_0)k_1}{k_1 + k_2} e^{-(k_1+k_2)t} + \frac{x_0 k_2 + y_0 k_1}{k_1 + k_2} \\ y(t) &= \frac{(y_0 - x_0)k_2}{k_1 + k_2} e^{-(k_1+k_2)t} + \frac{x_0 k_2 + y_0 k_1}{k_1 + k_2}\,. \end{aligned}$$

As $t \to \infty$, $x(t)$ and $y(t)$ approach the common limit

$$\begin{aligned} \frac{x_0 k_2 + y_0 k_1}{k_1 + k_2} = \frac{x_0 \kappa/V_B + y_0 \kappa/V_A}{\kappa/V_A + \kappa/V_B} &= \frac{x_0 V_A + y_0 V_B}{V_A + V_B} \\ &= x_0 \frac{V_A}{V_A + V_B} + y_0 \frac{V_B}{V_A + V_B}\,. \end{aligned}$$

This makes intuitive sense because the limiting concentration is seen to be a weighted average of the two initial concentrations.

**21.** Since there are initially 25 pounds of salt in tank $A$ and none in tank $B$, and since furthermore only pure water is being pumped into tank $A$, we would expect that $x_1(t)$ would steadily decrease over time. On the other hand, since salt is being added to tank $B$ from tank $A$, we would expect $x_2(t)$ to increase over time. However, since pure water is being added to the system at a constant



rate and a mixed solution is being pumped out of the system, it makes sense that the amount of salt in both tanks would approach 0 over time.

**22.** We assume here that the temperature, $T(t)$, of the metal bar does not affect the temperature, $T_A(t)$, of the medium in container $A$. By Newton's law of cooling, then, the differential equations for $T_A(t)$ and $T(t)$ are

$$\frac{dT_A}{dt} = k_A(T_A - T_B), \quad k_A < 0$$
$$\frac{dT}{dt} = k(T - T_A), \quad k < 0,$$

subject to the initial conditions $T(0) = T_0$ and $T_A(0) = T_1$. Separating variables in the first equation, we find $T_A(t) = T_B + c_1 e^{k_A t}$. Using $T_A(0) = T_1$ we find $c_1 = T_1 - T_B$, so

$$T_A(t) = T_B + (T_1 - T_B)e^{k_A t}.$$

Substituting into the second differential equation, we have

$$\frac{dT}{dt} = k(T - T_A) = kT - kT_A = kT - k[T_B + (T_1 - T_B)e^{k_A t}]$$
$$\frac{dT}{dt} - kT = -kT_B - k(T_1 - T_B)e^{k_A t}.$$

This is a linear differential equation with integrating factor $e^{\int -k\,dt} = e^{-kt}$. Then

$$\frac{d}{dt}[e^{-kt}T] = -kT_B e^{-kt} - k(T_1 - T_B)e^{(k_A - k)t}$$
$$e^{-kt}T = T_B e^{-kt} - \frac{k}{k_A - k}(T_1 - T_B)e^{(k_A - k)t} + c_2$$
$$T = T_B - \frac{k}{k_A - k}(T_1 - T_B)e^{k_A t} + c_2 e^{kt}.$$

Using $T(0) = T_0$ we find $c_2 = T_0 - T_B + \dfrac{k}{k_A - k}(T_1 - T_B)$, so

$$T(t) = T_B - \frac{k}{k_A - k}(T_1 - T_B)e^{k_A t} + \left[T_0 - T_B + \frac{k}{k_A - k}(T_1 - T_B)\right] e^{kt}.$$

## Chapter 3 in Review

**1.** The differential equation is $dP/dt = 0.15P$.

**2.** True. From $dA/dt = kA$, $A(0) = A_0$, we have $A(t) = A_0 e^{kt}$ and $A'(t) = kA_0 e^{kt}$, so $A'(0) = kA_0$. At $T = -(\ln 2)k$,

$$A'(-(\ln 2)/k) = kA(-(\ln 2)/k) = kA_0 e^{k[-(\ln 2)/k]} = kA_0 e^{-\ln 2} = \frac{1}{2}kA_0.$$

**3.** From $\dfrac{dP}{dt} = 0.018P$ and $P(0) = 4$ billion we obtain $P = 4e^{0.018t}$ so that $P(45) = 8.99$ billion.

**4.** Let $A = A(t)$ be the volume of $CO_2$ at time $t$. From $dA/dt = 1.2 - A/4$ and $A(0) = 16\,\text{ft}^3$ we obtain $A = 4.8 + 11.2e^{-t/4}$. Since $A(10) = 5.7\,\text{ft}^3$, the concentration is 0.071%. As $t \to \infty$ we have $A \to 4.8\,\text{ft}^3$ or 0.06%.

**5.** The starting point is $A(t) = A_0e^{-0.00012097t}$. With $A(t) = 0.53A_0$ we have

$$-0.00012097t = \ln 0.53 \quad \text{or} \quad t = \frac{\ln 0.53}{-0.00012097} \approx 5248 \text{ years}.$$

This represents the iceman's age in 1991, so the approximate date of his death would be

$$1991 - 5248 = -3257 \qquad \text{or} \qquad 3257 \text{ BC}.$$

**6. (a)** We assume that the rate of disintegration of Iodine-131 is proportional to the amount remaining. If $A(t)$ is the amount of Iodine-131 remaining at time $t$ then

$$\frac{dA}{dt} = kA, \quad A(0) = A_0,$$

where $k$ is the constant of proportionality and $A_0$ is the initial amount. The solution of the initial-value problem is $A(t) = A_0e^{kt}$. After one day the amount of Iodine-131 is

$$(1 - 0.083)A_0 = 0.917A_0 \qquad \text{so} \qquad A(1) = A_0e^k = 0.917A_0$$

and $e^k = 0.917$. After eight days

$$A(8) = A_0e^{8k} = A_0(e^k)^8 = A_0(0.917)^8 \approx 0.49998A_0.$$

**(b)** Since $A(8) \approx \frac{1}{2}A_0$, the half-life of Iodine-131 is approximately 8 days.

**7.** Separating variables, we have

$$\frac{\sqrt{a^2 - y^2}}{y}\,dy = -dx.$$

Substituting $y = s\sin\theta$, this becomes

$$\begin{aligned}
\frac{\sqrt{a^2 - a^2\sin^2\theta}}{a\sin\theta}(a\cos\theta)\,d\theta &= -dx \\
a\int \frac{\cos^2\theta}{\sin\theta}\,d\theta &= -\int dx \\
a\int \frac{1 - \sin^2\theta}{\sin\theta}\,d\theta &= -x + c \\
a\int (\csc\theta - \sin\theta)\,d\theta &= -x + c \\
a\ln|\csc\theta - \cot\theta| + a\cos\theta &= -x + c \\
a\ln\left|\frac{a}{y} - \frac{\sqrt{a^2 - y^2}}{y}\right| + a\,\frac{\sqrt{a^2 - y^2}}{a} &= -x + c.
\end{aligned}$$

Letting $s = 10$, this is

$$10\ln\left|\frac{10}{y} - \frac{\sqrt{100-y^2}}{y}\right| + \sqrt{100-y^2} = -x + c.$$

Letting $x = 0$ and $y = 10$ we determine that $c = 0$, so the solution is

$$10\ln\left|\frac{10}{y} - \frac{\sqrt{100-y^2}}{y}\right| + \sqrt{100-y^2} = -x.$$

**8.** From $V\,dC/dt = kA(C_s - C)$ and $C(0) = C_0$ we obtain $C = C_s + (C_0 - C_s)e^{-kAt/V}$.

**9. (a)** The differential equation

$$\begin{aligned}\frac{dT}{dt} &= k(T - T_m) = k[T - T_2 - B(T_1 - T)]\\ &= k[(1+B)T - (BT_1 + T_2)] = k(1+B)\left(T - \frac{BT_1 + T_2}{1+B}\right)\end{aligned}$$

is autonomous and has the single critical point $(BT_1 + T_2)/(1+B)$. Since $k < 0$ and $B > 0$, by phase-line analysis it is found that the critical point is an attractor and

$$\lim_{t\to\infty} T(t) = \frac{BT_1 + T_2}{1+B}.$$

Moreover,

$$\lim_{t\to\infty} T_m(t) = \lim_{t\to\infty}[T_2 + B(T_1 - T)] = T_2 + B\left(T_1 - \frac{BT_1 + T_2}{1+B}\right) = \frac{BT_1 + T_2}{1+B}.$$

**(b)** The differential equation is

$$\frac{dT}{dt} = k(T - T_m) = k(T - T_2 - BT_1 + BT)$$

or

$$\frac{dT}{dt} - k(1+B)T = -k(BT_1 + T_2).$$

This is linear and has integrating factor $e^{-\int k(1+B)dt} = e^{-k(1+B)t}$. Thus,

$$\begin{aligned}\frac{d}{dt}[e^{-k(1+B)t}T] &= -k(BT_1 + T_2)e^{-k(1+B)t}\\ e^{-k(1+B)t}T &= \frac{BT_1 + T_2}{1+B}e^{-k(1+B)t} + c\\ T(t) &= \frac{BT_1 + T_2}{1+B} + ce^{k(1+B)t}.\end{aligned}$$

Since $T(0) = T_1$ we find, $T(t) = \dfrac{BT_1 + T_2}{1+B} + \dfrac{T_1 - T_2}{1+B}e^{k(1+B)t}$.

**(c)** The temperature $T(t)$ decreases to $(BT_1+T_2)/(1+B)$, whereas $T_m(t)$ increases to $(BT_1+T_2)/(1+B)$ as $t \to \infty$. Thus, the temperature $(BT_1+T_2)/(1+B)$, (which is a weighted average,

$$\frac{B}{1+B}T_1 + \frac{1}{1+B}T_2,$$

of the two initial temperatures), can be interpreted as an equilibrium temperature. The body cannot get cooler than this value whereas the medium cannot get hotter than this value.

**10. (a)** By separation of variables and partial fractions,

$$\ln\left|\frac{T-T_m}{T+T_m}\right| - 2\tan^{-1}\left(\frac{T}{T_m}\right) = 4T_m^3kt + c.$$

Then rewrite the right-hand side of the differential equation as

$$\begin{aligned}\frac{dT}{dt} &= k(T^4 - T_m^4) = [(T_m + (T-T_m))^4 - T_m^4]\\ &= kT_m^4\left[\left(1 + \frac{T-T_m}{T_m}\right)^4 - 1\right]\\ &= kT_m^4\left[\left(1 + 4\,\frac{T-T_m}{T_m} + 6\left(\frac{T-T_m}{T_m}\right)^2 + \cdots\right) - 1\right] \leftarrow \text{binomial expansion}\end{aligned}$$

**(b)** When $T - T_m$ is small compared to $T_m$, every term in the expansion after the first two can be ignored, giving

$$\frac{dT}{dt} \approx k_1(T - T_m), \quad \text{where} \quad k_1 = 4kT_m^3.$$

**11.** Separating variables, we obtain

$$\begin{aligned}\frac{dq}{E_0 - q/C} &= \frac{dt}{k_1 + k_2t}\\ -C\ln\left|E_0 - \frac{q}{C}\right| &= \frac{1}{k_2}\ln|k_1 + k_2t| + c_1\\ \frac{(E_0 - q/C)^{-C}}{(k_1+k_2t)^{1/k_2}} &= c_2.\end{aligned}$$

Setting $q(0) = q_0$ we find $c_2 = (E_0 - q_0/C)^{-C}/k_1^{1/k_2}$, so

$$\frac{(E_0 - q/C)^{-C}}{(k_1 + k_2 t)^{1/k_2}} = \frac{(E_0 - q_0/C)^{-C}}{k_1^{1/k_2}}$$

$$\left(E_0 - \frac{q}{C}\right)^{-C} = \left(E_0 - \frac{q_0}{C}\right)^{-C}\left(\frac{k_1}{k_1 + k_2 t}\right)^{-1/k_2}$$

$$E_0 - \frac{q}{C} = \left(E_0 - \frac{q_0}{C}\right)\left(\frac{k_1}{k_1 + k_2 t}\right)^{1/Ck_2}$$

$$q = E_0 C + (q_0 - E_0 C)\left(\frac{k_1}{k_1 + k_2 t}\right)^{1/Ck_2}.$$

**12.** From $y\left[1 + (y')^2\right] = k$ we obtain $dx = (\sqrt{y}/\sqrt{k - y})dy$. If $y = k\sin^2\theta$ then

$$dy = 2k\sin\theta\cos\theta\,d\theta, \quad dx = 2k\left(\frac{1}{2} - \frac{1}{2}\cos 2\theta\right)d\theta, \quad \text{and} \quad x = k\theta - \frac{k}{2}\sin 2\theta + c.$$

If $x = 0$ when $\theta = 0$ then $c = 0$.

**13.** From $dx/dt = k_1 x(\alpha - x)$ we obtain

$$\left(\frac{1/\alpha}{x} + \frac{1/\alpha}{\alpha - x}\right)dx = k_1\,dt$$

so that $x = \alpha c_1 e^{\alpha k_1 t}/(1 + c_1 e^{\alpha k_1 t})$. From $dy/dt = k_2 xy$ we obtain

$$\ln|y| = \frac{k_2}{k_1}\ln\left|1 + c_1 e^{\alpha k_1 t}\right| + c \quad \text{or} \quad y = c_2\left(1 + c_1 e^{\alpha k_1 t}\right)^{k_2/k_1}.$$

**14.** In tank $A$ the salt input is

$$\left(7\,\frac{\text{gal}}{\text{min}}\right)\left(2\,\frac{\text{lb}}{\text{gal}}\right) + \left(1\,\frac{\text{gal}}{\text{min}}\right)\left(\frac{x_2}{100}\,\frac{\text{lb}}{\text{gal}}\right) = \left(14 + \frac{1}{100}x_2\right)\frac{\text{lb}}{\text{min}}.$$

The salt output is

$$\left(3\,\frac{\text{gal}}{\text{min}}\right)\left(\frac{x_1}{100}\,\frac{\text{lb}}{\text{gal}}\right) + \left(5\,\frac{\text{gal}}{\text{min}}\right)\left(\frac{x_1}{100}\,\frac{\text{lb}}{\text{gal}}\right) = \frac{2}{25}\,x_1\,\frac{\text{lb}}{\text{min}}.$$

In tank $B$ the salt input is

$$\left(5\,\frac{\text{gal}}{\text{min}}\right)\left(\frac{x_1}{100}\,\frac{\text{lb}}{\text{gal}}\right) = \frac{1}{20}\,x_1\,\frac{\text{lb}}{\text{min}}.$$

The salt output is

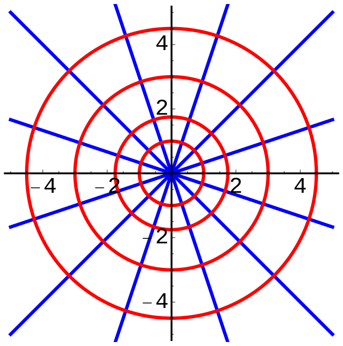
$$\left(1\,\frac{\text{gal}}{\text{min}}\right)\left(\frac{x_2}{100}\,\frac{\text{lb}}{\text{gal}}\right) + \left(4\,\frac{\text{gal}}{\text{min}}\right)\left(\frac{x_2}{100}\,\frac{\text{lb}}{\text{gal}}\right) = \frac{1}{20}\,x_2\,\frac{\text{lb}}{\text{min}}.$$

The system of differential equations is then

$$\frac{dx_1}{dt} = 14 + \frac{1}{100}x_2 - \frac{2}{25}x_1$$

$$\frac{dx_2}{dt} = \frac{1}{20}x_1 - \frac{1}{20}x_2.$$

**15.**



$$y = c_1 x$$

$$\frac{dy}{dx} = c_1$$

$$\frac{dy}{dx} = \frac{y}{x}$$

Therefore the differential equation of the orthogonal family is

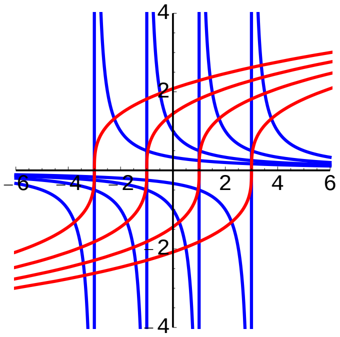
$$\frac{dy}{dx} = -\frac{x}{y}$$

$$y\,dy + x\,dx = 0$$

$$x^2 + y^2 = c_2$$

which is a family of circles ($c_2 > 0$) centered at the origin.

**16.**



$$x^2 - 2y^2 = c_1$$

$$2x - 4y\,\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{x}{2y}$$

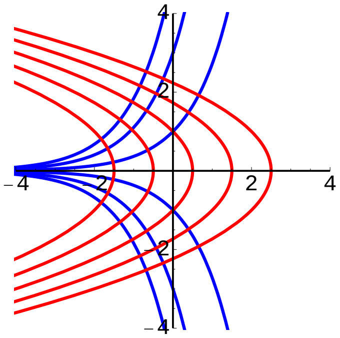
Therefore the differential equation of the orthogonal family is

$$\frac{dy}{dx} = -\frac{2y}{x}$$

$$\frac{1}{y}\,dy = -\frac{2}{x}\,dx$$

$$y = \frac{c_2}{x^2}$$

**17.** From $y = c_1 e^x$ we obtain $y' = y$ so that the differential equation of the orthogonal family is



$$\frac{dy}{dx} = -\frac{1}{y}.$$

Separating variables and integrating we get

$$y\,dy = -\,dx$$

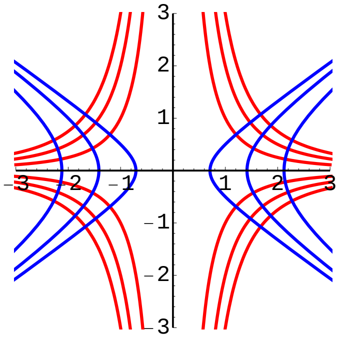
$$\frac{1}{2}y^2 = -x + c$$

$$y^2 + 2x = c_2$$

**18.** Differentiating the family of curves, we have

$$y' = -\frac{1}{(x+c_1)^2} = -y^2\,.$$

The differential equation for the family of orthogonal trajectories is then $y' = \dfrac{1}{y^2}$. Separating variables and integrating we get

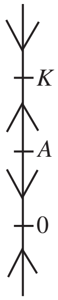


$$\begin{aligned} y^2\,dy &= dx \\ \frac{1}{3}y^3 &= x + c_2 \\ y^3 &= 3x + c_3\,. \end{aligned}$$

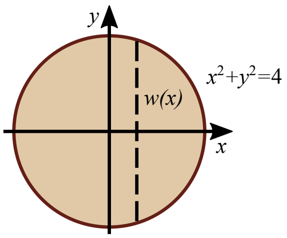
**19.** Critical points of the equation

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)\left(\frac{P}{A} - 1\right) \qquad r > 0,$$

are 0, $A$, and $K$. Here $A$ is called the **Allee threshold** and satisfies $0 < A < K$. From the accompanying phase portrait we see that $K$ and 0 are attractors, or asymptotically stable, but $A$ is a repeller, or unstable. Thus, for an initial value $P_0 < A$ the population decreases over time, that is, $P \to 0$ as $t \to \infty$.



**20.** **(a)** From the cross section on the right we see in this case that $w(x) = 2\sqrt{4-x^2}$ and the initial-value problem is then



$$2\sqrt{4-x^2}\,\frac{dx}{dt} = 1, \qquad x(0) = -2.$$

Solving the differential equation by separation of variables gives $x\sqrt{4-x^2} + 4\sin^{-1}\left(\frac{1}{2}x\right) = t + c$. Using $x(0) = -2$ we have

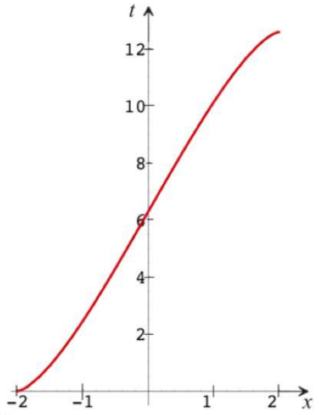
$$c = 4\sin^{-1}(-1) = 4\left(-\frac{\pi}{2}\right) = -2\pi\,.$$

Therefore an implicit solution is $x\sqrt{4-x^2} + 4\sin^{-1}\left(\frac{1}{2}x\right) = t - 2\pi$.

**(b)** The graph of $t(x) = 2\pi + x\sqrt{4-x^2} + 4\sin^{-1}\left(\frac{1}{2}x\right)$ on the $x$-interval $[-2, 2]$ is given on the right. From the graph we see that the time corresponding to $x = 2$ is approximately $t = 12.5$. The exact time to cut throug the piece of wood is

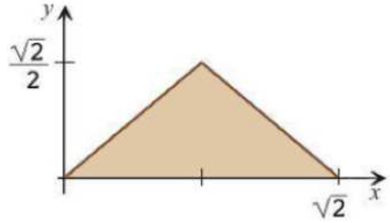
$$t(2) = 2\pi + 2\sqrt{4-2^2} + 4\sin^{-1}(1) = 2\pi + 4\left(\frac{\pi}{2}\right)$$

or $t = 4\pi$. In other words, the solution $x(t)$ defined implicitely by the equation in part (a) is defined on the $t$-interval $[0, 4\pi]$.



**21.** The piecewise-defined function $w(x)$ is now

$$w(x) = \begin{cases} x, & 0 \le x \le \dfrac{\sqrt{2}}{2} \\[2ex] \sqrt{2} - x, & \dfrac{\sqrt{}}{2} < x \le \sqrt{2} \end{cases}$$



First, we solve

$$x\,\frac{dx}{dt} = 1, \qquad x(0) = 0$$

by separation of variables. This yield $x(t) = \sqrt{2t}$. The time interval corresponding to $0 \le x \le \frac{\sqrt{2}}{2}$ is defined by $0 \le t \le \frac{1}{4}$. Second, we solve

$$\left(\sqrt{2} - x\right)\frac{dx}{dt} = 1, \qquad x\left(\frac{1}{4}\right) = \frac{\sqrt{2}}{2}.$$

This gives $x^2 - 2\sqrt{2}\,x + 2t + 1 = 0$. Using the quadratic formula, we have $x(t) = \sqrt{2} - \sqrt{1-2t}$. The time interval corresponding to $\frac{\sqrt{2}}{2} < x \le \sqrt{2}$ is defined by $\frac{1}{4} \le t \le \frac{1}{2}$. Thus,

$$x(t) = \begin{cases} \sqrt{2t}, & 0 \le t \le \frac{1}{4} \\ \sqrt{2} - \sqrt{1-2t}, & \frac{1}{4} < t \le \frac{1}{2}\,. \end{cases}$$

The time that it takes the saw to cut through the piece of wood is then $t = \frac{1}{2}$.

**22.** **(a)** By separation of variables

$$\begin{aligned} \frac{dA}{dt} &= -kA^2 \\ A^{-2}\,dA &= -k\,dt \\ -A^{-1} &= -kt + c_1 \\ A^{-1} &= c_2 + kt \\ A(t) &= \frac{1}{c_2 + kt} \end{aligned}$$

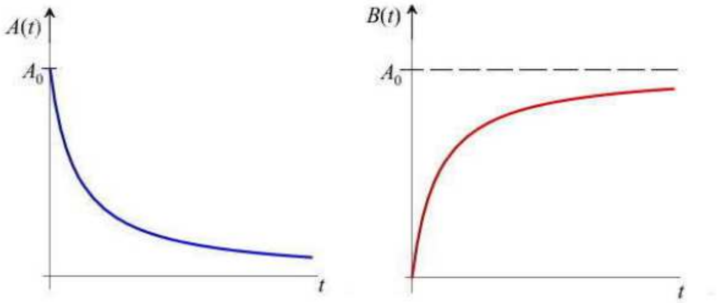
Then $A(0) = \dfrac{1}{c_2}$ and thus $c_2 = \dfrac{1}{A_0}$. Therefore $A(t) = \dfrac{A_0}{1 + A_0 kt}$.

**(b)** $A(t) + B(t) = A_0$

$$B(t) = A_0 - A(t) = A_0 - \frac{A_0}{1 + A_0 kt}$$

$$B(t) = \frac{kA_0^2 t}{1 + A_0 kt}\,.$$

**(c)**



# Chapter 4

# Higher-Order Differential Equations

## 4.1 Preliminary Theory - Linear Equations

**1.** From $y = c_1e^x + c_2e^{-x}$ we find $y' = c_1e^x - c_2e^{-x}$. Then $y(0) = c_1 + c_2 = 0$, $y'(0) = c_1 - c_2 = 1$ so that $c_1 = \frac{1}{2}$ and $c_2 = -\frac{1}{2}$. The solution is $y = \frac{1}{2}e^x - \frac{1}{2}e^{-x}$.

**2.** From $y = c_1e^{4x} + c_2e^{-x}$ we find $y' = 4c_1e^{4x} - c_2e^{-x}$. Then $y(0) = c_1 + c_2 = 1$, $y'(0) = 4c_1 - c_2 = 2$ so that $c_1 = \frac{3}{5}$ and $c_2 = \frac{2}{5}$. The solution is $y = \frac{3}{5}e^{4x} + \frac{2}{5}e^{-x}$.

**3.** From $y = c_1x + c_2x\ln x$ we find $y' = c_1 + c_2(1 + \ln x)$. Then $y(1) = c_1 = 3$, $y'(1) = c_1 + c_2 = -1$ so that $c_1 = 3$ and $c_2 = -4$. The solution is $y = 3x - 4x\ln x$.

**4.** From $y = c_1 + c_2\cos x + c_3\sin x$ we find $y' = -c_2\sin x + c_3\cos x$ and $y'' = -c_2\cos x - c_3\sin x$. Then $y(\pi) = c_1 - c_2 = 0$, $y'(\pi) = -c_3 = 2$, $y''(\pi) = c_2 = -1$ so that $c_1 = -1$, $c_2 = -1$, and $c_3 = -2$. The solution is $y = -1 - \cos x - 2\sin x$.

**5.** From $y = c_1 + c_2x^2$ we find $y' = 2c_2x$. Then $y(0) = c_1 = 0$, $y'(0) = 2c_2 \cdot 0 = 0$ and hence $y'(0) = 1$ is not possible. Since $a_2(x) = x$ is 0 at $x = 0$, Theorem 4.1.1 is not violated.

**6.** In this case we have $y(0) = c_1 = 0$, $y'(0) = 2c_2 \cdot 0 = 0$ so $c_1 = 0$ and $c_2$ is arbitrary. Two solutions are $y = x^2$ and $y = 2x^2$.

**7.** From $x(0) = x_0 = c_1$ we see that $x(t) = x_0\cos\omega t + c_2\sin\omega t$ and $x'(t) = -x_0\sin\omega t + c_2\omega\cos\omega t$. Then $x'(0) = x_1 = c_2\omega$ implies $c_2 = x_1/\omega$. Thus

$$x(t) = x_0\cos\omega t + \frac{x_1}{\omega}\sin\omega t.$$

**8.** Solving the system

$$x(t_0) = c_1\cos\omega t_0 + c_2\sin\omega t_0 = x_0$$

$$x'(t_0) = -c_1\omega\sin\omega t_0 + c_2\omega\cos\omega t_0 = x_1$$

for $c_1$ and $c_2$ gives

$$c_1 = \frac{\omega x_0\cos\omega t_0 - x_1\sin\omega t_0}{\omega} \quad \text{and} \quad c_2 = \frac{x_1\cos\omega t_0 + \omega x_0\sin\omega t_0}{\omega}.$$

Thus

$$\begin{aligned}
x(t) &= \frac{\omega x_0 \cos \omega t_0 - x_1 \sin \omega t_0}{\omega} \cos \omega t + \frac{x_1 \cos \omega t_0 + \omega x_0 \sin \omega t_0}{\omega} \sin \omega t \\
&= x_0 (\cos \omega t \cos \omega t_0 + \sin \omega t \sin \omega t_0) + \frac{x_1}{\omega} (\sin \omega t \cos \omega t_0 - \cos \omega t \sin \omega t_0) \\
&= x_0 \cos \omega (t - t_0) + \frac{x_1}{\omega} \sin \omega (t - t_0).
\end{aligned}$$

**9.** Since $a_2(x) = x - 2$ and $x_0 = 0$ the problem has a unique solution for $-\infty < x < 2$.

**10.** Since $a_0(x) = \tan x$ and $x_0 = 0$ the problem has a unique solution for $-\pi/2 < x < \pi/2$.

**11.** **(a)** We have $y(0) = c_1 + c_2 = 0$, $y''(1) = c_1 e + c_2 e^{-1} = 1$ so that $c_1 = e/\left(e^2 - 1\right)$ and $c_2 = -e/\left(e^2 - 1\right)$. The solution is $y = e\left(e^x - e^{-x}\right)/\left(e^2 - 1\right)$.

**(b)** We have $y(0) = c_3 \cosh 0 + c_4 \sinh 0 = c_3 = 0$ and $y(1) = c_3 \cosh 1 + c_4 \sinh 1 = c_4 \sinh 1 = 1$, so $c_3 = 0$ and $c_4 = 1/\sinh 1$. The solution is $y = (\sinh x)/(\sinh 1)$.

**(c)** Starting with the solution in part (b) we have

$$y = \frac{1}{\sinh 1} \sinh x = \frac{2}{e^1 - e^{-1}} \frac{e^x - e^{-x}}{2} = \frac{e^x - e^{-x}}{e - 1/e} = \frac{e}{e^2 - 1}(e^x - e^{-x}).$$

**12.** In this case we have $y(0) = c_1 = 1$, $y'(1) = 2c_2 = 6$ so that $c_1 = 1$ and $c_2 = 3$. The solution is $y = 1 + 3x^2$.

**13.** From $y = c_1 e^x \cos x + c_2 e^x \sin x$ we find $y' = c_1 e^x(-\sin x + \cos x) + c_2 e^x(\cos x + \sin x)$.

**(a)** We have $y(0) = c_1 = 1$, $y'(0) = c_1 + c_2 = 0$ so that $c_1 = 1$ and $c_2 = -1$. The solution is $y = e^x \cos x - e^x \sin x$.

**(b)** We have $y(0) = c_1 = 1$, $y(\pi) = -e^\pi = -1$, which is not possible.

**(c)** We have $y(0) = c_1 = 1$, $y(\pi/2) = c_2 e^{\pi/2} = 1$ so that $c_1 = 1$ and $c_2 = e^{-\pi/2}$. The solution is $y = e^x \cos x + e^{-\pi/2} e^x \sin x$.

**(d)** We have $y(0) = c_1 = 0$, $y(\pi) = c_2 e^\pi \sin \pi = 0$ so that $c_1 = 0$ and $c_2$ is arbitrary. Solutions are $y = c_2 e^x \sin x$, for any real numbers $c_2$.

**14.** **(a)** We have $y(-1) = c_1 + c_2 + 3 = 0$, $y(1) = c_1 + c_2 + 3 = 4$, which is not possible.

**(b)** We have $y(0) = c_1 \cdot 0 + c_2 \cdot 0 + 3 = 1$, which is not possible.

**(c)** We have $y(0) = c_1 \cdot 0 + c_2 \cdot 0 + 3 = 3$, $y(1) = c_1 + c_2 + 3 = 0$ so that $c_1$ is arbitrary and $c_2 = -3 - c_1$. Solutions are $y = c_1 x^2 - (c_1 + 3)x^4 + 3$.

**(d)** We have $y(1) = c_1 + c_2 + 3 = 3$, $y(2) = 4c_1 + 16c_2 + 3 = 15$ so that $c_1 = -1$ and $c_2 = 1$. The solution is $y = -x^2 + x^4 + 3$.

**15.** Since $(-4)x + (3)x^2 + (1)(4x - 3x^2) = 0$ the set of functions is linearly dependent.

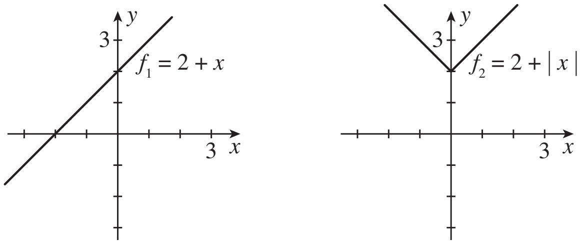
**16.** Since $(1)0 + (0)x + (0)e^x = 0$ the set of functions is linearly dependent. A similar argument shows that any set of functions containing $f(x) = 0$ will be linearly dependent.

**17.** Since $(-1/5)5 + (1)\cos^2 x + (1)\sin^2 x = 0$ the set of functions is linearly dependent.

**18.** Since $(1)\cos 2x + (1)1 + (-2)\cos^2 x = 0$ the set of functions is linearly dependent.

**19.** Since $(-4)x + (3)(x - 1) + (1)(x + 3) = 0$ the set of functions is linearly dependent.

**20.** From the graphs of $f_1(x) = 2 + x$ and $f_2(x) = 2 + |x|$ we see that the set of functions is linearly independent since they cannot be multiples of each other.



**21.** Suppose $c_1(1 + x) + c_2x + c_3x^2 = 0$. Then $c_1 + (c_1 + c_2)x + c_3x^2 = 0$ and so $c_1 = 0$, $c_1 + c_2 = 0$, and $c_3 = 0$. Since $c_1 = 0$ we also have $c_2 = 0$. Thus, the set of functions is linearly independent.

**22.** Since $(-1/2)e^x + (1/2)e^{-x} + (1)\sinh x = 0$ the set of functions is linearly dependent.

**23.** The functions satisfy the differential equation and are linearly independent since

$$W\left(e^{-3x}, e^{4x}\right) = 7e^x \neq 0$$

for $-\infty < x < \infty$. The general solution is

$$y = c_1e^{-3x} + c_2e^{4x}.$$

**24.** The functions satisfy the differential equation and are linearly independent since

$$W(\cosh 2x, \sinh 2x) = 2$$

for $-\infty < x < \infty$. The general solution is

$$y = c_1 \cosh 2x + c_2 \sinh 2x.$$

**25.** The functions satisfy the differential equation and are linearly independent since

$$W\left(e^x \cos 2x, e^x \sin 2x\right) = 2e^{2x} \neq 0$$

for $-\infty < x < \infty$. The general solution is $y = c_1e^x \cos 2x + c_2e^x \sin 2x$.

**26.** The functions satisfy the differential equation and are linearly independent since

$$W\left(e^{x/2}, xe^{x/2}\right) = e^x \neq 0$$

for $-\infty < x < \infty$. The general solution is

$$y = c_1 e^{x/2} + c_2 x e^{x/2}.$$

**27.** The functions satisfy the differential equation and are linearly independent since

$$W\left(x^3, x^4\right) = x^6 \neq 0$$

for $0 < x < \infty$. The general solution is

$$y = c_1 x^3 + c_2 x^4.$$

**28.** The functions satisfy the differential equation and are linearly independent since

$$W\left(\cos(\ln x), \sin(\ln x)\right) = 1/x \neq 0$$

for $0 < x < \infty$. The general solution is

$$y = c_1 \cos(\ln x) + c_2 \sin(\ln x).$$

**29.** The functions satisfy the differential equation and are linearly independent since

$$W\left(x, x^{-2}, x^{-2}\ln x\right) = 9x^{-6} \neq 0$$

for $0 < x < \infty$. The general solution is

$$y = c_1 x + c_2 x^{-2} + c_3 x^{-2} \ln x.$$

**30.** The functions satisfy the differential equation and are linearly independent since

$$W(1, x, \cos x, \sin x) = 1$$

for $-\infty < x < \infty$. The general solution is

$$y = c_1 + c_2 x + c_3 \cos x + c_4 \sin x.$$

**31.** The functions $y_1 = e^{2x}$ and $y_2 = e^{5x}$ form a fundamental set of solutions of the associated homogeneous equation, and $y_p = 6e^x$ is a particular solution of the nonhomogeneous equation.

**32.** The functions $y_1 = \cos x$ and $y_2 = \sin x$ form a fundamental set of solutions of the associated homogeneous equation, and $y_p = x\sin x + (\cos x)\ln(\cos x)$ is a particular solution of the nonhomogeneous equation.

**33.** The functions $y_1 = e^{2x}$ and $y_2 = xe^{2x}$ form a fundamental set of solutions of the associated homogeneous equation, and $y_p = x^2e^{2x} + x - 2$ is a particular solution of the nonhomogeneous equation.

**34.** The functions $y_1 = x^{-1/2}$ and $y_2 = x^{-1}$ form a fundamental set of solutions of the associated homogeneous equation, and $y_p = \frac{1}{15}x^2 - \frac{1}{6}x$ is a particular solution of the nonhomogeneous equation.

**35.** **(a)** We have $y'_{p_1} = 6e^{2x}$ and $y''_{p_1} = 12e^{2x}$, so

$$y''_{p_1} - 6y'_{p_1} + 5y_{p_1} = 12e^{2x} - 36e^{2x} + 15e^{2x} = -9e^{2x}.$$

Also, $y'_{p_2} = 2x + 3$ and $y''_{p_2} = 2$, so

$$y''_{p_2} - 6y'_{p_2} + 5y_{p_2} = 2 - 6(2x + 3) + 5(x^2 + 3x) = 5x^2 + 3x - 16.$$

**(b)** By the superposition principle for nonhomogeneous equations a particular solution of $y'' - 6y' + 5y = 5x^2 + 3x - 16 - 9e^{2x}$ is $y_p = x^2 + 3x + 3e^{2x}$. A particular solution of the second equation is

$$y_p = -2y_{p_2} - \frac{1}{9}y_{p_1} = -2x^2 - 6x - \frac{1}{3}e^{2x}.$$

**36.** **(a)** $y_{p_1} = 5$

**(b)** $y_{p_2} = -2x$

**(c)** $y_p = y_{p_1} + y_{p_2} = 5 - 2x$

**(d)** $y_p = \frac{1}{2}y_{p_1} - 2y_{p_2} = \frac{5}{2} + 4x$

**37.** **(a)** Since $D^2x = 0$, $x$ and 1 are solutions of $y'' = 0$. Since they are linearly independent, the general solution is $y = c_1x + c_2$.

**(b)** Since $D^3x^2 = 0$, $x^2$, $x$, and 1 are solutions of $y''' = 0$. Since they are linearly independent, the general solution is $y = c_1x^2 + c_2x + c_3$.

**(c)** Since $D^4x^3 = 0$, $x^3$, $x^2$, $x$, and 1 are solutions of $y^{(4)} = 0$. Since they are linearly independent, the general solution is $y = c_1x^3 + c_2x^2 + c_3x + c_4$.

**(d)** By part (a), the general solution of $y'' = 0$ is $y_c = c_1x + c_2$. Since $D^2x^2 = 2! = 2$, $y_p = x^2$ is a particular solution of $y'' = 2$. Thus, the general solution is $y = c_1x + c_2 + x^2$.

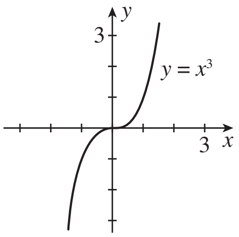
**(e)** By part (b), the general solution of $y''' = 0$ is $y_c = c_1x^2 + c_2x + c_3$. Since $D^3x^3 = 3! = 6$, $y_p = x^3$ is a particular solution of $y''' = 6$. Thus, the general solution is $y = c_1x^2 + c_2x + c_3 + x^3$.

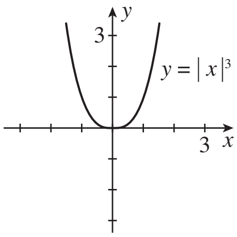
**(f)** By part (c), the general solution of $y^{(4)} = 0$ is $y_c = c_1x^3 + c_2x^2 + c_3x + c_4$. Since $D^4x^4 = 4! = 24$, $y_p = x^4$ is a particular solution of $y^{(4)} = 24$. Thus, the general solution is $y = c_1x^3 + c_2x^2 + c_3x + c_4 + x^4$.

**38.** By the superposition principle, if $y_1 = e^x$ and $y_2 = e^{-x}$ are both solutions of a homogeneous linear differential equation, then so are

$$\frac{1}{2}(y_1 + y_2) = \frac{e^x + e^{-x}}{2} = \cosh x \quad \text{and} \quad \frac{1}{2}(y_1 - y_2) = \frac{e^x - e^{-x}}{2} = \sinh x.$$

**39.** **(a)** From the graphs of $y_1 = x^3$ and $y_2 = |x|^3$ we see that the functions are linearly independent since they cannot be multiples of each other. It is easily shown that $y_1 = x^3$ is a solution of $x^2y'' - 4xy' + 6y = 0$. To show that $y_2 = |x|^3$ is a solution let $y_2 = x^3$ for $x \geq 0$ and let $y_2 = -x^3$ for $x < 0$.





**(b)** If $x \geq 0$ then $y_2 = x^3$ and

$$W(y_1, y_2) = \begin{vmatrix} x^3 & x^3 \\ 3x^2 & 3x^2 \end{vmatrix} = 0$$

If $x < 0$ then $y_2 = -x^3$ and

$$W(y_1, y_2) = \begin{vmatrix} x^3 & -x^3 \\ 3x^2 & -3x^2 \end{vmatrix} = 0$$

This does not violate Theorem 4.1.3 since $a_2(x) = x^2$ is zero at $x = 0$.

**(c)** The functions $Y_1 = x^3$ and $Y_2 = x^2$ are solutions of $x^2y'' - 4xy' + 6y = 0$ on the interval $(-\infty, \infty)$ because we have, in turn,

$$\begin{aligned} x^2Y_1'' - 4xY_1' + 6Y_1 &= x^2\,(6x) - 4x\left(3x^2\right) + 6x^3 = 0 \\ x^2Y_2'' - 4xY_2' + 6Y_2 &= x^2\,(2) - 4x\,(2x) + 6x^2 = 0. \end{aligned}$$

The solutions $Y_1 = x^3$ and $Y_2 = x^2$ are also linearly independent on the interval $(-\infty, \infty)$. In order for $c_1x^3 + c_2x^2 = 0$ for *every* real number $x$ it is necessary that $c_1 = c_2 = 0$. To see this, observe that if either $c_1$ or $c_2$ were not 0, then the equation $c_1c^3 + c_2x^2 = 0$ or $x^2\,(c_1x + c_2) = 0$ would hold for at most two real numbers.

**(d)** Since the linear differential equation is homogeneous, the superposition principle indicates that $y = x^3 + x^2$ is a solution of the equation. It is also clear that $y = x^3 + x^2$ satisfies the initial conditions $y(0) = 0$, $y'(0) = 0$.

**(e)** Neither is the general solution on $(-\infty, \infty)$ since we form a general solution on an interval for which $a_2(x) \neq 0$ for every $x$ in the interval.

**40.** Since $e^{x-3} = e^{-3}e^x = (e^{-5}e^2)e^x = e^{-5}e^{x+2}$, we see that $e^{x-3}$ is a constant multiple of $e^{x+2}$ and the set of functions is linearly dependent.

**41.** Since $0y_1 + 0y_2 + \cdots + 0y_k + 1y_{k+1} = 0$, the set of solutions is linearly dependent.

**42.** The set of solutions is linearly dependent. Suppose $n$ of the solutions are linearly independent (if not, then the set of $n+1$ solutions is linearly dependent). Without loss of generality, let this set be $y_1, y_2, \ldots, y_n$. Then $y = c_1y_1 + c_2y_2 + \cdots + c_ny_n$ is the general solution of the $n$th-order differential equation and for some choice, $c_1^*, c_2^*, \ldots, c_n^*$, of the coefficients $y_{n+1} = c_1^*y_1 + c_2^*y_2 + \cdots + c_n^*y_n$. But then the set $y_1, y_2, \ldots, y_n, y_{n+1}$ is linearly dependent.

## 4.2 Reduction of Order

*In Problems 1–8 we use reduction of order to find a second solution. In Problems 9–16 we use formula (5) from the text.*

**1.** Define $y = u(x)e^{2x}$ so

$$y' = 2ue^{2x} + u'e^{2x}, \quad y'' = e^{2x}u'' + 4e^{2x}u' + 4e^{2x}u, \quad \text{and} \quad y'' - 4y' + 4y = e^{2x}u'' = 0.$$

Therefore $u'' = 0$ and $u = c_1x + c_2$. Taking $c_1 = 1$ and $c_2 = 0$ we see that a second solution is $y_2 = xe^{2x}$.

**2.** Define $y = u(x)xe^{-x}$ so

$$y' = (1-x)e^{-x}u + xe^{-x}u', \quad y'' = xe^{-x}u'' + 2(1-x)e^{-x}u' - (2-x)e^{-x}u,$$

and

$$y'' + 2y' + y = e^{-x}(xu'' + 2u') = 0 \quad \text{or} \quad u'' + \frac{2}{x}u' = 0.$$

If $w = u'$ we obtain the linear first-order equation $w' + \frac{2}{x}w = 0$ which has the integrating factor $e^{2\int dx/x} = x^2$. Now

$$\frac{d}{dx}[x^2w] = 0 \quad \text{gives} \quad x^2w = c.$$

Therefore $w = u' = c/x^2$ and $u = c_1/x$. A second solution is $y_2 = \frac{1}{x}xe^{-x} = e^{-x}$.

**3.** Define $y = u(x)\cos 4x$ so

$$y' = -4u\sin 4x + u'\cos 4x, \quad y'' = u''\cos 4x - 8u'\sin 4x - 16u\cos 4x$$

and

$$y'' + 16y = (\cos 4x)u'' - 8(\sin 4x)u' = 0 \quad \text{or} \quad u'' - 8(\tan 4x)u' = 0.$$

If $w = u'$ we obtain the linear first-order equation $w' - 8(\tan 4x)w = 0$ which has the integrating factor $e^{-8\int \tan 4x\,dx} = \cos^2 4x$. Now

$$\frac{d}{dx}\left[(\cos^2 4x)w\right] = 0 \quad \text{gives} \quad (\cos^2 4x)w = c.$$

Therefore $w = u' = c\sec^2 4x$ and $u = c_1 \tan 4x$. A second solution is
$y_2 = \tan 4x \cos 4x = \sin 4x$.

**4.** Define $y = u(x)\sin 3x$ so

$$y' = 3u\cos 3x + u'\sin 3x, \quad y'' = u''\sin 3x + 6u'\cos 3x - 9u\sin 3x,$$

and

$$y'' + 9y = (\sin 3x)u'' + 6(\cos 3x)u' = 0 \quad \text{or} \quad u'' + 6(\cot 3x)u' = 0.$$

If $w = u'$ we obtain the linear first-order equation $w' + 6(\cot 3x)w = 0$ which has the integrating factor $e^{6\int \cot 3x\,dx} = \sin^2 3x$. Now

$$\frac{d}{dx}[(\sin^2 3x)w] = 0 \quad \text{gives} \quad (\sin^2 3x)w = c.$$

Therefore $w = u' = c\csc^2 3x$ and $u = c_1 \cot 3x$. A second solution is
$y_2 = \cot 3x \sin 3x = \cos 3x$.

**5.** Define $y = u(x)\cosh x$ so

$$y' = u\sinh x + u'\cosh x, \quad y'' = u''\cosh x + 2u'\sinh x + u\cosh x$$

and

$$y'' - y = (\cosh x)u'' + 2(\sinh x)u' = 0 \quad \text{or} \quad u'' + 2(\tanh x)u' = 0.$$

If $w = u'$ we obtain the linear first-order equation $w' + 2(\tanh x)w = 0$ which has the integrating factor $e^{2\int \tanh x\,dx} = \cosh^2 x$. Now

$$\frac{d}{dx}\left[(\cosh^2 x)w\right] = 0 \quad \text{gives} \quad (\cosh^2 x)w = c.$$

Therefore $w = u' = c\,\mathrm{sech}^2 x$ and $u = c\tanh x$. A second solution is
$y_2 = \tanh x \cosh x = \sinh x$.

**6.** Define $y = u(x)e^{5x}$ so

$$y' = 5e^{5x}u + e^{5x}u', \quad y'' = e^{5x}u'' + 10e^{5x}u' + 25e^{5x}u$$

and

$$y'' - 25y = e^{5x}(u'' + 10u') = 0 \quad \text{or} \quad u'' + 10u' = 0.$$

If $w = u'$ we obtain the linear first-order equation $w' + 10w = 0$ which has the integrating factor $e^{10\int dx} = e^{10x}$. Now

$$\frac{d}{dx}\left[e^{10x}w\right] = 0 \quad \text{gives} \quad e^{10x}w = c.$$

Therefore $w = u' = ce^{-10x}$ and $u = c_1e^{-10x}$. A second solution is $y_2 = e^{-10x}e^{5x} = e^{-5x}$.

**7.** Define $y = u(x)e^{2x/3}$ so

$$y' = \frac{2}{3}e^{2x/3}u + e^{2x/3}u', \quad y'' = e^{2x/3}u'' + \frac{4}{3}e^{2x/3}u' + \frac{4}{9}e^{2x/3}u$$

and

$$9y'' - 12y' + 4y = 9e^{2x/3}u'' = 0.$$

Therefore $u'' = 0$ and $u = c_1x + c_2$. Taking $c_1 = 1$ and $c_2 = 0$ we see that a second solution is $y_2 = xe^{2x/3}$.

**8.** Define $y = u(x)e^{x/3}$ so

$$y' = \frac{1}{3}e^{x/3}u + e^{x/3}u', \quad y'' = e^{x/3}u'' + \frac{2}{3}e^{x/3}u' + \frac{1}{9}e^{x/3}u$$

and

$$6y'' + y' - y = e^{x/3}(6u'' + 5u') = 0 \quad \text{or} \quad u'' + \frac{5}{6}u' = 0.$$

If $w = u'$ we obtain the linear first-order equation $w' + \frac{5}{6}w = 0$ which has the integrating factor $e^{(5/6)\int dx} = e^{5x/6}$. Now

$$\frac{d}{dx}\left[e^{5x/6}w\right] = 0 \quad \text{gives} \quad e^{5x/6}w = c.$$

Therefore $w = u' = ce^{-5x/6}$ and $u = c_1e^{-5x/6}$. A second solution is $y_2 = e^{-5x/6}e^{x/3} = e^{-x/2}$.

**9.** Identifying $P(x) = -7/x$ we have

$$y_2 = x^4\int\frac{e^{-\int(-7/x)\,dx}}{x^8}\,dx = x^4\int\frac{1}{x}\,dx = x^4\ln|x|.$$

A second solution is $y_2 = x^4\ln|x|$.

**10.** Identifying $P(x) = 2/x$ we have

$$y_2 = x^2\int\frac{e^{-\int(2/x)\,dx}}{x^4}\,dx = x^2\int x^{-6}\,dx = -\frac{1}{5}x^{-3}.$$

A second solution is $y_2 = x^{-3}$.

**11.** Identifying $P(x) = 1/x$ we have

$$y_2 = \ln x\int\frac{e^{-\int dx/x}}{(\ln x)^2}\,dx = \ln x\int\frac{dx}{x(\ln x)^2} = \ln x\left(-\frac{1}{\ln x}\right) = -1.$$

A second solution is $y_2 = 1$.

**12.** Identifying $P(x) = 0$ we have

$$y_2 = x^{1/2} \ln x \int \frac{e^{-\int 0\, dx}}{x(\ln x)^2}\, dx = x^{1/2} \ln x \left( -\frac{1}{\ln x} \right) = -x^{1/2}.$$

A second solution is $y_2 = x^{1/2}$.

**13.** Identifying $P(x) = -1/x$ we have

$$\begin{aligned} y_2 &= x \sin(\ln x) \int \frac{e^{-\int -dx/x}}{x^2 \sin^2(\ln x)}\, dx = x \sin(\ln x) \int \frac{x}{x^2 \sin^2(\ln x)}\, dx \\ &= x \sin(\ln x) \int \frac{\csc^2(\ln x)}{x}\, dx = [x \sin(\ln x)]\,[-\cot(\ln x)] = -x \cos(\ln x). \end{aligned}$$

A second solution is $y_2 = x \cos(\ln x)$.

**14.** Identifying $P(x) = -3/x$ we have

$$\begin{aligned} y_2 &= x^2 \cos(\ln x) \int \frac{e^{-\int -3\, dx/x}}{x^4 \cos^2(\ln x)}\, dx = x^2 \cos(\ln x) \int \frac{x^3}{x^4 \cos^2(\ln x)}\, dx \\ &= x^2 \cos(\ln x) \int \frac{\sec^2(\ln x)}{x}\, dx = x^2 \cos(\ln x) \tan(\ln x) = x^2 \sin(\ln x). \end{aligned}$$

A second solution is $y_2 = x^2 \sin(\ln x)$.

**15.** Identifying $P(x) = 2(1+x)/\left(1 - 2x - x^2\right)$ we have

$$\begin{aligned} y_2 &= (x+1) \int \frac{e^{-\int 2(1+x)\, dx/\left(1-2x-x^2\right)}}{(x+1)^2}\, dx = (x+1) \int \frac{e^{\ln\left(1-2x-x^2\right)}}{(x+1)^2}\, dx \\ &= (x+1) \int \frac{1 - 2x - x^2}{(x+1)^2}\, dx = (x+1) \int \left[ \frac{2}{(x+1)^2} - 1 \right] dx \\ &= (x+1) \left[ -\frac{2}{x+1} - x \right] = -2 - x^2 - x. \end{aligned}$$

A second solution is $y_2 = x^2 + x + 2$.

**16.** Identifying $P(x) = 2x/\left(1 - x^2\right)$ we have

$$y_2 = \int e^{-\int (2x)\, dx/\left(1-x^2\right)}\, dx = \int e^{\ln\left(1-x^2\right)}\, dx = \int \left(1 - x^2\right) dx = x - \frac{1}{3} x^3.$$

A second solution is $y_2 = \frac{1}{3} x^3 - x$.

**17.** Define $y = u(x)e^{-2x}$ so

$$y' = -2ue^{-2x} + u'e^{-2x}, \quad y'' = u''e^{-2x} - 4u'e^{-2x} + 4ue^{-2x}$$

and

$$y'' - 4y = e^{-2x}u'' - 4e^{-2x}u' = \left(u'' - 4u'\right)e^{-2x} = 2.$$

If $w = u'$ we obtain the linear first-order equation $w' - 4w = 2e^{2x}$ which has the integrating factor $e^{-4\int dx} = e^{-4x}$. Now

$$\frac{d}{dx}[e^{-4x}w] = 2e^{-2x} \quad \text{gives} \quad e^{-4x}w = -e^{-2x} + c_1.$$

Therefore $w = u' = -e^{2x} + c_1e^{4x}$ and

$$\begin{aligned} u &= -\frac{1}{2}e^{2x} + \frac{1}{4}c_1e^{4x} + c_2 \\ y &= -\frac{1}{2} + \frac{1}{4}c_1e^{2x} + c_2e^{-2x}. \end{aligned}$$

From the last equation we see that a second solution is $y_2 = e^{2x}$ and $y_p = -\frac{1}{2}$.

**18.** Define $y = u(x) \cdot 1 = u(x)$ so

$$y' = u', \quad y'' = u'' \quad \text{and} \quad y'' + y' = u'' + u' = 1.$$

If $w = u'$ we obtain the linear first-order equation $w' + w = 1$ which has the integrating factor $e^{\int dx} = e^x$. Now

$$\frac{d}{dx}[e^xw] = e^x \quad \text{gives} \quad e^xw = e^x + c_1.$$

Therefore $w = u' = 1 + c_1e^{-x}$ and

$$y = u = x - c_1e^{-x} + c_2$$

From the last equation we see that a second solution is $y_2 = e^{-x}$ and $y_p = x$.

**19.** Define $y = u(x)e^x$ so

$$y' = ue^x + u'e^x, \quad y'' = u''e^x + 2u'e^x + ue^x$$

and

$$y'' - 3y' + 2y = e^xu'' - e^xu' = 5e^{3x}.$$

If $w = u'$ we obtain the linear first-order equation $w' - w = 5e^{2x}$ which has the integrating factor $e^{-\int dx} = e^{-x}$. Now

$$\frac{d}{dx}[e^{-x}w] = 5e^x \quad \text{gives} \quad e^{-x}w = 5e^x + c_1.$$

Therefore $w = u' = 5e^{2x} + c_1e^x$ and

$$\begin{aligned} u &= \frac{5}{2}e^{2x} + c_1e^x + c_2 \\ y &= \frac{5}{2}e^{3x} + c_1e^{2x} + c_2e^x \end{aligned}$$

From the last equation we see that a second solution is $y_2 = e^{2x}$ and $y_p = \frac{5}{2}e^{3x}$.

**20.** Define $y = u(x)e^x$ so

$$y' = ue^x + u'e^x, \quad y'' = u''e^x + 2u'e^x + ue^x$$

and

$$y'' - 4y' + 3y = e^xu'' - e^xu' = x.$$

If $w = u'$ we obtain the linear first-order equation $w' - 2w = xe^{-x}$ which has the integrating factor $e^{-\int 2dx} = e^{-2x}$. Now

$$\frac{d}{dx}[e^{-2x}w] = xe^{-3x} \quad \text{gives} \quad e^{-2x}w = -\frac{1}{3}\,xe^{-3x} - \frac{1}{9}e^{-3x} + c_1.$$

Therefore $w = u' = -\frac{1}{3}\,xe^{-x} - \frac{1}{9}e^{-x} + c_1e^{2x}$ and

$$\begin{aligned} u &= \frac{1}{3}xe^{-x} + \frac{4}{9}e^{-x} + \frac{1}{2}c_1e^{2x} + c_2 \\ y &= \frac{1}{3}x + \frac{4}{9} + \frac{1}{2}c_1e^{3x} + c_2e^x \end{aligned}$$

From the last equation we see that a second solution is $y_2 = e^{3x}$ and $y_p = \frac{1}{3}x + \frac{4}{9}$.

**21.** Dividing by $x^2$ we have

$$\begin{aligned} x^2y'' + \left(x^2 - x\right)y' + (1 - x)\,y = 0 \\ y'' + \left(1 - \frac{1}{x}\right)y' + \left(\frac{1}{x^2} - \frac{1}{x}\right)y = 0 \end{aligned}$$

Using $P(x) = 1 - \dfrac{1}{x}$ and formula (5) in the text we have

$$y_2(x) = x\int_{x_0}^{x} \frac{e^{-(1-1/t)\,dt}}{t^2}\,dt = x\int_{x_0}^{x} \frac{e^{-t+\ln t}}{t^2}\,dt = x\int_{x_0}^{x} \frac{e^{-t}e^{\ln t}}{t^2}\,dt = x\int_{x_0}^{x} \frac{e^{-t}t}{t^2}\,dt$$

Therefore $y_2(x) = x\displaystyle\int_{x_0}^{x} \frac{e^{-t}}{t}\,dt$, $x + 0 > 0$.

**22.** Dividing by $2x$ we have

$$2xy'' - (2x+1)\,y' + y = 0$$

$$y'' + \left(-1 - \frac{1}{2x}\right) y' + \frac{1}{2x} = 0$$

Using $P(x) = -1 - \dfrac{1}{2x}$ and formula (5) in the text we have

$$y_2(x) = e^x \int_{x_0}^{x} \frac{e^{-(1+1/2t)\,dt}}{e^{2t}}\,dt = e^x \int_{x_0}^{x} \frac{e^{t+\frac{1}{2}\ln t}}{e^{2t}}\,dt = e^x \int_{x_0}^{x} \frac{e^t\left(e^{\ln t}\right)^{1/2}}{e^{2t}}\,dt = e^x \int_{x_0}^{x} \frac{e^t\sqrt{t}}{e^{2t}}\,dt$$

Therefore $y_2(x) = e^x \displaystyle\int_{x_0}^{x} \sqrt{t}\, e^{-t}\,dt$, $x_0 \geq 0$.

**23.** **(a)** For $m_1$ constant, let $y_1 = e^{m_1 x}$. Then $y_1' = m_1 e^{m_1 x}$ and $y_1'' = m_1^2 e^{m_1 x}$. Substituting into the differential equation we obtain

$$\begin{aligned} ay_1'' + by_1' + cy_1 &= am_1^2 e^{m_1 x} + bm_1 e^{m_1 x} + ce^{m_1 x} \\ &= e^{m_1 x}(am_1^2 + bm_1 + c) = 0. \end{aligned}$$

Thus, $y_1 = e^{m_1 x}$ will be a solution of the differential equation whenever $am_1^2 + bm_1 + c = 0$. Since a quadratic equation always has at least one real or complex root, the differential equation must have a solution of the form $y_1 = e^{m_1 x}$.

**(b)** Write the differential equation in the form

$$y'' + \frac{b}{a}y' + \frac{c}{a}y = 0,$$

and let $y_1 = e^{m_1 x}$ be a solution. Then a second solution is given by

$$\begin{aligned} y_2 &= e^{m_1 x} \int \frac{e^{-bx/a}}{e^{2m_1 x}}\,dx \\ &= e^{m_1 x} \int e^{-(b/a+2m_1)x}\,dx \\ &= -\frac{1}{b/a + 2m_1}\, e^{m_1 x} e^{-(b/a+2m_1)x} \qquad (m_1 \neq -b/2a) \\ &= -\frac{1}{b/a + 2m_1}\, e^{-(b/a+m_1)x}. \end{aligned}$$

Thus, when $m_1 \neq -b/2a$, a second solution is given by $y_2 = e^{m_2 x}$ where $m_2 = -b/a - m_1$. When $m_1 = -b/2a$ a second solution is given by

$$y_2 = e^{m_1 x} \int dx = xe^{m_1 x}.$$

**(c)** The functions

$$\sin x = \frac{1}{2i}(e^{ix} - e^{-ix}) \qquad \cos x = \frac{1}{2}(e^{ix} + e^{-ix})$$

$$\sinh x = \frac{1}{2}(e^{x} - e^{-x}) \qquad \cosh x = \frac{1}{2}(e^{x} + e^{-x})$$

are all expressible in terms of exponential functions.

**24.** We have $y_1' = 1$ and $y_1'' = 0$, so $xy_1'' - xy_1' + y_1 = 0 - x + x = 0$ and $y_1(x) = x$ is a solution of the differential equation. Letting $y = u(x)y_1(x) = xu(x)$ we get

$$y' = xu'(x) + u(x) \quad \text{and} \quad y'' = xu''(x) + 2u'(x).$$

Then $xy'' - xy' + y = x^2u'' + 2xu' - x^2u' - xu + xu = x^2u'' - (x^2 - 2x)u' = 0$. If we make the substitution $w = u'$, the linear first-order differential equation becomes $x^2w' - (x^2 - x)w = 0$, which is separable:

$$\begin{aligned}
\frac{dw}{dx} &= \left(1 - \frac{1}{x}\right) w \\
\frac{dw}{w} &= \left(1 - \frac{1}{x}\right) dx \\
\ln w &= x - \ln x + c \\
w &= c_1 \frac{e^x}{x}\,.
\end{aligned}$$

Then $u' = c_1e^x/x$ and $u = c_1\int e^x dx/x$. To integrate $e^x/x$ we use the series representation for $e^x$. Thus, a second solution is

$$\begin{aligned}
y_2 = xu(x) &= x\int \frac{e^x}{x}\,dx \\
&= x\int \frac{1}{x}\left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots\right) dx \\
&= x\int \left(\frac{1}{x} + 1 + \frac{1}{2!}x + \frac{1}{3!}x^2 + \cdots\right) dx \\
&= x\left(\ln x + x + \frac{1}{2(2!)}x^2 + \frac{1}{3(3!)}x^3 + \cdots\right) \\
&= x\ln x + x^2 + \frac{1}{2(2!)}x^3 + \frac{1}{3(3!)}x^4 + \cdots\,.
\end{aligned}$$

An interval of definition is probably $(0, \infty)$ because of the $\ln x$ term.

**25.** **(a)** We have $y' = y'' = e^x$, so

$$xy'' - (x + 10)y' + 10y = xe^x - (x + 10)e^x + 10e^x = 0,$$

and $y = e^x$ is a solution of the differential equation.

**(b)** By (5) in the text a second solution is

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{-\int P(x)\,dx}}{y_1^2}\,dx = e^x \int \frac{e^{\int \frac{x+10}{x}\,dx}}{e^{2x}}\,dx = e^x \int \frac{e^{\int (1+10/x)\,dx}}{e^{2x}}\,dx \\ &= e^x \int \frac{e^{x+\ln x^{10}}}{e^{2x}}\,dx = e^x \int x^{10}e^{-x}\,dx \\ &= e^x(-3{,}628{,}800 - 3{,}628{,}800x - 1{,}814{,}400x^2 - 604{,}800x^3 - 151{,}200x^4 \\ &\qquad - 30{,}240x^5 - 5{,}040x^6 - 720x^7 - 90x^8 - 10x^9 - x^{10})e^{-x} \\ &= -3{,}628{,}800 - 3{,}628{,}800x - 1{,}814{,}400x^2 - 604{,}800x^3 - 151{,}200x^4 \\ &\qquad - 30{,}240x^5 - 5{,}040x^6 - 720x^7 - 90x^8 - 10x^9 - x^{10}. \end{aligned}$$

**(c)** By Corollary (a) of Theorem 3.1.2, $-\dfrac{1}{10!}y_2 = \displaystyle\sum_{n=0}^{10} \frac{1}{n!}x^n$ is a solution.

## 4.3 Homogeneous Linear Equations with Constant Coefficients

**1.** From $4m^2 + m = 0$ we obtain $m_1 = 0$ and $m_2 = -1/4$ so that $y = c_1 + c_2e^{-x/4}$.

**2.** From $m^2 - 36 = 0$ we obtain $m_1 = 6$ and $m_1 = -6$ so that $y = c_1e^{6x} + c_2e^{-6x}$.

**3.** From $m^2 - m - 6 = 0$ we obtain $m_1 = 3$ and $m_2 = -2$ so that $y = c_1e^{3x} + c_2e^{-2x}$.

**4.** From $m^2 - 3m + 2 = 0$ we obtain $m_1 = 1$ and $m_2 = 2$ so that $y = c_1e^{x} + c_2e^{2x}$.

**5.** From $m^2 + 8m + 16 = 0$ we obtain $m_1 = -4$ and $m_2 = -4$ so that $y = c_1e^{-4x} + c_2xe^{-4x}$.

**6.** From $m^2 - 10m + 25 = 0$ we obtain $m_1 = 5$ and $m_2 = 5$ so that $y = c_1e^{5x} + c_2xe^{5x}$.

**7.** From $12m^2 - 5m - 2 = 0$ we obtain $m_1 = -1/4$ and $m_2 = 2/3$ so that $y = c_1e^{-x/4} + c_2e^{2x/3}$.

**8.** From $m^2 + 4m - 1 = 0$ we obtain $m = -2 \pm \sqrt{5}$ so that $y = c_1e^{(-2+\sqrt{5}\,)x} + c_2e^{(-2-\sqrt{5}\,)x}$.

**9.** From $m^2 + 9 = 0$ we obtain $m_1 = 3i$ and $m_2 = -3i$ so that $y = c_1\cos 3x + c_2\sin 3x$.

**10.** From $3m^2 + 1 = 0$ we obtain $m_1 = i/\sqrt{3}$ and $m_2 = -i/\sqrt{3}$ so that

$$y = c_1\cos\,(x/\sqrt{3}\,) + c_2\sin\,(x/\sqrt{3}).$$

**11.** From $m^2 - 4m + 5 = 0$ we obtain $m = 2 \pm i$ so that $y = e^{2x}(c_1\cos x + c_2\sin x)$.

**12.** From $2m^2 + 2m + 1 = 0$ we obtain $m = -1/2 \pm i/2$ so that

$$y = e^{-x/2}[c_1\cos\,(x/2) + c_2\sin\,(x/2)].$$

**13.** From $3m^2+2m+1=0$ we obtain $m=-1/3\pm\sqrt{2}\,i/3$ so that

$$y=e^{-x/3}[c_1\cos(\sqrt{2}\,x/3)+c_2\sin(\sqrt{2}\,x/3)].$$

**14.** From $2m^2-3m+4=0$ we obtain $m=3/4\pm\sqrt{23}\,i/4$ so that

$$y=e^{3x/4}[c_1\cos(\sqrt{23}\,x/4)+c_2\sin(\sqrt{23}\,x/4)].$$

**15.** From $m^3-4m^2-5m=0$ we obtain $m_1=0$, $m_2=5$, and $m_3=-1$ so that

$$y=c_1+c_2e^{5x}+c_3e^{-x}.$$

**16.** From $m^3-1=0$ we obtain $m_1=1$ and $m_2=-1/2\pm\sqrt{3}\,i/2$ so that

$$y=c_1e^{x}+e^{-x/2}[c_2\cos(\sqrt{3}\,x/2)+c_3\sin(\sqrt{3}\,x/2)].$$

**17.** From $m^3-5m^2+3m+9=0$ we obtain $m_1=-1$, $m_2=3$, and $m_3=3$ so that

$$y=c_1e^{-x}+c_2e^{3x}+c_3xe^{3x}.$$

**18.** From $m^3+3m^2-4m-12=0$ we obtain $m_1=-2$, $m_2=2$, and $m_3=-3$ so that

$$y=c_1e^{-2x}+c_2e^{2x}+c_3e^{-3x}.$$

**19.** From $m^3+m^2-2=0$ we obtain $m_1=1$ and $m_2=-1\pm i$ so that

$$u=c_1e^{t}+e^{-t}(c_2\cos t+c_3\sin t).$$

**20.** From $m^3-m^2-4=0$ we obtain $m_1=2$ and $m_2=-1/2\pm\sqrt{7}\,i/2$ so that

$$x=c_1e^{2t}+e^{-t/2}[c_2\cos(\sqrt{7}\,t/2)+c_3\sin(\sqrt{7}\,t/2)].$$

**21.** From $m^3+3m^2+3m+1=0$ we obtain $m_1=-1$, $m_2=-1$, and $m_3=-1$ so that

$$y=c_1e^{-x}+c_2xe^{-x}+c_3x^2e^{-x}.$$

**22.** From $m^3-6m^2+12m-8=0$ we obtain $m_1=2$, $m_2=2$, and $m_3=2$ so that

$$y=c_1e^{2x}+c_2xe^{2x}+c_3x^2e^{2x}.$$

**23.** From $m^4+m^3+m^2=0$ we obtain $m_1=0$, $m_2=0$, and $m_3=-1/2\pm\sqrt{3}\,i/2$ so that

$$y=c_1+c_2x+e^{-x/2}[c_3\cos(\sqrt{3}\,x/2)+c_4\sin(\sqrt{3}\,x/2)].$$

**24.** From $m^4-2m^2+1=0$ we obtain $m_1=1$, $m_2=1$, $m_3=-1$, and $m_4=-1$ so that

$$y=c_1e^{x}+c_2xe^{x}+c_3e^{-x}+c_4xe^{-x}.$$

**25.** From $16m^4 + 24m^2 + 9 = 0$ we obtain $m_1 = \pm\sqrt{3}\,i/2$ and $m_2 = \pm\sqrt{3}\,i/2$ so that

$$y = c_1 \cos(\sqrt{3}\,x/2) + c_2 \sin(\sqrt{3}\,x/2) + c_3 x \cos(\sqrt{3}\,x/2) + c_4 x \sin(\sqrt{3}\,x/2).$$

**26.** From $m^4 - 7m^2 - 18 = 0$ we obtain $m_1 = 3$, $m_2 = -3$, and $m_3 = \pm\sqrt{2}\,i$ so that

$$y = c_1 e^{3x} + c_2 e^{-3x} + c_3 \cos\sqrt{2}\,x + c_4 \sin\sqrt{2}\,x.$$

**27.** From $m^5 + 5m^4 - 2m^3 - 10m^2 + m + 5 = 0$ we obtain $m_1 = -1$, $m_2 = -1$, $m_3 = 1$, and $m_4 = 1$, and $m_5 = -5$ so that

$$u = c_1 e^{-r} + c_2 r e^{-r} + c_3 e^{r} + c_4 r e^{r} + c_5 e^{-5r}.$$

**28.** From $2m^5 - 7m^4 + 12m^3 + 8m^2 = 0$ we obtain $m_1 = 0$, $m_2 = 0$, $m_3 = -1/2$, and $m_4 = 2 \pm 2i$ so that

$$x = c_1 + c_2 s + c_3 e^{-s/2} + e^{2s}(c_4 \cos 2s + c_5 \sin 2s).$$

**29.** From $m^2 + 16 = 0$ we obtain $m = \pm 4i$ so that $y = c_1 \cos 4x + c_2 \sin 4x$. If $y(0) = 2$ and $y'(0) = -2$ then $c_1 = 2$, $c_2 = -1/2$, and $y = 2\cos 4x - \frac{1}{2}\sin 4x$.

**30.** From $m^2 + 1 = 0$ we obtain $m = \pm i$ so that $y = c_1 \cos\theta + c_2 \sin\theta$. If $y(\pi/3) = 0$ and $y'(\pi/3) = 2$ then

$$\begin{aligned} \frac{1}{2}c_1 + \frac{\sqrt{3}}{2}c_2 &= 0 \\ -\frac{\sqrt{3}}{2}c_1 + \frac{1}{2}c_2 &= 2, \end{aligned}$$

so $c_1 = -\sqrt{3}$, $c_2 = 1$, and $y = -\sqrt{3}\,\cos\theta + \sin\theta$.

**31.** From $m^2 - 4m - 5 = 0$ we obtain $m_1 = -1$ and $m_1 = 5$, so that $y = c_1 e^{-t} + c_2 e^{5t}$. If $y(1) = 0$ and $y'(1) = 2$, then $c_1 e^{-1} + c_2 e^{5} = 0$, $-c_1 e^{-1} + 5c_2 e^{5} = 2$, so $c_1 = -e/3$, $c_2 = e^{-5}/3$, and $y = -\frac{1}{3}e^{1-t} + \frac{1}{3}e^{5t-5}$.

**32.** From $4m^2 - 4m - 3 = 0$ we obtain $m_1 = -1/2$ and $m_2 = 3/2$ so that $y = c_1 e^{-x/2} + c_2 e^{3x/2}$. If $y(0) = 1$ and $y'(0) = 5$ then $c_1 + c_2 = 1$, $-\frac{1}{2}c_1 + \frac{1}{2}32c_2 = 5$, so $c_1 = -7/4$, $c_2 = 11/4$, and $y = -\frac{7}{4}e^{-x/2} + \frac{11}{4}e^{3x/2}$.

**33.** From $m^2 + m + 2 = 0$ we obtain $m = -1/2 \pm \sqrt{7}\,i/2$ so that
$y = e^{-x/2}[c_1 \cos(\sqrt{7}\,x/2) + c_2 \sin(\sqrt{7}\,x/2)]$. If $y(0) = 0$ and $y'(0) = 0$ then $c_1 = 0$ and $c_2 = 0$ so that $y = 0$.

**34.** From $m^2 - 2m + 1 = 0$ we obtain $m_1 = 1$ and $m_2 = 1$ so that $y = c_1 e^{x} + c_2 x e^{x}$. If $y(0) = 5$ and $y'(0) = 10$ then $c_1 = 5$, $c_1 + c_2 = 10$ so $c_1 = 5$, $c_2 = 5$, and $y = 5e^{x} + 5xe^{x}$.

**35.** From $m^3 + 12m^2 + 36m = 0$ we obtain $m_1 = 0$, $m_2 = -6$, and $m_3 = -6$ so that $y = c_1 + c_2e^{-6x} + c_3xe^{-6x}$. If $y(0) = 0$, $y'(0) = 1$, and $y''(0) = -7$ then

$$c_1 + c_2 = 0, \quad -6c_2 + c_3 = 1, \quad 36c_2 - 12c_3 = -7,$$

so $c_1 = 5/36$, $c_2 = -5/36$, $c_3 = 1/6$, and $y = \frac{5}{36} - \frac{5}{36}e^{-6x} + \frac{1}{6}xe^{-6x}$.

**36.** From $m^3 + 2m^2 - 5m - 6 = 0$ we obtain $m_1 = -1$, $m_2 = 2$, and $m_3 = -3$ so that

$$y = c_1e^{-x} + c_2e^{2x} + c_3e^{-3x}.$$

If $y(0) = 0$, $y'(0) = 0$, and $y''(0) = 1$ then

$$c_1 + c_2 + c_3 = 0, \quad -c_1 + 2c_2 - 3c_3 = 0, \quad c_1 + 4c_2 + 9c_3 = 1,$$

so $c_1 = -1/6$, $c_2 = 1/15$, $c_3 = 1/10$, and

$$y = -\frac{1}{6}e^{-x} + \frac{1}{15}e^{2x} + \frac{1}{10}e^{-3x}.$$

**37.** From $m^2 - 10m + 25 = 0$ we obtain $m_1 = 5$ and $m_2 = 5$ so that $y = c_1e^{5x} + c_2xe^{5x}$. If $y(0) = 1$ and $y(1) = 0$ then $c_1 = 1$, $c_1e^5 + c_2e^5 = 0$, so $c_1 = 1$, $c_2 = -1$, and $y = e^{5x} - xe^{5x}$.

**38.** From $m^2 + 4 = 0$ we obtain $m = \pm 2i$ so that $y = c_1 \cos 2x + c_2 \sin 2x$. If $y(0) = 0$ and $y(\pi) = 0$ then $c_1 = 0$ and $y = c_2 \sin 2x$.

**39.** From $m^2 + 1 = 0$ we obtain $m = \pm i$ so that $y = c_1 \cos x + c_2 \sin x$ and $y' = -c_1 \sin x + c_2 \cos x$. From $y'(0) = c_1(0) + c_2(1) = c_2 = 0$ and $y'(\pi/2) = -c_1(1) = 0$ we find $c_1 = c_2 = 0$. A solution of the boundary-value problem is $y = 0$.

**40.** From $m^2 - 2m + 2 = 0$ we obtain $m = 1 \pm i$ so that $y = e^x(c_1 \cos x + c_2 \sin x)$. If $y(0) = 1$ and $y(\pi) = 1$ then $c_1 = 1$ and $y(\pi) = e^\pi \cos \pi = -e^\pi$. Since $-e^\pi \neq 1$, the boundary-value problem has no solution.

**41.** The auxiliary equation is $m^2 - 3 = 0$ which has roots $-\sqrt{3}$ and $\sqrt{3}$. By (10) the general solution is $y = c_1e^{\sqrt{3}\,x} + c_2e^{-\sqrt{3}\,x}$. By (11) the general solution is $y = c_1 \cosh \sqrt{3}\,x + c_2 \sinh \sqrt{3}\,x$. For $y = c_1e^{\sqrt{3}\,x} + c_2e^{-\sqrt{3}\,x}$ the initial conditions imply $c_1 + c_2 = 1$, $\sqrt{3}\,c_1 - \sqrt{3}\,c_2 = 5$. Solving for $c_1$ and $c_2$ we find $c_1 = \frac{1}{2}(1 + 5\sqrt{3})$ and $c_2 = \frac{1}{2}(1 - 5\sqrt{3})$ so $y = \frac{1}{2}(1 + 5\sqrt{3})e^{\sqrt{3}\,x} + \frac{1}{2}(1 - 5\sqrt{3}\,)e^{-\sqrt{3}\,x}$. For $y = c_1 \cosh \sqrt{3}\,x + c_2 \sinh \sqrt{3}\,x$ the initial conditions imply $c_1 = 1$, $\sqrt{3}\,c_2 = 5$. Solving for $c_1$ and $c_2$ we find $c_1 = 1$ and $c_2 = \frac{5}{3}\sqrt{3}$ so $y = \cosh \sqrt{3}\,x + \frac{5}{3}\sqrt{3} \sinh \sqrt{3}\,x$.

**42.** The auxiliary equation is $m^2 - 1 = 0$ which has roots $-1$ and $1$. By (10) the general solution is $y = c_1e^x + c_2e^{-x}$. By (11) the general solution is $y = c_1 \cosh x + c_2 \sinh x$. For $y = c_1e^x + c_2e^{-x}$ the boundary conditions imply $c_1 + c_2 = 1$, $c_1e - c_2e^{-1} = 0$. Solving for $c_1$ and $c_2$ we find $c_1 = 1/(1 + e^2)$ and $c_2 = e^2/(1 + e^2)$ so $y = e^x/(1 + e^2) + e^2e^{-x}/(1 + e^2)$. For $y = c_1 \cosh x + c_2 \sinh x$ the boundary conditions imply $c_1 = 1$, $c_2 = -\tanh 1$, so $y = \cosh x - (\tanh 1) \sinh x$.

**43.** The auxiliary equation should have two positive roots, so that the solution has the form $y = c_1e^{k_1x} + c_2e^{k_2x}$. Thus, the differential equation is (f).

**44.** The auxiliary equation should have one positive and one negative root, so that the solution has the form $y = c_1e^{k_1x} + c_2e^{-k_2x}$. Thus, the differential equation is (a).

**45.** The auxiliary equation should have a pair of complex roots $\alpha \pm \beta i$ where $\alpha < 0$, so that the solution has the form $e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$. Thus, the differential equation is (e).

**46.** The auxiliary equation should have a repeated negative root, so that the solution has the form $y = c_1e^{-x} + c_2xe^{-x}$. Thus, the differential equation is (c).

**47.** The differential equation should have the form $y'' + k^2y = 0$ where $k = 1$ so that the period of the solution is $2\pi$. Thus, the differential equation is (d).

**48.** The differential equation should have the form $y'' + k^2y = 0$ where $k = 2$ so that the period of the solution is $\pi$. Thus, the differential equation is (b).

**49.** We have $(m - 1)(m - 5) = m^2 - 6m + 5$, so the differential equation is $y'' - 6y' + 5y = 0$.

**50.** We have $(m + 4)(m + 3) = m^2 + 7m + 12$, so the differential equation is $y'' + 7y' + 12y = 0$.

**51.** We have $m(m - 2) = m^2 - 2m$, so the differential equation is $y'' - 2y' = 0$.

**52.** We have $(m - 10)^2 = m^2 - 20m + 100$, so the differential equation is $y'' - 20y' + 100y = 0$.

**53.** We have $(m - 3i)(m + 3i) = m^2 + 9$, so the differential equation is $y'' + 9y = 0$.

**54.** We have $(m - 7)(m + 7) = m^2 - 49$, so the differential equation is $y'' - 49y = 0$.

**55.** We have $[m - (-1 + i)]\,[m - (-1 - i)] = m^2 + 2m + 2$, so the differential equation is $y'' + 2y' + 2y = 0$.

**56.** We have $m\,[m - (2 + 5i)]\,[m - (2 - 5i)] = m^3 - 4m^2 + 29m$, so the differential equation is $y''' - 4y'' + 29y' = 0$.

**57.** We have $m^2(m - 8) = m^3 - 8m^2$, so the differential equation is $y''' - 8y'' = 0$.

**58.** We have $(m^2 + 1)(m^2 + 4) = m^4 + 5m^2 + 29$, so the differential equation is $y^{(4)} + 5y'' + 4y = 0$.

**59.** A third root must be $m_3 = 3 - i$ and the auxiliary equation is

$$\left(m + \frac{1}{2}\right)[m - (3 + i)][m - (3 - i)] = \left(m + \frac{1}{2}\right)(m^2 - 6m + 10) = m^3 - \frac{11}{2}m^2 + 7m + 5.$$

The differential equation is

$$y''' - \frac{11}{2}y'' + 7y' + 5y = 0.$$

**60.** The auxiliary equation of $2y''' + 7y'' + 4y' = 0$ is $2m^3 + 7m^2 + 4m - 4 = 0$. Because $m_1 = \frac{1}{2}$ is a root of the equation it follows from the Factor Theorem of algebra that $m - \frac{1}{2}$ is a factor of $2m^3 + 7m^2 + 4m - 4$. By synthetic division we find that

$$2m^3 + 7m^2 + 4m - 4 = \left(m - \frac{1}{2}\right)\left(2m^2 + 8m + 8\right)$$

or

$$2m^3 + 7m^2 + 4m - 4 = (2m - 1)\left(m^2 + 4m + 4\right) = (2m - 1)(m + 2)^2$$

Thus the roots of the auxiliary equation are $m_1 = \frac{1}{2}$, $m_2 = m_3 = -2$, and the general solution of the differential equation is

$$y = c_1 e^{x/2} + c_2 e^{-2x} + c_3 x e^{-2x}.$$

**61.** From the solution $y_1 = e^{-4x} \cos x$ we conclude that $m_1 = -4 + i$ and $m_2 = -4 - i$ are roots of the auxiliary equation. Hence another solution must be $y_2 = e^{-4x} \sin x$. Now dividing the polynomial $m^3 + 6m^2 + m - 34$ by $[\,m - (-4 + i)]\,[m - (-4 - i)] = m^2 + 8m + 17$ gives $m - 2$. Therefore $m_3 = 2$ is the third root of the auxiliary equation, and the general solution of the differential equation is

$$y = c_1 e^{-4x} \cos x + c_2 e^{-4x} \sin x + c_3 e^{2x}.$$

**62.** Factoring the difference of two squares we obtain

$$m^4 + 1 = (m^2 + 1)^2 - 2m^2 = (m^2 + 1 - \sqrt{2}\,m)(m^2 + 1 + \sqrt{2}\,m) = 0.$$

Using the quadratic formula on each factor we get $m = \pm\sqrt{2}/2 \pm \sqrt{2}\,i/2$. The solution of the differential equation is

$$y(x) = e^{\sqrt{2}\,x/2}\left(c_1 \cos \frac{\sqrt{2}}{2}x + c_2 \sin \frac{\sqrt{2}}{2}x\right) + e^{-\sqrt{2}\,x/2}\left(c_3 \cos \frac{\sqrt{2}}{2}x + c_4 \sin \frac{\sqrt{2}}{2}x\right).$$

**63.** Using the definition of $\sinh x$ and the formula for the cosine of the sum of two angles, we have

$$\begin{aligned} y &= \sinh x - 2\cos(x + \pi/6) \\ &= \frac{1}{2}e^x - \frac{1}{2}e^{-x} - 2\left[(\cos x)\left(\cos\frac{\pi}{6}\right) - (\sin x)\left(\sin\frac{\pi}{6}\right)\right] \\ &= \frac{1}{2}e^x - \frac{1}{2}e^{-x} - 2\left(\frac{\sqrt{3}}{2}\cos x - \frac{1}{2}\sin x\right) \\ &= \frac{1}{2}e^x - \frac{1}{2}e^{-x} - \sqrt{3}\,\cos x + \sin x. \end{aligned}$$

This form of the solution can be obtained from the general solution
$y = c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x$ by choosing $c_1 = \frac{1}{2}$, $c_2 = -\frac{1}{2}$, $c_3 = -\sqrt{3}$, and $c_4 = 1$.

**64.** The auxiliary equation is $m^2 + \lambda = 0$ and we consider three cases where $\lambda = 0$, $\lambda = \alpha^2 > 0$, and $\lambda = -\alpha^2 < 0$:

**Case I**

When $\alpha = 0$ the general solution of the differential equation is $y = c_1 + c_2 x$. The boundary conditions imply $0 = y(0) = c_1$ and $0 = y(\pi/2) = c_2\pi/2$, so that $c_1 = c_2 = 0$ and the problem possesses only the trivial solution.

**Case II**

When $\lambda = -\alpha^2 < 0$ the general solution of the differential equation is $y = c_1 e^{\alpha x} + c_2 e^{-\alpha x}$, or alternatively, $y = c_1 \cosh(\alpha x) + c_2 \sinh(\alpha x)$. Again, $y(0) = 0$ implies $c_1 = 0$ so $y = c_2 \sinh(\alpha x)$. The second boundary condition implies $0 = y(\pi/2) = c_2 \sinh(\alpha\pi/2)$ or $c_2 = 0$. In this case also, the problem possesses only the trivial solution.

**Case III**

When $\lambda = \alpha^2 > 0$ the general solution of the differential equation is
$y = c_1 \cos(\alpha x) + c_2 \sin(\alpha x)$. In this case also, $y(0) = 0$ yields $c_1 = 0$, so that $y = c_2 \sin(\alpha x)$. The second boundary condition implies $0 = c_2 \sin(\alpha\pi/2)$. When $\alpha\,\pi/2$ is an integer multiple of $\pi$, that is, when $\alpha = 2k$ for $k$ a nonzero integer, the problem will have nontrivial solutions. Thus, for $\lambda = \alpha^2 = 4k^2$ the boundary-value problem will have nontrivial solutions $y = c_2 \sin(2kx)$, where $k$ is a nonzero integer. On the other hand, when $\alpha$ is not an even integer, the boundary-value problem will have only the trivial solution.

**65.** Using a CAS to solve the auxiliary equation $m^3 - 6m^2 + 2m + 1$ we find $m_1 = -0.270534$, $m_2 = 0.658675$, and $m_3 = 5.61186$. The general solution is

$$y = c_1 e^{-0.270534x} + c_2 e^{0.658675x} + c_3 e^{5.61186x}.$$

**66.** Using a CAS to solve the auxiliary equation $6.11m^3 + 8.59m^2 + 7.93m + 0.778 = 0$ we find $m_1 = -0.110241$, $m_2 = -0.647826 + 0.857532i$, and $m_3 = -0.647826 - 0.857532i$. The general solution is

$$y = c_1 e^{-0.110241x} + e^{-0.647826x}(c_2 \cos 0.857532x + c_3 \sin 0.857532x).$$

**67.** Using a CAS to solve the auxiliary equation $3.15m^4 - 5.34m^2 + 6.33m - 2.03 = 0$ we find $m_1 = -1.74806$, $m_2 = 0.501219$, $m_3 = 0.62342 + 0.588965i$, and $m_4 = 0.62342 - 0.588965i$. The general solution is

$$y = c_1 e^{-1.74806x} + c_2 e^{0.501219x} + e^{0.62342x}(c_3 \cos 0.588965x + c_4 \sin 0.588965x).$$

**68.** Using a CAS to solve the auxiliary equation $m^4 + 2m^2 - m + 2 = 0$ we find $m_1 = 1/2 + \sqrt{3}\,i/2$, $m_2 = 1/2 - \sqrt{3}\,i/2$, $m_3 = -1/2 + \sqrt{7}\,i/2$, and $m_4 = -1/2 - \sqrt{7}\,i/2$. The general solution is

$$y = e^{x/2}\left(c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right) + e^{-x/2}\left(c_3 \cos\frac{\sqrt{7}}{2}x + c_4 \sin\frac{\sqrt{7}}{2}x\right).$$

**69.** From $2m^4+3m^3-16m^2+15m-4=0$ we obtain $m_1=-4$, $m_2=\frac{1}{2}$, $m_3=1$, and $m_4=1$, so that $y=c_1e^{-4x}+c_2e^{x/2}+c_3e^x+c_4xe^x$. If $y(0)=-2$, $y'(0)=6$, $y''(0)=3$, and $y'''(0)=\frac{1}{2}$, then

$$\begin{aligned} c_1+c_2+c_3&=-2\\ -4c_1+\frac{1}{2}c_2+c_3+c_4&=6\\ 16c_1+\frac{1}{4}c_2+c_3+2c_4&=3\\ -64c_1+\frac{1}{8}c_2+c_3+3c_4&=\frac{1}{2}, \end{aligned}$$

so $c_1=-\frac{4}{75}$, $c_2=-\frac{116}{3}$, $c_3=\frac{918}{25}$, $c_4=-\frac{58}{5}$, and

$$y=-\frac{4}{75}e^{-4x}-\frac{116}{3}e^{x/2}+\frac{918}{25}e^x-\frac{58}{5}xe^x.$$

**70.** From $m^4-3m^3+3m^2-m=0$ we obtain $m_1=0$, $m_2=1$, $m_3=1$, and $m_4=1$ so that $y=c_1+c_2e^x+c_3xe^x+c_4x^2e^x$. If $y(0)=0$, $y'(0)=0$, $y''(0)=1$, and $y'''(0)=1$ then

$$c_1+c_2=0,\quad c_2+c_3=0,\quad c_2+2c_3+2c_4=1,\quad c_2+3c_3+6c_4=1,$$

so $c_1=2$, $c_2=-2$, $c_3=2$, $c_4=-1/2$, and

$$y=2-2e^x+2xe^x-\frac{1}{2}x^2e^x.$$

## 4.4 Undetermined Coefficients - Superposition Approach

**1.** From $m^2+3m+2=0$ we find $m_1=-1$ and $m_2=-2$. Then $y_c=c_1e^{-x}+c_2e^{-2x}$ and we assume $y_p=A$. Substituting into the differential equation we obtain $2A=6$. Then $A=3$, $y_p=3$ and

$$y=c_1e^{-x}+c_2e^{-2x}+3.$$

**2.** From $4m^2+9=0$ we find $m_1=-\frac{3}{2}i$ and $m_2=\frac{3}{2}i$. Then $y_c=c_1\cos\frac{3}{2}x+c_2\sin\frac{3}{2}x$ and we assume $y_p=A$. Substituting into the differential equation we obtain $9A=15$. Then $A=\frac{5}{3}$, $y_p=\frac{5}{3}$ and

$$y=c_1\cos\frac{3}{2}x+c_2\sin\frac{3}{2}x+\frac{5}{3}.$$

**3.** From $m^2-10m+25=0$ we find $m_1=m_2=5$. Then $y_c=c_1e^{5x}+c_2xe^{5x}$ and we assume $y_p=Ax+B$. Substituting into the differential equation we obtain $25A=30$ and $-10A+25B=3$. Then $A=\frac{6}{5}$, $B=\frac{3}{5}$, $y_p=\frac{6}{5}x+\frac{3}{5}$, and

$$y=c_1e^{5x}+c_2xe^{5x}+\frac{6}{5}x+\frac{3}{5}.$$

**4.** From $m^2+m-6=0$ we find $m_1=-3$ and $m_2=2$. Then $y_c=c_1e^{-3x}+c_2e^{2x}$ and we assume $y_p=Ax+B$. Substituting into the differential equation we obtain $-6A=2$ and $A-6B=0$. Then $A=-\frac{1}{3}$, $B=-\frac{1}{18}$, $y_p=-\frac{1}{3}x-\frac{1}{18}$, and

$$y=c_1e^{-3x}+c_2e^{2x}-\frac{1}{3}x-\frac{1}{18}\,.$$

**5.** From $\frac{1}{4}m^2+m+1=0$ we find $m_1=m_2=-2$. Then $y_c=c_1e^{-2x}+c_2xe^{-2x}$ and we assume $y_p=Ax^2+Bx+C$. Substituting into the differential equation we obtain $A=1$, $2A+B=-2$, and $\frac{1}{2}A+B+C=0$. Then $A=1$, $B=-4$, $C=\frac{7}{2}$, $y_p=x^2-4x+\frac{7}{2}$, and

$$y=c_1e^{-2x}+c_2xe^{-2x}+x^2-4x+\frac{7}{2}\,.$$

**6.** From $m^2-8m+20=0$ we find $m_1=4+2i$ and $m_2=4-2i$. Then $y_c=e^{4x}(c_1\cos 2x+c_2\sin 2x)$ and we assume $y_p=Ax^2+Bx+C+(Dx+E)e^x$. Substituting into the differential equation we obtain

$$\begin{aligned}2A-8B+20C&=0\\ -6D+13E&=0\\ -16A+20B&=0\\ 13D&=-26\\ 20A&=100.\end{aligned}$$

Then $A=5$, $B=4$, $C=\frac{11}{10}$, $D=-2$, $E=-\frac{12}{13}$, $y_p=5x^2+4x+\frac{11}{10}+\left(-2x-\frac{12}{13}\right)e^x$ and

$$y=e^{4x}(c_1\cos 2x+c_2\sin 2x)+5x^2+4x+\frac{11}{10}+\left(-2x-\frac{12}{13}\right)e^x.$$

**7.** From $m^2+3=0$ we find $m_1=\sqrt{3}\,i$ and $m_2=-\sqrt{3}\,i$. Then $y_c=c_1\cos\sqrt{3}\,x+c_2\sin\sqrt{3}\,x$ and we assume $y_p=(Ax^2+Bx+C)e^{3x}$. Substituting into the differential equation we obtain $2A+6B+12C=0$, $12A+12B=0$, and $12A=-48$. Then $A=-4$, $B=4$, $C=-\frac{4}{3}$, $y_p=\left(-4x^2+4x-\frac{4}{3}\right)e^{3x}$ and

$$y=c_1\cos\sqrt{3}\,x+c_2\sin\sqrt{3}\,x+\left(-4x^2+4x-\frac{4}{3}\right)e^{3x}.$$

**8.** From $4m^2-4m-3=0$ we find $m_1=\frac{3}{2}$ and $m_2=-\frac{1}{2}$. Then $y_c=c_1e^{3x/2}+c_2e^{-x/2}$ and we assume $y_p=A\cos 2x+B\sin 2x$. Substituting into the differential equation we obtain $-19-8B=1$ and $8A-19B=0$. Then $A=-\frac{19}{425}$, $B=-\frac{8}{425}$, $y_p=-\frac{19}{425}\cos 2x-\frac{8}{425}\sin 2x$, and

$$y=c_1e^{3x/2}+c_2e^{-x/2}-\frac{19}{425}\cos 2x-\frac{8}{425}\sin 2x.$$

**9.** From $m^2-m=0$ we find $m_1=1$ and $m_2=0$. Then $y_c=c_1e^x+c_2$ and we assume $y_p=Ax$. Substituting into the differential equation we obtain $-A=-3$. Then $A=3$, $y_p=3x$ and $y=c_1e^x+c_2+3x$.

**10.** From $m^2 + 2m = 0$ we find $m_1 = -2$ and $m_2 = 0$. Then $y_c = c_1e^{-2x} + c_2$ and we assume $y_p = Ax^2 + Bx + Cxe^{-2x}$. Substituting into the differential equation we obtain $2A + 2B = 5$, $4A = 2$, and $-2C = -1$. Then $A = \frac{1}{2}$, $B = 2$, $C = \frac{1}{2}$, $y_p = \frac{1}{2}x^2 + 2x + \frac{1}{2}xe^{-2x}$, and

$$y = c_1e^{-2x} + c_2 + \frac{1}{2}x^2 + 2x + \frac{1}{2}xe^{-2x}.$$

**11.** From $m^2 - m + \frac{1}{4} = 0$ we find $m_1 = m_2 = \frac{1}{2}$. Then $y_c = c_1e^{x/2} + c_2xe^{x/2}$ and we assume $y_p = A + Bx^2e^{x/2}$. Substituting into the differential equation we obtain $\frac{1}{4}A = 3$ and $2B = 1$. Then $A = 12$, $B = \frac{1}{2}$, $y_p = 12 + \frac{1}{2}x^2e^{x/2}$, and

$$y = c_1e^{x/2} + c_2xe^{x/2} + 12 + \frac{1}{2}x^2e^{x/2}.$$

**12.** From $m^2 - 16 = 0$ we find $m_1 = 4$ and $m_2 = -4$. Then $y_c = c_1e^{4x} + c_2e^{-4x}$ and we assume $y_p = Axe^{4x}$. Substituting into the differential equation we obtain $8A = 2$. Then $A = \frac{1}{4}$, $y_p = \frac{1}{4}xe^{4x}$ and

$$y = c_1e^{4x} + c_2e^{-4x} + \frac{1}{4}xe^{4x}.$$

**13.** From $m^2 + 4 = 0$ we find $m_1 = 2i$ and $m_2 = -2i$. Then $y_c = c_1\cos 2x + c_2\sin 2x$ and we assume $y_p = Ax\cos 2x + Bx\sin 2x$. Substituting into the differential equation we obtain $4B = 0$ and $-4A = 3$. Then $A = -\frac{3}{4}$, $B = 0$, $y_p = -\frac{3}{4}x\cos 2x$, and

$$y = c_1\cos 2x + c_2\sin 2x - \frac{3}{4}x\cos 2x.$$

**14.** From $m^2 - 4 = 0$ we find $m_1 = 2$ and $m_2 = -2$. Then $y_c = c_1e^{2x} + c_2e^{-2x}$ and we assume that $y_p = (Ax^2 + Bx + C)\cos 2x + (Dx^2 + Ex + F)\sin 2x$. Substituting into the differential equation we obtain

$$\begin{aligned} -8A &= 0 \\ -8B + 8D &= 0 \\ 2A - 8C + 4E &= 0 \\ -8D &= 1 \\ -8A - 8E &= 0 \\ -4B + 2D - 8F &= -3 \end{aligned}$$

Then $A = 0$, $B = -\frac{1}{8}$, $C = 0$, $D = -\frac{1}{8}$, $E = 0$, $F = \frac{13}{32}$, so
$y_p = -\frac{1}{8}\,x\cos 2x + \left(-\frac{1}{8}\,x^2 + \frac{13}{32}\right)\sin 2x$, and

$$y = c_1e^{2x} + c_2e^{-2x} - \frac{1}{8}\,x\cos 2x + \left(-\frac{1}{8}\,x^2 + \frac{13}{32}\right)\sin 2x.$$

**15.** From $m^2+1=0$ we find $m_1=i$ and $m_2=-i$. Then $y_c=c_1\cos x+c_2\sin x$ and we assume $y_p=(Ax^2+Bx)\cos x+(Cx^2+Dx)\sin x$. Substituting into the differential equation we obtain $4C=0$, $2A+2D=0$, $-4A=2$, and $-2B+2C=0$. Then $A=-\frac{1}{2}$, $B=0$, $C=0$, $D=\frac{1}{2}$, $y_p=-\frac{1}{2}x^2\cos x+\frac{1}{2}x\sin x$, and

$$y=c_1\cos x+c_2\sin x-\frac{1}{2}x^2\cos x+\frac{1}{2}x\sin x.$$

**16.** From $m^2-5m=0$ we find $m_1=5$ and $m_2=0$. Then $y_c=c_1e^{5x}+c_2$ and we assume $y_p=Ax^4+Bx^3+Cx^2+Dx$. Substituting into the differential equation we obtain $-20A=2$, $12A-15B=-4$, $6B-10C=-1$, and $2C-5D=6$. Then $A=-\frac{1}{10}$, $B=\frac{14}{75}$, $C=\frac{53}{250}$, $D=-\frac{697}{625}$, $y_p=-\frac{1}{10}x^4+\frac{14}{75}x^3+\frac{53}{250}x^2-\frac{697}{625}x$, and

$$y=c_1e^{5x}+c_2-\frac{1}{10}x^4+\frac{14}{75}x^3+\frac{53}{250}x^2-\frac{697}{625}x.$$

**17.** From $m^2-2m+5=0$ we find $m_1=1+2i$ and $m_2=1-2i$. Then $y_c=e^x(c_1\cos 2x+c_2\sin 2x)$ and we assume $y_p=Axe^x\cos 2x+Bxe^x\sin 2x$. Substituting into the differential equation we obtain $4B=1$ and $-4A=0$. Then $A=0$, $B=\frac{1}{4}$, $y_p=\frac{1}{4}xe^x\sin 2x$, and

$$y=e^x(c_1\cos 2x+c_2\sin 2x)+\frac{1}{4}xe^x\sin 2x.$$

**18.** From $m^2-2m+2=0$ we find $m_1=1+i$ and $m_2=1-i$. Then $y_c=e^x(c_1\cos x+c_2\sin x)$ and we assume $y_p=Ae^{2x}\cos x+Be^{2x}\sin x$. Substituting into the differential equation we obtain $A+2B=1$ and $-2A+B=-3$. Then $A=\frac{7}{5}$, $B=-\frac{1}{5}$, $y_p=\frac{7}{5}e^{2x}\cos x-\frac{1}{5}e^{2x}\sin x$ and

$$y=e^x(c_1\cos x+c_2\sin x)+\frac{7}{5}e^{2x}\cos x-\frac{1}{5}e^{2x}\sin x.$$

**19.** From $m^2+2m+1=0$ we find $m_1=m_2=-1$. Then $y_c=c_1e^{-x}+c_2xe^{-x}$ and we assume $y_p=A\cos x+B\sin x+C\cos 2x+D\sin 2x$. Substituting into the differential equation we obtain $2B=0$, $-2A=1$, $-3C+4D=3$, and $-4C-3D=0$. Then $A=-\frac{1}{2}$, $B=0$, $C=-\frac{9}{25}$, $D=\frac{12}{25}$, $y_p=-\frac{1}{2}\cos x-\frac{9}{25}\cos 2x+\frac{12}{25}\sin 2x$, and

$$y=c_1e^{-x}+c_2xe^{-x}-\frac{1}{2}\cos x-\frac{9}{25}\cos 2x+\frac{12}{25}\sin 2x.$$

**20.** From $m^2+2m-24=0$ we find $m_1=-6$ and $m_2=4$. Then $y_c=c_1e^{-6x}+c_2e^{4x}$ and we assume $y_p=A+(Bx^2+Cx)e^{4x}$. Substituting into the differential equation we obtain $-24A=16$, $2B+10C=-2$, and $20B=-1$. Then $A=-\frac{2}{3}$, $B=-\frac{1}{20}$, $C=-\frac{19}{100}$, $y_p=-\frac{2}{3}-\left(\frac{1}{20}x^2+\frac{19}{100}x\right)e^{4x}$, and

$$y=c_1e^{-6x}+c_2e^{4x}-\frac{2}{3}-\left(\frac{1}{20}x^2+\frac{19}{100}x\right)e^{4x}.$$

**21.** From $m^3 - 6m^2 = 0$ we find $m_1 = m_2 = 0$ and $m_3 = 6$. Then $y_c = c_1 + c_2x + c_3e^{6x}$ and we assume $y_p = Ax^2 + B\cos x + C\sin x$. Substituting into the differential equation we obtain $-12A = 3$, $6B - C = -1$, and $B + 6C = 0$. Then $A = -\frac{1}{4}$, $B = -\frac{6}{37}$, $C = \frac{1}{37}$, $y_p = -\frac{1}{4}x^2 - \frac{6}{37}\cos x + \frac{1}{37}\sin x$, and

$$y = c_1 + c_2x + c_3e^{6x} - \frac{1}{4}x^2 - \frac{6}{37}\cos x + \frac{1}{37}\sin x.$$

**22.** From $m^3 - 2m^2 - 4m + 8 = 0$ we find $m_1 = m_2 = 2$ and $m_3 = -2$. Then $y_c = c_1e^{2x} + c_2xe^{2x} + c_3e^{-2x}$ and we assume $y_p = (Ax^3 + Bx^2)e^{2x}$. Substituting into the differential equation we obtain $24A = 6$ and $6A + 8B = 0$. Then $A = \frac{1}{4}$, $B = -\frac{3}{16}$, $y_p = \left(\frac{1}{4}x^3 - \frac{3}{16}x^2\right)e^{2x}$, and

$$y = c_1e^{2x} + c_2xe^{2x} + c_3e^{-2x} + \left(\frac{1}{4}x^3 - \frac{3}{16}x^2\right)e^{2x}.$$

**23.** From $m^3 - 3m^2 + 3m - 1 = 0$ we find $m_1 = m_2 = m_3 = 1$. Then $y_c = c_1e^x + c_2xe^x + c_3x^2e^x$ and we assume $y_p = Ax + B + Cx^3e^x$. Substituting into the differential equation we obtain $-A = 1$, $3A - B = 0$, and $6C = -4$. Then $A = -1$, $B = -3$, $C = -\frac{2}{3}$, $y_p = -x - 3 - \frac{2}{3}x^3e^x$, and

$$y = c_1e^x + c_2xe^x + c_3x^2e^x - x - 3 - \frac{2}{3}x^3e^x.$$

**24.** From $m^3 - m^2 - 4m + 4 = 0$ we find $m_1 = 1$, $m_2 = 2$, and $m_3 = -2$. Then $y_c = c_1e^x + c_2e^{2x} + c_3e^{-2x}$ and we assume $y_p = A + Bxe^x + Cxe^{2x}$. Substituting into the differential equation we obtain $4A = 5$, $-3B = -1$, and $4C = 1$. Then $A = \frac{5}{4}$, $B = \frac{1}{3}$, $C = \frac{1}{4}$, $y_p = \frac{5}{4} + \frac{1}{3}xe^x + \frac{1}{4}xe^{2x}$, and

$$y = c_1e^x + c_2e^{2x} + c_3e^{-2x} + \frac{5}{4} + \frac{1}{3}xe^x + \frac{1}{4}xe^{2x}.$$

**25.** From $m^4 + 2m^2 + 1 = 0$ we find $m_1 = m_3 = i$ and $m_2 = m_4 = -i$. Then $y_c = c_1\cos x + c_2\sin x + c_3x\cos x + c_4x\sin x$ and we assume $y_p = Ax^2 + Bx + C$. Substituting into the differential equation we obtain $A = 1$, $B = -2$, and $4A + C = 1$. Then $A = 1$, $B = -2$, $C = -3$, $y_p = x^2 - 2x - 3$, and

$$y = c_1\cos x + c_2\sin x + c_3x\cos x + c_4x\sin x + x^2 - 2x - 3.$$

**26.** From $m^4 - m^2 = 0$ we find $m_1 = m_2 = 0$, $m_3 = 1$, and $m_4 = -1$. Then $y_c = c_1 + c_2x + c_3e^x + c_4e^{-x}$ and we assume $y_p = Ax^3 + Bx^2 + (Cx^2 + Dx)e^{-x}$. Substituting into the differential equation we obtain $-6A = 4$, $-2B = 0$, $10C - 2D = 0$, and $-4C = 2$. Then $A = -\frac{2}{3}$, $B = 0$, $C = -\frac{1}{2}$, $D = -\frac{5}{2}$, $y_p = -\frac{2}{3}x^3 - \left(\frac{1}{2}x^2 + \frac{5}{2}x\right)e^{-x}$, and

$$y = c_1 + c_2x + c_3e^x + c_4e^{-x} - \frac{2}{3}x^3 - \left(\frac{1}{2}x^2 + \frac{5}{2}x\right)e^{-x}.$$

**27.** We have $y_c = c_1 \cos 2x + c_2 \sin 2x$ and we assume $y_p = A$. Substituting into the differential equation we find $A = -\frac{1}{2}$. Thus $y = c_1 \cos 2x + c_2 \sin 2x - \frac{1}{2}$. From the initial conditions we obtain $c_1 = 0$ and $c_2 = \sqrt{2}$, so

$$y = \sqrt{2}\,\sin 2x - \frac{1}{2}\,.$$

**28.** We have $y_c = c_1 e^{-2x} + c_2 e^{x/2}$ and we assume $y_p = Ax^2 + Bx + C$. Substituting into the differential equation we find $A = -7$, $B = -19$, and $C = -37$. Thus $y = c_1 e^{-2x} + c_2 e^{x/2} - 7x^2 - 19x - 37$. From the initial conditions we obtain $c_1 = -\frac{1}{5}$ and $c_2 = \frac{186}{5}$, so

$$y = -\frac{1}{5}e^{-2x} + \frac{186}{5}\,e^{x/2} - 7x^2 - 19x - 37.$$

**29.** We have $y_c = c_1 e^{-x/5} + c_2$ and we assume $y_p = Ax^2 + Bx$. Substituting into the differential equation we find $A = -3$ and $B = 30$. Thus $y = c_1 e^{-x/5} + c_2 - 3x^2 + 30x$. From the initial conditions we obtain $c_1 = 200$ and $c_2 = -200$, so

$$y = 200e^{-x/5} - 200 - 3x^2 + 30x.$$

**30.** We have $y_c = c_1 e^{-2x} + c_2 x e^{-2x}$ and we assume $y_p = (Ax^3 + Bx^2)e^{-2x}$. Substituting into the differential equation we find $A = \frac{1}{6}$ and $B = \frac{3}{2}$. Thus $y = c_1 e^{-2x} + c_2 x e^{-2x} + \left(\frac{1}{6}x^3 + \frac{3}{2}x^2\right) e^{-2x}$. From the initial conditions we obtain $c_1 = 2$ and $c_2 = 9$, so

$$y = 2e^{-2x} + 9xe^{-2x} + \left(\frac{1}{6}x^3 + \frac{3}{2}x^2\right) e^{-2x}.$$

**31.** We have $y_c = e^{-2x}(c_1 \cos x + c_2 \sin x)$ and we assume $y_p = Ae^{-4x}$. Substituting into the differential equation we find $A = 7$. Thus $y = e^{-2x}(c_1 \cos x + c_2 \sin x) + 7e^{-4x}$. From the initial conditions we obtain $c_1 = -10$ and $c_2 = 9$, so

$$y = e^{-2x}(-10 \cos x + 9 \sin x) + 7e^{-4x}.$$

**32.** We have $y_c = c_1 \cosh x + c_2 \sinh x$ and we assume $y_p = Ax \cosh x + Bx \sinh x$. Substituting into the differential equation we find $A = 0$ and $B = \frac{1}{2}$. Thus

$$y = c_1 \cosh x + c_2 \sinh x + \frac{1}{2}x \sinh x.$$

From the initial conditions we obtain $c_1 = 2$ and $c_2 = 12$, so

$$y = 2 \cosh x + 12 \sinh x + \frac{1}{2}x \sinh x.$$

**33.** We have $x_c = c_1 \cos \omega t + c_2 \sin \omega t$ and we assume $x_p = At \cos \omega t + Bt \sin \omega t$. Substituting into the differential equation we find $A = -F_0/2\omega$ and $B = 0$. Thus $x = c_1 \cos \omega t + c_2 \sin \omega t - (F_0/2\omega)t \cos \omega t$. From the initial conditions we obtain $c_1 = 0$ and $c_2 = F_0/2\omega^2$, so

$$x = (F_0/2\omega^2) \sin \omega t - (F_0/2\omega)t \cos \omega t.$$

**34.** We have $x_c = c_1 \cos \omega t + c_2 \sin \omega t$ and we assume $x_p = A\cos\gamma t + B\sin\gamma t$, where $\gamma \neq \omega$. Substituting into the differential equation we find $A = F_0/(\omega^2 - \gamma^2)$ and $B = 0$. Thus

$$x = c_1 \cos \omega t + c_2 \sin \omega t + \frac{F_0}{\omega^2 - \gamma^2} \cos \gamma t.$$

From the initial conditions we obtain $c_1 = -F_0/(\omega^2 - \gamma^2)$ and $c_2 = 0$, so

$$x = -\frac{F_0}{\omega^2 - \gamma^2} \cos \omega t + \frac{F_0}{\omega^2 - \gamma^2} \cos \gamma t.$$

**35.** We have $y_c = c_1 + c_2 e^x + c_3 x e^x$ and we assume $y_p = Ax + Bx^2 e^x + Ce^{5x}$. Substituting into the differential equation we find $A = 2$, $B = -12$, and $C = \frac{1}{2}$. Thus

$$y = c_1 + c_2 e^x + c_3 x e^x + 2x - 12x^2 e^x + \frac{1}{2} e^{5x}.$$

From the initial conditions we obtain $c_1 = 11$, $c_2 = -11$, and $c_3 = 9$, so

$$y = 11 - 11e^x + 9xe^x + 2x - 12x^2 e^x + \frac{1}{2} e^{5x}.$$

**36.** We have $y_c = c_1 e^{-2x} + e^x(c_2 \cos \sqrt{3}\, x + c_3 \sin \sqrt{3}\, x)$ and we assume $y_p = Ax + B + Cxe^{-2x}$. Substituting into the differential equation we find $A = \frac{1}{4}$, $B = -\frac{5}{8}$, and $C = \frac{2}{3}$. Thus

$$y = c_1 e^{-2x} + e^x(c_2 \cos \sqrt{3}\, x + c_3 \sin \sqrt{3}\, x) + \frac{1}{4}x - \frac{5}{8} + \frac{2}{3} x e^{-2x}.$$

From the initial conditions we obtain $c_1 = -\frac{23}{12}$, $c_2 = -\frac{59}{24}$, and $c_3 = \frac{17}{72}\sqrt{3}$, so

$$y = -\frac{23}{12} e^{-2x} + e^x \left( -\frac{59}{24} \cos \sqrt{3}\, x + \frac{17}{72} \sqrt{3} \sin \sqrt{3}\, x \right) + \frac{1}{4}x - \frac{5}{8} + \frac{2}{3} x e^{-2x}.$$

**37.** We have $y_c = c_1 \cos x + c_2 \sin x$ and we assume $y_p = Ax^2 + Bx + C$. Substituting into the differential equation we find $A = 1$, $B = 0$, and $C = -1$. Thus $y = c_1 \cos x + c_2 \sin x + x^2 - 1$. From $y(0) = 5$ and $y(1) = 0$ we obtain

$$\begin{aligned} c_1 - 1 &= 5 \\ (\cos 1)c_1 + (\sin 1)c_2 &= 0. \end{aligned}$$

Solving this system we find $c_1 = 6$ and $c_2 = -6 \cot 1$. The solution of the boundary-value problem is

$$y = 6 \cos x - 6(\cot 1) \sin x + x^2 - 1.$$

**38.** We have $y_c = e^x(c_1 \cos x + c_2 \sin x)$ and we assume $y_p = Ax + B$. Substituting into the differential equation we find $A = 1$ and $B = 0$. Thus $y = e^x(c_1 \cos x + c_2 \sin x) + x$. From $y(0) = 0$ and $y(\pi) = \pi$ we obtain

$$\begin{aligned} c_1 &= 0 \\ \pi - e^{\pi} c_1 &= \pi. \end{aligned}$$

Solving this system we find $c_1 = 0$ and $c_2$ is any real number. The solution of the boundary-value problem is

$$y = c_2 e^x \sin x + x.$$

**39.** The general solution of the differential equation $y'' + 3y = 6x$ is $y = c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x + 2x$. The condition $y(0) = 0$ implies $c_1 = 0$ and so $y = c_2 \sin\sqrt{3}x + 2x$. The condition $y(1) + y'(1) = 0$ implies $c_2 \sin\sqrt{3} + 2 + c_2\sqrt{3}\cos\sqrt{3} + 2 = 0$ so $c_2 = -4/(\sin\sqrt{3} + \sqrt{3}\cos\sqrt{3})$. The solution is

$$y = \frac{-4\sin\sqrt{3}x}{\sin\sqrt{3} + \sqrt{3}\cos\sqrt{3}} + 2x.$$

**40.** Using the general solution $y = c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x + 2x$, the boundary conditions $y(0) + y'(0) = 0$, $y(1) = 0$ yield the system

$$\begin{aligned} c_1 + \sqrt{3}\,c_2 + 2 &= 0 \\ c_1 \cos\sqrt{3} + c_2 \sin\sqrt{3} + 2 &= 0. \end{aligned}$$

Solving gives

$$c_1 = \frac{2\left(-\sqrt{3} + \sin\sqrt{3}\right)}{\sqrt{3}\cos\sqrt{3} - \sin\sqrt{3}} \qquad \text{and} \qquad c_2 = \frac{2\left(1 - \cos\sqrt{3}\right)}{\sqrt{3}\cos\sqrt{3} - \sin\sqrt{3}}.$$

Thus,

$$y = \frac{2\left(-\sqrt{3} + \sin\sqrt{3}\right)\cos\sqrt{3}\,x}{\sqrt{3}\,\cos\sqrt{3} - \sin\sqrt{3}} + \frac{2\left(1 - \cos\sqrt{3}\right)\sin\sqrt{3}\,x}{\sqrt{3}\cos\sqrt{3} - \sin\sqrt{3}} + 2x.$$

**41.** We have $y_c = c_1 \cos 2x + c_2 \sin 2x$ and we assume $y_p = A\cos x + B\sin x$ on $[0, \pi/2]$. Substituting into the differential equation we find $A = 0$ and $B = \frac{1}{3}$. Thus $y = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{3}\sin x$ on $[0, \pi/2]$. On $(\pi/2, \infty)$ we have $y = c_3 \cos 2x + c_4 \sin 2x$. From $y(0) = 1$ and $y'(0) = 2$ we obtain

$$\begin{aligned} c_1 &= 1 \\ \frac{1}{3} + 2c_2 &= 2. \end{aligned}$$

Solving this system we find $c_1 = 1$ and $c_2 = \frac{5}{6}$. Thus $y = \cos 2x + \frac{5}{6}\sin 2x + \frac{1}{3}\sin x$ on $[0, \pi/2]$. Now continuity of $y$ at $x = \pi/2$ implies

$$\cos\pi + \frac{5}{6}\sin\pi + \frac{1}{3}\sin\frac{\pi}{2} = c_3 \cos\pi + c_4 \sin\pi$$

or $-1 + \frac{1}{3} = -c_3$. Hence $c_3 = \frac{2}{3}$. Continuity of $y'$ at $x = \pi/2$ implies

$$-2\sin\pi + \frac{5}{3}\cos\pi + \frac{1}{3}\cos\frac{\pi}{2} = -2c_3 \sin\pi + 2c_4 \cos\pi$$

or $-\frac{5}{3} = -2c_4$. Then $c_4 = \frac{5}{6}$ and so

$$y(x) = \begin{cases} \cos 2x + \dfrac{5}{6}\sin 2x + \dfrac{1}{3}\sin x, & 0 \le x \le \pi/2 \\ \dfrac{2}{3}\cos 2x + \dfrac{5}{6}\sin 2x, & x > \pi/2 \end{cases}$$

**42.** We have $y_c = e^x(c_1 \cos 3x + c_2 \sin 3x)$ and we assume $y_p = A$ on $[0, \pi]$. Substituting into the differential equation we find $A = 2$. Thus, $y = e^x(c_1 \cos 3x + c_2 \sin 3x) + 2$ on $[0, \pi]$. On $(\pi, \infty)$ we have $y = e^x(c_3 \cos 3x + c_4 \sin 3x)$. From $y(0) = 0$ and $y'(0) = 0$ we obtain

$$c_1 = -2, \qquad c_1 + 3c_2 = 0.$$

Solving this system, we find $c_1 = -2$ and $c_2 = \frac{2}{3}$. Thus $y = e^x(-2\cos 3x + \frac{2}{3}\sin 3x) + 2$ on $[0, \pi]$. Now, continuity of $y$ at $x = \pi$ implies

$$e^{\pi}(-2\cos 3\pi + \frac{2}{3}\sin 3\pi) + 2 = e^{\pi}(c_3 \cos 3\pi + c_4 \sin 3\pi)$$

or $2 + 2e^{\pi} = -c_3 e^{\pi}$ or $c_3 = -2e^{-\pi}(1 + e^{\pi})$. Continuity of $y'$ at $\pi$ implies

$$\frac{20}{3} e^{\pi} \sin 3\pi = e^{\pi}[(c_3 + 3c_4)\cos 3\pi + (-3c_3 + c_4)\sin 3\pi]$$

or $-c_3 e^{\pi} - 3c_4 e^{\pi} = 0$. Since $c_3 = -2e^{-\pi}(1 + e^{\pi})$ we have $c_4 = \frac{2}{3}e^{-\pi}(1 + e^{\pi})$. Therefore

$$y(x) = \begin{cases} e^x(-2\cos 3x + \dfrac{2}{3}\sin 3x) + 2, & 0 \le x \le \pi \\ (1 + e^{\pi})e^{x-\pi}(-2\cos 3x + \dfrac{2}{3}\sin 3x), & x > \pi. \end{cases}$$

**43.** **(a)** From $y_p = Ae^{kx}$ we find $y_p' = Ake^{kx}$ and $y_p'' = Ak^2 e^{kx}$. Substituting into the differential equation we get

$$aAk^2 e^{kx} + bAke^{kx} + cAe^{kx} = (ak^2 + bk + c)Ae^{kx} = e^{kx},$$

so $(ak^2 + bk + c)A = 1$. Since $k$ is not a root of $am^2 + bm + c = 0$, $A = 1/(ak^2 + bk + c)$.

**(b)** From $y_p = Axe^{kx}$ we find $y_p' = Akxe^{kx} + Ae^{kx}$ and $y_p'' = Ak^2 xe^{kx} + 2Ake^{kx}$. Substituting into the differential equation we get

$$\begin{aligned} &aAk^2 xe^{kx} + 2aAke^{kx} + bAkxe^{kx} + bAe^{kx} + cAxe^{kx} \\ &\qquad = (ak^2 + bk + c)Axe^{kx} + (2ak + b)Ae^{kx} \\ &\qquad = (0)Axe^{kx} + (2ak + b)Ae^{kx} = (2ak + b)Ae^{kx} = e^{kx} \end{aligned}$$

where $ak^2 + bk + c = 0$ because $k$ is a root of the auxiliary equation. Now, the roots of the auxiliary equation are $-b/2a \pm \sqrt{b^2 - 4ac}\,/2a$, and since $k$ is a root of multiplicity one, $k \neq -b/2a$ and $2ak + b \neq 0$. Thus $(2ak + b)A = 1$ and $A = 1/(2ak + b)$.

**(c)** If $k$ is a root of multiplicity two, then, as we saw in part (b), $k = -b/2a$ and $2ak + b = 0$. From $y_p = Ax^2 e^{kx}$ we find $y_p' = Akx^2 e^{kx} + 2Axe^{kx}$ and $y_p'' = Ak^2 x^2 e^{kx} + 4Akxe^{kx} = 2Ae^{kx}$. Substituting into the differential equation, we get

$$\begin{aligned} &aAk^2 x^2 e^{kx} + 4aAkxe^{kx} + 2aAe^{kx} + bAkx^2 e^{kx} + 2bAxe^{kx} + cAx^2 e^{kx} \\ &\qquad = (ak^2 + bk + c)Ax^2 e^{kx} + 2(2ak + b)Axe^{kx} + 2aAe^{kx} \\ &\qquad = (0)Ax^2 e^{kx} + 2(0)Axe^{kx} + 2aAe^{kx} = 2aAe^{kx} = e^{kx}. \end{aligned}$$

Since the differential equation is second order, $a \neq 0$ and $A = 1/(2a)$.

**44.** Using the double-angle formula for the cosine, we have

$$\sin x\cos 2x = \sin x(\cos^2 x - \sin^2 x) = \sin x(1 - 2\sin^2 x) = \sin x - 2\sin^3 x.$$

Since $\sin x$ is a solution of the related homogeneous differential equation we look for a particular solution of the form $y_p = Ax\sin x + Bx\cos x + C\sin^3 x$.

Substituting into the differential equation we obtain

$$2A\cos x + (6C - 2B)\sin x - 8C\sin^3 x = \sin x - 2\sin^3 x.$$

Equating coefficients we find $A = 0$, $C = \frac{1}{4}$, and $B = \frac{1}{4}$. Thus, a particular solution is

$$y_p = \frac{1}{4}x\cos x + \frac{1}{4}\sin^3 x.$$

**45.** $f(x) = e^x \sin x$. We see that $y_p \to \infty$ as $x \to \infty$ and $y_p \to 0$ as $x \to -\infty$.

**46.** $f(x) = e^{-x}$. We see that $y_p \to 0$ as $x \to \infty$ and $y_p \to \infty$ as $x \to -\infty$.

**47.** $f(x) = \sin 2x$. We see that $y_p$ is sinusoidal.

**48.** $f(x) = 1$. We see that $y_p$ is constant and simply translates $y_c$ vertically.

**49.** The complementary function is $y_c = e^{2x}(c_1\cos 2x + c_2\sin 2x)$. We assume a particular solution of the form $y_p = (Ax^3 + Bx^2 + Cx)e^{2x}\cos 2x + (Dx^3 + Ex^2 + F)e^{2x}\sin 2x$. Substituting into the differential equation and using a CAS to simplify yields

$$\begin{aligned}&[12Dx^2 + (6A + 8E)x + (2B + 4F)]e^{2x}\cos 2x\\ &\qquad + [-12Ax^2 + (-8B + 6D)x + (-4C + 2E)]e^{2x}\sin 2x\\ &\qquad\qquad = (2x^2 - 3x)e^{2x}\cos 2x + (10x^2 - x - 1)e^{2x}\sin 2x\end{aligned}$$

This gives the system of equations

$$\begin{aligned}12D &= 2, & 6A + 8E &= -3 & 2B + 4F &= 0,\\ -12A &= 10, & -8B + 6D &= -1 & -4C + 2E &= -1,\end{aligned}$$

from which we find $A = -\frac{5}{6}$, $B = \frac{1}{4}$, $C = \frac{3}{8}$, $D = \frac{1}{6}$, $E = \frac{1}{4}$, and $F = -\frac{1}{8}$. Thus, a particular solution of the differential equation is

$$y_p = \left(-\frac{5}{6}x^3 + \frac{1}{4}x^2 + \frac{3}{8}x\right)e^{2x}\cos 2x + \left(\frac{1}{6}x^3 + \frac{1}{4}x^2 - \frac{1}{8}x\right)e^{2x}\sin 2x.$$

**50.** The complementary function is $y_c = c_1\cos x + c_2\sin x + c_3x\cos x + c_4x\sin x$. We assume a particular solution of the form $y_p = Ax^2\cos x + Bx^3\sin x$. Substituting into the differential equation and using a CAS to simplify yields

$$(-8A + 24B)\cos x + 3Bx\sin x = 2\cos x - 3x\sin x.$$

This implies $-8A + 24B = 2$ and $-24B = -3$. Thus $B = \frac{1}{8}$, $A = \frac{1}{8}$, and
$y_p = \frac{1}{8}x^2\cos x + \frac{1}{8}x^3\sin x$.

## 4.5 Undetermined Coefficients - Superposition Approach

**1.** $(9D^2-4)y=(3D-2)(3D+2)y=\sin x$

**2.** $(D^2-5)y=(D-\sqrt{5}\,)(D+\sqrt{5}\,)y=x^2-2x$

**3.** $(D^2-4D-12)y=(D-6)(D+2)y=x-6$

**4.** $(2D^2-3D-2)y=(2D+1)(D-2)y=1$

**5.** $(D^3+10D^2+25D)y=D(D+5)^2y=e^x$

**6.** $(D^3+4D)y=D(D^2+4)y=e^x\cos 2x$

**7.** $(D^3+2D^2-13D+10)y=(D-1)(D-2)(D+5)y=xe^{-x}$

**8.** $(D^3+4D^2+3D)y=D(D+1)(D+3)y=x^2\cos x-3x$

**9.** $(D^4+8D)y=D(D+2)(D^2-2D+4)y=4$

**10.** $(D^4-8D^2+16)y=(D-2)^2(D+2)^2y=(x^3-2x)e^{4x}$

**11.** $D^4y=D^4(10x^3-2x)=D^3(30x^2-2)=D^2(60x)=D(60)=0$

**12.** $(2D-1)y=(2D-1)4e^{x/2}=8De^{x/2}-4e^{x/2}=4e^{x/2}-4e^{x/2}=0$

**13.** $(D-2)(D+5)(e^{2x}+3e^{-5x})=(D-2)(2e^{2x}-15e^{-5x}+5e^{2x}+15e^{-5x})=(D-2)7e^{2x}=14e^{2x}-14e^{2x}=0$

**14.** $$\begin{aligned}(D^2+64)(2\cos 8x-5\sin 8x)&=D(-16\sin 8x-40\cos 8x)+64(2\cos 8x-5\sin 8x)\\&=-128\cos 8x+320\sin 8x+128\cos 8x-320\sin 8x=0\end{aligned}$$

**15.** $D^4$ because of $x^3$

**16.** $D^5$ because of $x^4$

**17.** $D(D-2)$ because of 1 and $e^{2x}$

**18.** $D^2(D-6)^2$ because of $x$ and $xe^{6x}$

**19.** $D^2+4$ because of $\cos 2x$

**20.** $D(D^2+1)$ because of 1 and $\sin x$

**21.** $D^3(D^2+16)$ because of $x^2$ and $\sin 4x$

**22.** $D^2(D^2+1)(D^2+25)$ because of $x$, $\sin x$, and $\cos 5x$

**23.** $(D+1)(D-1)^3$ because of $e^{-x}$ and $x^2e^x$

**24.** $D(D-1)(D-2)$ because of 1, $e^x$, and $e^{2x}$

**25.** $D(D^2-2D+5)$ because of 1 and $e^x\cos 2x$

**26.** $(D^2+2D+2)(D^2-4D+5)$ because of $e^{-x}\sin x$ and $e^{2x}\cos x$

**27.** $1,\ x,\ x^2,\ x^3,\ x^4$

**28.** $D^2 + 4D = D(D + 4);\quad 1,\ e^{-4x}$

**29.** $e^{6x},\ e^{-3x/2}$

**30.** $D^2 - 9D - 36 = (D - 12)(D + 3);\quad e^{12x},\ e^{-3x}$

**31.** $\cos\sqrt{5}\,x,\ \sin\sqrt{5}\,x$

**32.** $D^2 - 6D + 10 = D^2 - 2(3)D + (3^2 + 1^2);\quad e^{3x}\cos x,\ e^{3x}\sin x$

**33.** $D^3 - 10D^2 + 25D = D(D - 5)^2;\quad 1,\ e^{5x},\ xe^{5x}$

**34.** $1,\ x,\ e^{5x},\ e^{7x}$

**35.** Applying $D$ to the differential equation we obtain

$$D(D^2 - 9)y = 0.$$

Then

$$y = \underbrace{c_1e^{3x} + c_2e^{-3x}}_{y_c} + c_3$$

and $y_p = A$. Substituting $y_p$ into the differential equation yields $-9A = 54$ or $A = -6$. The general solution is

$$y = c_1e^{3x} + c_2e^{-3x} - 6.$$

**36.** Applying $D$ to the differential equation we obtain

$$D(2D^2 - 7D + 5)y = 0.$$

Then

$$y = \underbrace{c_1e^{5x/2} + c_2e^{x}}_{y_c} + c_3$$

and $y_p = A$. Substituting $y_p$ into the differential equation yields $5A = -29$ or $A = -29/5$. The general solution is

$$y = c_1e^{5x/2} + c_2e^{x} - \frac{29}{5}\,.$$

**37.** Applying $D$ to the differential equation we obtain

$$D(D^2 + D)y = D^2(D + 1)y = 0.$$

Then

$$y = \underbrace{c_1 + c_2e^{-x}}_{y_c} + c_3x$$

and $y_p = Ax$. Substituting $y_p$ into the differential equation yields $A = 3$. The general solution is

$$y = c_1 + c_2e^{-3x} + 3x.$$

**38.** Applying $D$ to the differential equation we obtain

$$D(D^3+2D^2+D)y=D^2(D+1)^2y=0.$$

Then

$$y=\underbrace{c_1+c_2e^{-x}+c_3xe^{-x}}_{y_c}+c_4x$$

and $y_p=Ax$. Substituting $y_p$ into the differential equation yields $A=10$. The general solution is

$$y=c_1+c_2e^{-x}+c_3xe^{-x}+10x.$$

**39.** Applying $D^2$ to the differential equation we obtain

$$D^2(D^2+4D+4)y=D^2(D+2)^2y=0.$$

Then

$$y=\underbrace{c_1e^{-2x}+c_2xe^{-2x}}_{y_c}+c_3+c_4x$$

and $y_p=Ax+B$. Substituting $y_p$ into the differential equation yields $4Ax+(4A+4B)=2x+6$. Equating coefficients gives

$$\begin{aligned}4A&=2\\4A+4B&=6.\end{aligned}$$

Then $A=1/2$, $B=1$, and the general solution is

$$y=c_1e^{-2x}+c_2xe^{-2x}+\frac{1}{2}x+1.$$

**40.** Applying $D^2$ to the differential equation we obtain

$$D^2(D^2+3D)y=D^3(D+3)y=0.$$

Then

$$y=\underbrace{c_1+c_2e^{-3x}}_{y_c}+c_3x^2+c_4x$$

and $y_p=Ax^2+Bx$. Substituting $y_p$ into the differential equation yields $6Ax+(2A+3B)=4x-5$. Equating coefficients gives

$$\begin{aligned}6A&=4\\2A+3B&=-5.\end{aligned}$$

Then $A=2/3$, $B=-19/9$, and the general solution is

$$y=c_1+c_2e^{-3x}+\frac{2}{3}x^2-\frac{19}{9}x.$$

**41.** Applying $D^3$ to the differential equation we obtain

$$D^3(D^3 + D^2)y = D^5(D + 1)y = 0.$$

Then

$$y = \underbrace{c_1 + c_2x + c_3e^{-x}}_{y_c} + c_4x^4 + c_5x^3 + c_6x^2$$

and $y_p = Ax^4 + Bx^3 + Cx^2$. Substituting $y_p$ into the differential equation yields

$$12Ax^2 + (24A + 6B)x + (6B + 2C) = 8x^2.$$

Equating coefficients gives

$$\begin{aligned} 12A &= 8 \\ 24A + 6B &= 0 \\ 6B + 2C &= 0. \end{aligned}$$

Then $A = 2/3$, $B = -8/3$, $C = 8$, and the general solution is

$$y = c_1 + c_2x + c_3e^{-x} + \frac{2}{3}x^4 - \frac{8}{3}x^3 + 8x^2.$$

**42.** Applying $D^4$ to the differential equation we obtain

$$D^4(D^2 - 2D + 1)y = D^4(D - 1)^2y = 0.$$

Then

$$y = \underbrace{c_1e^x + c_2xe^x}_{y_c} + c_3x^3 + c_4x^2 + c_5x + c_6$$

and $y_p = Ax^3 + Bx^2 + Cx + E$. Substituting $y_p$ into the differential equation yields

$$Ax^3 + (B - 6A)x^2 + (6A - 4B + C)x + (2B - 2C + E) = x^3 + 4x.$$

Equating coefficients gives

$$\begin{aligned} A &= 1 \\ B - 6A &= 0 \\ 6A - 4B + C &= 4 \\ 2B - 2C + E &= 0. \end{aligned}$$

Then $A = 1$, $B = 6$, $C = 22$, $E = 32$ , and the general solution is

$$y = c_1e^x + c_2xe^x + x^3 + 6x^2 + 22x + 32.$$

**43.** Applying $D-4$ to the differential equation we obtain

$$(D-4)(D^2-D-12)y=(D-4)^2(D+3)y=0.$$

Then

$$y=\underbrace{c_1e^{4x}+c_2e^{-3x}}_{y_c}+c_3xe^{4x}$$

and $y_p=Axe^{4x}$. Substituting $y_p$ into the differential equation yields $7Ae^{4x}=e^{4x}$. Equating coefficients gives $A=1/7$. The general solution is

$$y=c_1e^{4x}+c_2e^{-3x}+\frac{1}{7}xe^{4x}.$$

**44.** Applying $D-6$ to the differential equation we obtain

$$(D-6)(D^2+2D+2)y=0.$$

Then

$$y=\underbrace{e^{-x}(c_1\cos x+c_2\sin x)}_{y_c}+c_3e^{6x}$$

and $y_p=Ae^{6x}$. Substituting $y_p$ into the differential equation yields $50Ae^{6x}=5e^{6x}$. Equating coefficients gives $A=1/10$. The general solution is

$$y=e^{-x}(c_1\cos x+c_2\sin x)+\frac{1}{10}e^{6x}.$$

**45.** Applying $D(D-1)$ to the differential equation we obtain

$$D(D-1)(D^2-2D-3)y=D(D-1)(D+1)(D-3)y=0.$$

Then

$$y=\underbrace{c_1e^{3x}+c_2e^{-x}}_{y_c}+c_3e^{x}+c_4$$

and $y_p=Ae^x+B$. Substituting $y_p$ into the differential equation yields $-4Ae^x-3B=4e^x-9$. Equating coefficients gives $A=-1$ and $B=3$. The general solution is

$$y=c_1e^{3x}+c_2e^{-x}-e^x+3.$$

**46.** Applying $D^2(D+2)$ to the differential equation we obtain

$$D^2(D+2)(D^2+6D+8)y=D^2(D+2)^2(D+4)y=0.$$

Then

$$y=\underbrace{c_1e^{-2x}+c_2e^{-4x}}_{y_c}+c_3xe^{-2x}+c_4x+c_5$$

and $y_p = Axe^{-2x} + Bx + C$. Substituting $y_p$ into the differential equation yields

$$2Ae^{-2x} + 8Bx + (6B + 8C) = 3e^{-2x} + 2x.$$

Equating coefficients gives

$$\begin{aligned} 2A &= 3 \\ 8B &= 2 \\ 6B + 8C &= 0. \end{aligned}$$

Then $A = 3/2$, $B = 1/4$, $C = -3/16$ , and the general solution is

$$y = c_1e^{-2x} + c_2e^{-4x} + \frac{3}{2}xe^{-2x} + \frac{1}{4}x - \frac{3}{16}.$$

**47.** Applying $D^2 + 1$ to the differential equation we obtain

$$(D^2 + 1)(D^2 + 25)y = 0.$$

Then

$$y = \underbrace{c_1 \cos 5x + c_2 \sin 5x}_{y_c} + c_3 \cos x + c_4 \sin x$$

and $y_p = A\cos x + B\sin x$. Substituting $y_p$ into the differential equation yields

$$24A\cos x + 24B\sin x = 6\sin x.$$

Equating coefficients gives $A = 0$ and $B = 1/4$. The general solution is

$$y = c_1 \cos 5x + c_2 \sin 5x + \frac{1}{4}\sin x.$$

**48.** Applying $D(D^2 + 1)$ to the differential equation we obtain

$$D(D^2 + 1)(D^2 + 4)y = 0.$$

Then

$$y = \underbrace{c_1 \cos 2x + c_2 \sin 2x}_{y_c} + c_3 \cos x + c_4 \sin x + c_5$$

and $y_p = A\cos x + B\sin x + C$. Substituting $y_p$ into the differential equation yields

$$3A\cos x + 3B\sin x + 4C = 4\cos x + 3\sin x - 8.$$

Equating coefficients gives $A = 4/3$, $B = 1$, and $C = -2$. The general solution is

$$y = c_1 \cos 2x + c_2 \sin 2x + \frac{4}{3}\cos x + \sin x - 2.$$

**49.** Applying $(D-4)^2$ to the differential equation we obtain

$$(D-4)^2(D^2+6D+9)y=(D-4)^2(D+3)^2y=0.$$

Then

$$y=\underbrace{c_1e^{-3x}+c_2xe^{-3x}}_{y_c}+c_3xe^{4x}+c_4e^{4x}$$

and $y_p=Axe^{4x}+Be^{4x}$. Substituting $y_p$ into the differential equation yields

$$49Axe^{4x}+(14A+49B)e^{4x}=-xe^{4x}.$$

Equating coefficients gives

$$\begin{aligned}49A&=-1\\14A+49B&=0.\end{aligned}$$

Then $A=-1/49$, $B=2/343$, and the general solution is

$$y=c_1e^{-3x}+c_2xe^{-3x}-\frac{1}{49}xe^{4x}+\frac{2}{343}e^{4x}.$$

**50.** Applying $D^2(D-1)^2$ to the differential equation we obtain

$$D^2(D-1)^2(D^2+3D-10)y=D^2(D-1)^2(D-2)(D+5)y=0.$$

Then

$$y=\underbrace{c_1e^{2x}+c_2e^{-5x}}_{y_c}+c_3xe^{x}+c_4e^{x}+c_5x+c_6$$

and $y_p=Axe^x+Be^x+Cx+E$. Substituting $y_p$ into the differential equation yields

$$-6Axe^x+(5A-6B)e^x-10Cx+(3C-10E)=xe^x+x.$$

Equating coefficients gives

$$\begin{aligned}-6A&=1\\5A-6B&=0\\-10C&=1\\3C-10E&=0.\end{aligned}$$

Then $A=-1/6$, $B=-5/36$, $C=-1/10$, $E=-3/100$, and the general solution is

$$y=c_1e^{2x}+c_2e^{-5x}-\frac{1}{6}xe^x-\frac{5}{36}e^x-\frac{1}{10}x-\frac{3}{100}.$$

**51.** Applying $D(D-1)^3$ to the differential equation we obtain

$$D(D-1)^3(D^2-1)y = D(D-1)^4(D+1)y = 0.$$

Then

$$y = \underbrace{c_1e^x + c_2e^{-x}}_{y_c} + c_3x^3e^x + c_4x^2e^x + c_5xe^x + c_6$$

and $y_p = Ax^3e^x + Bx^2e^x + Cxe^x + E$. Substituting $y_p$ into the differential equation yields

$$6Ax^2e^x + (6A+4B)xe^x + (2B+2C)e^x - E = x^2e^x + 5.$$

Equating coefficients gives

$$\begin{aligned} 6A &= 1 \\ 6A + 4B &= 0 \\ 2B + 2C &= 0 \\ -E &= 5. \end{aligned}$$

Then $A = 1/6$, $B = -1/4$, $C = 1/4$, $E = -5$, and the general solution is

$$y = c_1e^x + c_2e^{-x} + \frac{1}{6}x^3e^x - \frac{1}{4}x^2e^x + \frac{1}{4}xe^x - 5.$$

**52.** Applying $(D+1)^3$ to the differential equation we obtain

$$(D+1)^3(D^2+2D+1)y = (D+1)^5y = 0.$$

Then

$$y = \underbrace{c_1e^{-x} + c_2xe^{-x}}_{y_c} + c_3x^4e^{-x} + c_4x^3e^{-x} + c_5x^2e^{-x}$$

and $y_p = Ax^4e^{-x} + Bx^3e^{-x} + Cx^2e^{-x}$. Substituting $y_p$ into the differential equation yields

$$12Ax^2e^{-x} + 6Bxe^{-x} + 2Ce^{-x} = x^2e^{-x}.$$

Equating coefficients gives $A = \frac{1}{12}$, $B = 0$, and $C = 0$. The general solution is

$$y = c_1e^{-x} + c_2xe^{-x} + \frac{1}{12}x^4e^{-x}.$$

**53.** Applying $D^2 - 2D + 2$ to the differential equation we obtain

$$(D^2 - 2D + 2)(D^2 - 2D + 5)y = 0.$$

Then

$$y = \underbrace{e^x(c_1\cos 2x + c_2\sin 2x)}_{y_c} + e^x(c_3\cos x + c_4\sin x)$$

and $y_p = Ae^x \cos x + Be^x \sin x$. Substituting $y_p$ into the differential equation yields

$$3Ae^x \cos x + 3Be^x \sin x = e^x \sin x.$$

Equating coefficients gives $A = 0$ and $B = 1/3$. The general solution is

$$y = e^x(c_1 \cos 2x + c_2 \sin 2x) + \frac{1}{3}e^x \sin x.$$

**54.** Applying $D^2 - 2D + 10$ to the differential equation we obtain

$$(D^2 - 2D + 10)\left(D^2 + D + \frac{1}{4}\right)y = (D^2 - 2D + 10)\left(D + \frac{1}{2}\right)^2 y = 0.$$

Then

$$y = \underbrace{c_1 e^{-x/2} + c_2 x e^{-x/2}}_{y_c} + c_3 e^x \cos 3x + c_4 e^x \sin 3x$$

and $y_p = Ae^x \cos 3x + Be^x \sin 3x$. Substituting $y_p$ into the differential equation yields

$$(9B - 27A/4)e^x \cos 3x - (9A + 27B/4)e^x \sin 3x = -e^x \cos 3x + e^x \sin 3x.$$

Equating coefficients gives

$$-\frac{27}{4}A + 9B = -1$$

$$-9A - \frac{27}{4}B = 1.$$

Then $A = -4/225$, $B = -28/225$, and the general solution is

$$y = c_1 e^{-x/2} + c_2 x e^{-x/2} - \frac{4}{225}e^x \cos 3x - \frac{28}{225}e^x \sin 3x.$$

**55.** Applying $D^2 + 25$ to the differential equation we obtain

$$(D^2 + 25)(D^2 + 25) = (D^2 + 25)^2 = 0.$$

Then

$$y = \underbrace{c_1 \cos 5x + c_2 \sin 5x}_{y_c} + c_3 x \cos 5x + c_4 x \sin 5x$$

and $y_p = Ax \cos 5x + Bx \sin 5x$. Substituting $y_p$ into the differential equation yields

$$10B \cos 5x - 10A \sin 5x = 20 \sin 5x.$$

Equating coefficients gives $A = -2$ and $B = 0$. The general solution is

$$y = c_1 \cos 5x + c_2 \sin 5x - 2x \cos 5x.$$

**56.** Applying $D^2+1$ to the differential equation we obtain

$$(D^2+1)(D^2+1)=(D^2+1)^2=0.$$

Then

$$y=\underbrace{c_1\cos x+c_2\sin x}_{y_c}+c_3x\cos x+c_4x\sin x$$

and $y_p=Ax\cos x+Bx\sin x$. Substituting $y_p$ into the differential equation yields

$$2B\cos x-2A\sin x=4\cos x-\sin x.$$

Equating coefficients gives $A=1/2$ and $B=2$. The general solution is

$$y=c_1\cos x+c_2\sin x+\frac{1}{2}x\cos x+2x\sin x.$$

**57.** Applying $(D^2+1)^2$ to the differential equation we obtain

$$(D^2+1)^2(D^2+D+1)=0.$$

Then

$$y=\underbrace{e^{-x/2}\left[c_1\cos\frac{\sqrt{3}}{2}x+c_2\sin\frac{\sqrt{3}}{2}x\right]}_{y_c}+c_3\cos x+c_4\sin x+c_5x\cos x+c_6x\sin x$$

and $y_p=A\cos x+B\sin x+Cx\cos x+Ex\sin x$. Substituting $y_p$ into the differential equation yields

$$(B+C+2E)\cos x+Ex\cos x+(-A-2C+E)\sin x-Cx\sin x=x\sin x.$$

Equating coefficients gives

$$\begin{aligned}B+C+2E&=0\\E&=0\\-A-2C+E&=0\\-C&=1.\end{aligned}$$

Then $A=2$, $B=1$, $C=-1$, and $E=0$, and the general solution is

$$y=e^{-x/2}\left[c_1\cos\frac{\sqrt{3}}{2}x+c_2\sin\frac{\sqrt{3}}{2}x\right]+2\cos x+\sin x-x\cos x.$$

**58.** Writing $\cos^2 x = \frac{1}{2}(1+\cos 2x)$ and applying $D(D^2+4)$ to the differential equation we obtain

$$D(D^2+4)(D^2+4) = D(D^2+4)^2 = 0.$$

Then

$$y = \underbrace{c_1\cos 2x + c_2\sin 2x}_{y_c} + c_3x\cos 2x + c_4x\sin 2x + c_5$$

and $y_p = Ax\cos 2x + Bx\sin 2x + C$. Substituting $y_p$ into the differential equation yields

$$-4A\sin 2x + 4B\cos 2x + 4C = \frac{1}{2} + \frac{1}{2}\cos 2x.$$

Equating coefficients gives $A = 0$, $B = 1/8$, and $C = 1/8$. The general solution is

$$y = c_1\cos 2x + c_2\sin 2x + \frac{1}{8}x\sin 2x + \frac{1}{8}.$$

**59.** Applying $D^3$ to the differential equation we obtain

$$D^3(D^3+8D^2) = D^5(D+8) = 0.$$

Then

$$y = \underbrace{c_1 + c_2x + c_3e^{-8x}}_{y_c} + c_4x^2 + c_5x^3 + c_6x^4$$

and $y_p = Ax^2 + Bx^3 + Cx^4$. Substituting $y_p$ into the differential equation yields

$$16A + 6B + (48B + 24C)x + 96Cx^2 = 2 + 9x - 6x^2.$$

Equating coefficients gives

$$\begin{aligned} 16A + 6B &= 2 \\ 48B + 24C &= 9 \\ 96C &= -6. \end{aligned}$$

Then $A = 11/256$, $B = 7/32$, and $C = -1/16$, and the general solution is

$$y = c_1 + c_2x + c_3e^{-8x} + \frac{11}{256}x^2 + \frac{7}{32}x^3 - \frac{1}{16}x^4.$$

**60.** Applying $D(D-1)^2(D+1)$ to the differential equation we obtain

$$D(D-1)^2(D+1)(D^3-D^2+D-1) = D(D-1)^3(D+1)(D^2+1) = 0.$$

Then

$$y = \underbrace{c_1e^x + c_2\cos x + c_3\sin x}_{y_c} + c_4 + c_5e^{-x} + c_6xe^x + c_7x^2e^x$$

and $y_p = A + Be^{-x} + Cxe^x + Ex^2e^x$. Substituting $y_p$ into the differential equation yields

$$4Exe^x + (2C + 4E)e^x - 4Be^{-x} - A = xe^x - e^{-x} + 7.$$

Equating coefficients gives

$$\begin{aligned} 4E &= 1 \\ 2C + 4E &= 0 \\ -4B &= -1 \\ -A &= 7. \end{aligned}$$

Then $A = -7$, $B = 1/4$, $C = -1/2$, and $E = 1/4$, and the general solution is

$$y = c_1e^x + c_2\cos x + c_3\sin x - 7 + \frac{1}{4}e^{-x} - \frac{1}{2}xe^x + \frac{1}{4}x^2e^x.$$

**61.** Applying $D^2(D-1)$ to the differential equation we obtain

$$D^2(D-1)(D^3 - 3D^2 + 3D - 1) = D^2(D-1)^4 = 0.$$

Then

$$y = \underbrace{c_1e^x + c_2xe^x + c_3x^2e^x}_{y_c} + c_4 + c_5x + c_6x^3e^x$$

and $y_p = A + Bx + Cx^3e^x$. Substituting $y_p$ into the differential equation yields

$$(-A + 3B) - Bx + 6Ce^x = 16 - x + e^x.$$

Equating coefficients gives

$$\begin{aligned} -A + 3B &= 16 \\ -B &= -1 \\ 6C &= 1. \end{aligned}$$

Then $A = -13$, $B = 1$, and $C = 1/6$, and the general solution is

$$y = c_1e^x + c_2xe^x + c_3x^2e^x - 13 + x + \frac{1}{6}x^3e^x.$$

**62.** Writing $(e^x + e^{-x})^2 = 2 + e^{2x} + e^{-2x}$ and applying $D(D-2)(D+2)$ to the differential equation we obtain

$$D(D-2)(D+2)(2D^3 - 3D^2 - 3D + 2) = D(D-2)^2(D+2)(D+1)(2D-1) = 0.$$

Then

$$y = \underbrace{c_1e^{-x} + c_2e^{2x} + c_3e^{x/2}}_{y_c} + c_4 + c_5xe^{2x} + c_6e^{-2x}$$

and $y_p = A + Bxe^{2x} + Ce^{-2x}$. Substituting $y_p$ into the differential equation yields

$$2A + 9Be^{2x} - 20Ce^{-2x} = 2 + e^{2x} + e^{-2x}.$$

Equating coefficients gives $A = 1$, $B = 1/9$, and $C = -1/20$. The general solution is

$$y = c_1e^{-x} + c_2e^{2x} + c_3e^{x/2} + 1 + \frac{1}{9}xe^{2x} - \frac{1}{20}e^{-2x}.$$

**63.** Applying $D(D-1)$ to the differential equation we obtain

$$D(D-1)(D^4 - 2D^3 + D^2) = D^3(D-1)^3 = 0.$$

Then

$$y = \underbrace{c_1 + c_2x + c_3e^x + c_4xe^x}_{y_c} + c_5x^2 + c_6x^2e^x$$

and $y_p = Ax^2 + Bx^2e^x$. Substituting $y_p$ into the differential equation yields $2A + 2Be^x = 1 + e^x$. Equating coefficients gives $A = 1/2$ and $B = 1/2$. The general solution is

$$y = c_1 + c_2x + c_3e^x + c_4xe^x + \frac{1}{2}x^2 + \frac{1}{2}x^2e^x.$$

**64.** Applying $D^3(D-2)$ to the differential equation we obtain

$$D^3(D-2)(D^4 - 4D^2) = D^5(D-2)^2(D+2) = 0.$$

Then

$$y = \underbrace{c_1 + c_2x + c_3e^{2x} + c_4e^{-2x}}_{y_c} + c_5x^2 + c_6x^3 + c_7x^4 + c_8xe^{2x}$$

and $y_p = Ax^2 + Bx^3 + Cx^4 + Exe^{2x}$. Substituting $y_p$ into the differential equation yields

$$(-8A + 24C) - 24Bx - 48Cx^2 + 16Ee^{2x} = 5x^2 - e^{2x}.$$

Equating coefficients gives

$$\begin{aligned} -8A + 24C &= 0 \\ -24B &= 0 \\ -48C &= 5 \\ 16E &= -1. \end{aligned}$$

Then $A = -5/16$, $B = 0$, $C = -5/48$, and $E = -1/16$, and the general solution is

$$y = c_1 + c_2x + c_3e^{2x} + c_4e^{-2x} - \frac{5}{16}x^2 - \frac{5}{48}x^4 - \frac{1}{16}xe^{2x}.$$

**65.** The complementary function is $y_c = c_1e^{8x} + c_2e^{-8x}$. Using $D$ to annihilate 16 we find $y_p = A$. Substituting $y_p$ into the differential equation we obtain $-64A = 16$. Thus $A = -1/4$ and

$$y = c_1e^{8x} + c_2e^{-8x} - \frac{1}{4}$$
$$y' = 8c_1e^{8x} - 8c_2e^{-8x}.$$

The initial conditions imply

$$c_1 + c_2 = \frac{5}{4}$$
$$8c_1 - 8c_2 = 0.$$

Thus $c_1 = c_2 = 5/8$ and

$$y = \frac{5}{8}e^{8x} + \frac{5}{8}e^{-8x} - \frac{1}{4}.$$

**66.** The complementary function is $y_c = c_1 + c_2e^{-x}$. Using $D^2$ to annihilate $x$ we find $y_p = Ax + Bx^2$. Substituting $y_p$ into the differential equation we obtain $(A + 2B) + 2Bx = x$. Thus $A = -1$ and $B = 1/2$, and

$$y = c_1 + c_2e^{-x} - x + \frac{1}{2}x^2$$
$$y' = -c_2e^{-x} - 1 + x.$$

The initial conditions imply

$$c_1 + c_2 = 1$$
$$-c_2 = 1.$$

Thus $c_1 = 2$ and $c_2 = -1$, and

$$y = 2 - e^{-x} - x + \frac{1}{2}x^2.$$

**67.** The complementary function is $y_c = c_1 + c_2e^{5x}$. Using $D^2$ to annihilate $x - 2$ we find $y_p = Ax + Bx^2$. Substituting $y_p$ into the differential equation we obtain $(-5A + 2B) - 10Bx = -2 + x$. Thus $A = 9/25$ and $B = -1/10$, and

$$y = c_1 + c_2e^{5x} + \frac{9}{25}x - \frac{1}{10}x^2$$
$$y' = 5c_2e^{5x} + \frac{9}{25} - \frac{1}{5}x.$$

The initial conditions imply

$$c_1 + c_2 = 0$$
$$c_2 = \frac{41}{125}\,.$$

Thus $c_1 = -41/125$ and $c_2 = 41/125$, and

$$y = -\frac{41}{125} + \frac{41}{125}e^{5x} + \frac{9}{25}x - \frac{1}{10}x^2.$$

**68.** The complementary function is $y_c = c_1e^x + c_2e^{-6x}$. Using $D-2$ to annihilate $10e^{2x}$ we find $y_p = Ae^{2x}$. Substituting $y_p$ into the differential equation we obtain $8Ae^{2x} = 10e^{2x}$. Thus $A = 5/4$ and

$$\begin{aligned} y &= c_1e^x + c_2e^{-6x} + \frac{5}{4}e^{2x} \\ y' &= c_1e^x - 6c_2e^{-6x} + \frac{5}{2}e^{2x}. \end{aligned}$$

The initial conditions imply

$$\begin{aligned} c_1 + c_2 &= -\frac{1}{4} \\ c_1 - 6c_2 &= -\frac{3}{2}. \end{aligned}$$

Thus $c_1 = -3/7$ and $c_2 = 5/28$, and

$$y = -\frac{3}{7}e^x + \frac{5}{28}e^{-6x} + \frac{5}{4}e^{2x}$$

**69.** The complementary function is $y_c = c_1\cos x + c_2\sin x$. Using $(D^2+1)(D^2+4)$ to annihilate $8\cos 2x - 4\sin x$ we find $y_p = Ax\cos x + Bx\sin x + C\cos 2x + E\sin 2x$. Substituting $y_p$ into the differential equation we obtain $2B\cos x - 3C\cos 2x - 2A\sin x - 3E\sin 2x = 8\cos 2x - 4\sin x$. Thus $A = 2$, $B = 0$, $C = -8/3$, and $E = 0$, and

$$\begin{aligned} y &= c_1\cos x + c_2\sin x + 2x\cos x - \frac{8}{3}\cos 2x \\ y' &= -c_1\sin x + c_2\cos x + 2\cos x - 2x\sin x + \frac{16}{3}\sin 2x. \end{aligned}$$

The initial conditions imply

$$\begin{aligned} c_2 + \frac{8}{3} &= -1 \\ -c_1 - \pi &= 0. \end{aligned}$$

Thus $c_1 = -\pi$ and $c_2 = -11/3$, and

$$y = -\pi\cos x - \frac{11}{3}\sin x + 2x\cos x - \frac{8}{3}\cos 2x.$$

**70.** The complementary function is $y_c = c_1 + c_2e^x + c_3xe^x$. Using $D(D-1)^2$ to annihilate $xe^x + 5$ we find $y_p = Ax + Bx^2e^x + Cx^3e^x$. Substituting $y_p$ into the differential equation we obtain $A + (2B + 6C)e^x + 6Cxe^x = xe^x + 5$. Thus $A = 5$, $B = -1/2$, and $C = 1/6$, and

$$\begin{aligned} y &= c_1 + c_2e^x + c_3xe^x + 5x - \frac{1}{2}x^2e^x + \frac{1}{6}x^3e^x \\ y' &= c_2e^x + c_3(xe^x + e^x) + 5 - xe^x + \frac{1}{6}x^3e^x \\ y'' &= c_2e^x + c_3(xe^x + 2e^x) - e^x - xe^x + \frac{1}{2}x^2e^x + \frac{1}{6}x^3e^x. \end{aligned}$$

The initial conditions imply

$$\begin{aligned} c_1 + c_2 &= 2 \\ c_2 + c_3 + 5 &= 2 \\ c_2 + 2c_3 - 1 &= -1. \end{aligned}$$

Thus $c_1 = 8$, $c_2 = -6$, and $c_3 = 3$, and

$$y = 8 - 6e^x + 3xe^x + 5x - \frac{1}{2}x^2e^x + \frac{1}{6}x^3e^x.$$

**71.** The complementary function is $y_c = e^{2x}(c_1\cos 2x + c_2\sin 2x)$. Using $D^4$ to annihilate $x^3$ we find $y_p = A + Bx + Cx^2 + Ex^3$. Substituting $y_p$ into the differential equation we obtain $(8A - 4B + 2C) + (8B - 8C + 6E)x + (8C - 12E)x^2 + 8Ex^3 = x^3$. Thus $A = 0$, $B = 3/32$, $C = 3/16$, and $E = 1/8$, and

$$\begin{aligned} y &= e^{2x}(c_1\cos 2x + c_2\sin 2x) + \frac{3}{32}x + \frac{3}{16}x^2 + \frac{1}{8}x^3 \\ y' &= e^{2x}\left[c_1(2\cos 2x - 2\sin 2x) + c_2(2\cos 2x + 2\sin 2x)\right] + \frac{3}{32} + \frac{3}{8}x + \frac{3}{8}x^2. \end{aligned}$$

The initial conditions imply

$$\begin{aligned} c_1 &= 2 \\ 2c_1 + 2c_2 + \frac{3}{32} &= 4. \end{aligned}$$

Thus $c_1 = 2$, $c_2 = -3/64$, and

$$y = e^{2x}\left(2\cos 2x - \frac{3}{64}\sin 2x\right) + \frac{3}{32}x + \frac{3}{16}x^2 + \frac{1}{8}x^3.$$

**72.** The complementary function is $y_c = c_1 + c_2x + c_3x^2 + c_4e^x$. Using $D^2(D-1)$ to annihilate $x + e^x$ we find $y_p = Ax^3 + Bx^4 + Cxe^x$. Substituting $y_p$ into the differential equation we obtain $(-6A + 24B) - 24Bx + Ce^x = x + e^x$. Thus $A = -1/6$, $B = -1/24$, and $C = 1$, and

$$\begin{aligned} y &= c_1 + c_2x + c_3x^2 + c_4e^x - \frac{1}{6}x^3 - \frac{1}{24}x^4 + xe^x \\ y' &= c_2 + 2c_3x + c_4e^x - \frac{1}{2}x^2 - \frac{1}{6}x^3 + e^x + xe^x \\ y'' &= 2c_3 + c_4e^x - x - \frac{1}{2}x^2 + 2e^x + xe^x. \\ y''' &= c_4e^x - 1 - x + 3e^x + xe^x \end{aligned}$$

The initial conditions imply

$$\begin{aligned} c_1 + c_4 &= 0 \\ c_2 + c_4 + 1 &= 0 \\ 2c_3 + c_4 + 2 &= 0 \\ 2 + c_4 &= 0. \end{aligned}$$

Thus $c_1 = 2$, $c_2 = 1$, $c_3 = 0$, and $c_4 = -2$, and

$$y = 2 + x - 2e^x - \frac{1}{6}x^3 - \frac{1}{24}x^4 + xe^x.$$

**73.** To see in this case that the factors of $L$ do not commute consider the operators $(xD-1)(D+4)$ and $(D+4)(xD-1)$. Applying the operators to the function $x$ we find

$$\begin{aligned} (xD-1)(D+4)x &= (xD^2 + 4xD - D - 4)x \\ &= xD^2x + 4xDx - Dx - 4x \\ &= x(0) + 4x(1) - 1 - 4x = -1 \end{aligned}$$

and

$$\begin{aligned} (D+4)(xD-1)x &= (D+4)(xDx - x) \\ &= (D+4)(x \cdot 1 - x) = 0. \end{aligned}$$

Thus, the operators are not the same.

## 4.6 Variation of Parameters

*The particular solution, $y_p = u_1y_1 + u_2y_2$, in the following problems can take on a variety of forms, especially where trigonometric functions are involved. The validity of a particular form can best be checked by substituting it back into the differential equation.*

**1.** The auxiliary equation is $m^2 + 1 = 0$, so $y_c = c_1 \cos x + c_2 \sin x$ and

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

Identifying $f(x) = \sec x$ we obtain

$$u_1' = -\frac{\sin x \sec x}{1} = -\tan x$$
$$u_2' = \frac{\cos x \sec x}{1} = 1.$$

Then $u_1 = \ln|\cos x|$, $u_2 = x$, and

$$y = c_1 \cos x + c_2 \sin x + \cos x \ln|\cos x| + x \sin x.$$

**2.** The auxiliary equation is $m^2 + 1 = 0$, so $y_c = c_1 \cos x + c_2 \sin x$ and

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

Identifying $f(x) = \tan x$ we obtain

$$u_1' = -\sin x \tan x = \frac{\cos^2 x - 1}{\cos x} = \cos x - \sec x$$
$$u_2' = \sin x.$$

Then $u_1 = \sin x - \ln|\sec x + \tan x|$, $u_2 = -\cos x$, and

$$y = c_1 \cos x + c_2 \sin x + \cos x\,(\sin x - \ln|\sec x + \tan x|) - \cos x \sin x$$
$$= c_1 \cos x + c_2 \sin x - \cos x \ln|\sec x + \tan x|.$$

**3.** The auxiliary equation is $m^2 + 1 = 0$, so $y_c = c_1 \cos x + c_2 \sin x$ and

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

Identifying $f(x) = \sin x$ we obtain

$$u_1' = -\sin^2 x$$
$$u_2' = \cos x \sin x.$$

Then

$$\begin{aligned} u_1 &= \frac{1}{4}\sin 2x - \frac{1}{2}x = \frac{1}{2}\sin x\cos x - \frac{1}{2}x \\ u_2 &= -\frac{1}{2}\cos^2 x. \end{aligned}$$

and

$$\begin{aligned} y &= c_1\cos x + c_2\sin x + \frac{1}{2}\sin x\cos^2 x - \frac{1}{2}x\cos x - \frac{1}{2}\cos^2 x\sin x \\ &= c_1\cos x + c_2\sin x - \frac{1}{2}x\cos x. \end{aligned}$$

**4.** The auxiliary equation is $m^2+1=0$, so $y_c = c_1\cos x + c_2\sin x$ and

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

Identifying $f(x) = \sec x\tan x$ we obtain

$$\begin{aligned} u_1' &= -\sin x(\sec x\tan x) = -\tan^2 x = 1 - \sec^2 x \\ u_2' &= \cos x(\sec x\tan x) = \tan x. \end{aligned}$$

Then $u_1 = x - \tan x$, $u_2 = -\ln|\cos x|$, and

$$\begin{aligned} y &= c_1\cos x + c_2\sin x + x\cos x - \sin x - \sin x\ln|\cos x| \\ &= c_1\cos x + c_3\sin x + x\cos x - \sin x\ln|\cos x|. \end{aligned}$$

**5.** The auxiliary equation is $m^2+1=0$, so $y_c = c_1\cos x + c_2\sin x$ and

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

Identifying $f(x) = \cos^2 x$ we obtain

$$\begin{aligned} u_1' &= -\sin x\cos^2 x \\ u_2' &= \cos^3 x = \cos x\left(1 - \sin^2 x\right). \end{aligned}$$

Then $u_1 = \frac{1}{3}\cos^3 x$, $u_2 = \sin x - \frac{1}{3}\sin^3 x$, and

$$\begin{aligned} y &= c_1\cos x + c_2\sin x + \frac{1}{3}\cos^4 x + \sin^2 x - \frac{1}{3}\sin^4 x \\ &= c_1\cos x + c_2\sin x + \frac{1}{3}\left(\cos^2 x + \sin^2 x\right)\left(\cos^2 x - \sin^2 x\right) + \sin^2 x \\ &= c_1\cos x + c_2\sin x + \frac{1}{3}\cos^2 x + \frac{2}{3}\sin^2 x \\ &= c_1\cos x + c_2\sin x + \frac{1}{3} + \frac{1}{3}\sin^2 x. \end{aligned}$$

**6.** The auxiliary equation is $m^2+1=0$, so $y_c = c_1\cos x + c_2 \sin x$ and

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

Identifying $f(x) = \sec^2 x$ we obtain

$$\begin{aligned} u_1' &= -\frac{\sin x}{\cos^2 x} \\ u_2' &= \sec x. \end{aligned}$$

Then

$$\begin{aligned} u_1 &= -\frac{1}{\cos x} = -\sec x \\ u_2 &= \ln|\sec x + \tan x| \end{aligned}$$

and

$$\begin{aligned} y &= c_1\cos x + c_2 \sin x - \cos x \sec x + \sin x \ln|\sec x + \tan x| \\ &= c_1\cos x + c_2\sin x - 1 + \sin x \ln|\sec x + \tan x|. \end{aligned}$$

**7.** The auxiliary equation is $m^2-1=0$, so $y_c = c_1e^x + c_2e^{-x}$ and

$$W = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -2$$

Identifying $f(x) = \cosh x = \frac{1}{2}(e^{-x}+e^x)$ we obtain

$$\begin{aligned} u_1' &= \frac{1}{4}e^{-2x} + \frac{1}{4} \\ u_2' &= -\frac{1}{4} - \frac{1}{4}e^{2x}. \end{aligned}$$

Then

$$\begin{aligned} u_1 &= -\frac{1}{8}e^{-2x} + \frac{1}{4}x \\ u_2 &= -\frac{1}{8}e^{2x} - \frac{1}{4}x \end{aligned}$$

and

$$\begin{aligned} y &= c_1e^x + c_2e^{-x} - \frac{1}{8}e^{-x} + \frac{1}{4}xe^x - \frac{1}{8}e^x - \frac{1}{4}xe^{-x} \\ &= c_3e^x + c_4e^{-x} + \frac{1}{4}x(e^x - e^{-x}) \\ &= c_3e^x + c_4e^{-x} + \frac{1}{2}x\sinh x. \end{aligned}$$

**8.** The auxiliary equation is $m^2 - 1 = 0$, so $y_c = c_1 e^x + c_2 e^{-x}$ and

$$W = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -2$$

Identifying $f(x) = \sinh 2x$ we obtain

$$u_1' = -\frac{1}{4}e^{-3x} + \frac{1}{4}e^x u_2' \qquad = \frac{1}{4}e^{-x} - \frac{1}{4}e^{3x}.$$

Then

$$\begin{aligned} u_1 &= \frac{1}{12}e^{-3x} + \frac{1}{4}e^x \\ u_2 &= -\frac{1}{4}e^{-x} - \frac{1}{12}e^{3x}. \end{aligned}$$

and

$$\begin{aligned} y &= c_1 e^x + c_2 e^{-x} + \frac{1}{12}e^{-2x} + \frac{1}{4}e^{2x} - \frac{1}{4}e^{-2x} - \frac{1}{12}e^{2x} \\ &= c_1 e^x + c_2 e^{-x} + \frac{1}{6}\left(e^{2x} - e^{-2x}\right) \\ &= c_1 e^x + c_2 e^{-x} + \frac{1}{3}\sinh 2x. \end{aligned}$$

**9.** The auxiliary equation is $m^2 - 9 = 0$, so $y_c = c_1 e^{3x} + c_2 e^{-3x}$ and

$$W = \begin{vmatrix} e^{3x} & e^{-3x} \\ 3e^{3x} & -3e^{-3x} \end{vmatrix} = -6$$

Identifying $f(x) = 9x/e^{3x}$ we obtain $u_1' = \frac{3}{2}xe^{-6x}$ and $u_2' = -\frac{3}{2}x$. Then

$$\begin{aligned} u_1 &= -\frac{1}{24}e^{-6x} - \frac{1}{4}xe^{-6x}, \\ u_2 &= -\frac{3}{4}x^2 \end{aligned}$$

and

$$\begin{aligned} y &= c_1 e^{3x} + c_2 e^{-3x} - \frac{1}{24}e^{-3x} - \frac{1}{4}xe^{-3x} - \frac{3}{4}x^2 e^{-3x} \\ &= c_1 e^{3x} + c_3 e^{-3x} - \frac{1}{4}xe^{-3x}(1 + 3x). \end{aligned}$$

**10.** The auxiliary equation is $4m^2 - 1 = 0$, so $y_c = c_1 e^{-x/2} + c_2 e^{x/2}$ and

$$W = \begin{vmatrix} e^{-x/2} & e^{x/2} \\ -\frac{1}{2}e^{-x/2} & \frac{1}{2}e^{x/2} \end{vmatrix} = 1$$

Dividing by 4 we now identify $f(x) = \frac{1}{4}e^{x/2} + \frac{3}{4}$ and obtain $u_1' = -\frac{3}{4}e^{x/2} - \frac{1}{4}e^x$ and $u_2' = \frac{3}{4}e^{-x/2} + \frac{1}{4}$. Then

$$u_1 = -\frac{3}{2}e^{x/2} - \frac{1}{4}e^x,$$

$$u_2 = -\frac{3}{2}e^{-x/2} + \frac{1}{4}x$$

and

$$\begin{aligned} y &= c_1 e^{-x/2} + c_2 e^{x/2} - \frac{1}{4}e^{x/2} - \frac{3}{2} + \frac{1}{4}xe^{x/2} - \frac{3}{2} \\ &= c_1 e^{-x/2} + c_3 e^{x/2} + \frac{1}{4}xe^{x/2} - 3 \end{aligned}$$

**11.** The auxiliary equation is $m^2 + 3m + 2 = (m+1)(m+2) = 0$, so $y_c = c_1 e^{-x} + c_2 e^{-2x}$ and

$$W = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix} = -e^{-3x}$$

Identifying $f(x) = 1/(1 + e^x)$ we obtain

$$u_1' = \frac{e^x}{1 + e^x}$$

$$u_2' = -\frac{e^{2x}}{1 + e^x} = \frac{e^x}{1 + e^x} - e^x.$$

Then $u_1 = \ln(1 + e^x)$, $u_2 = \ln(1 + e^x) - e^x$, and

$$\begin{aligned} y &= c_1 e^{-x} + c_2 e^{-2x} + e^{-x}\ln(1 + e^x) + e^{-2x}\ln(1 + e^x) - e^{-x} \\ &= c_3 e^{-x} + c_2 e^{-2x} + (1 + e^{-x})e^{-x}\ln(1 + e^x). \end{aligned}$$

**12.** The auxiliary equation is $m^2 - 2m + 1 = (m-1)^2 = 0$, so $y_c = c_1 e^x + c_2 xe^x$ and

$$W = \begin{vmatrix} e^x & xe^x \\ e^x & xe^x + e^x \end{vmatrix} = e^{2x}$$

Identifying $f(x) = e^x / \left(1 + x^2\right)$ we obtain

$$u_1' = -\frac{xe^x e^x}{e^{2x}\left(1 + x^2\right)} = -\frac{x}{1 + x^2}$$

$$u_2' = \frac{e^x e^x}{e^{2x}\left(1 + x^2\right)} = \frac{1}{1 + x^2}.$$

Then $u_1 = -\frac{1}{2}\ln\left(1+x^2\right)$, $u_2 = \tan^{-1} x$, and

$$y = c_1 e^x + c_2 x e^x - \frac{1}{2} e^x \ln\left(1+x^2\right) + x e^x \tan^{-1} x.$$

**13.** The auxiliary equation is $m^2 + 3m + 2 = (m+1)(m+2) = 0$, so $y_c = c_1 e^{-x} + c_2 e^{-2x}$ and

$$W = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & = 2e^{-2} \end{vmatrix} = -e^{-3x}$$

Identifying $f(x) = \sin e^x$ we obtain

$$u_1' = \frac{e^{-2x}\sin e^x}{e^{-3x}} = e^x \sin e^x$$

$$u_2' = \frac{e^{-x}\sin e^x}{-e^{-3x}} = -e^{2x}\sin e^x.$$

Then $u_1 = -\cos e^x$, $u_2 = e^x \cos x - \sin e^x$, and

$$\begin{aligned} y &= c_1 e^{-x} + c_2 e^{-2x} - e^{-x}\cos e^x + e^{-x}\cos e^x - e^{-2x}\sin e^x \\ &= c_1 e^{-x} + c_2 e^{-2x} - e^{-2x}\sin e^x. \end{aligned}$$

**14.** The auxiliary equation is $m^2 - 2m + 1 = (m-1)^2 = 0$, so $y_c = c_1 e^t + c_2 t e^t$ and

$$W = \begin{vmatrix} e^t & te^t \\ e^t & te^t + e^t \end{vmatrix} = e^{2t}$$

Identifying $f(t) = e^t \tan^{-1} t$ we obtain

$$u_1' = -\frac{te^t e^t \tan^{-1} t}{e^{2t}} = -t\tan^{-1} t$$

$$u_2' = \frac{e^t e^t \tan^{-1} t}{e^{2t}} = \tan^{-1} t.$$

Then

$$u_1 = -\frac{1+t^2}{2}\tan^{-1} t + \frac{t}{2}$$

$$u_2 = t\tan^{-1} t - \frac{1}{2}\ln\left(1+t^2\right)$$

and

$$\begin{aligned} y &= c_1 e^t + c_2 t e^t + \left(-\frac{1+t^2}{2}\tan^{-1} t + \frac{t}{2}\right) e^t + \left(t\tan^{-1} t - \frac{1}{2}\ln\left(1+t^2\right)\right) te^t \\ &= c_1 e^t + c_3 t e^t + \frac{1}{2} e^t \left[\left(t^2 - 1\right)\tan^{-1} t - \ln\left(1+t^2\right)\right]. \end{aligned}$$

**15.** The auxiliary equation is $m^2 + 2m + 1 = (m+1)^2 = 0$, so $y_c = c_1e^{-t} + c_2te^{-t}$ and

$$W = \begin{vmatrix} e^{-t} & te^{-t} \\ -e^{-t} & -te^{-t} + e^{-t} \end{vmatrix} = e^{-2t}$$

Identifying $f(t) = e^{-t}\ln t$ we obtain

$$\begin{aligned} u_1' &= -\frac{te^{-t}e^{-t}\ln t}{e^{-2t}} = -t\ln t \\ u_2' &= \frac{e^{-t}e^{-t}\ln t}{e^{-2t}} = \ln t. \end{aligned}$$

Then

$$\begin{aligned} u_1 &= -\frac{1}{2}t^2\ln t + \frac{1}{4}t^2 \\ u_2 &= t\ln t - t \end{aligned}$$

and

$$\begin{aligned} y &= c_1e^{-t} + c_2te^{-t} - \frac{1}{2}t^2e^{-t}\ln t + \frac{1}{4}t^2e^{-t} + t^2e^{-t}\ln t - t^2e^{-t} \\ &= c_1e^{-t} + c_2te^{-t} + \frac{1}{2}t^2e^{-t}\ln t - \frac{3}{4}t^2e^{-t}. \end{aligned}$$

**16.** The auxiliary equation is $2m^2 + m = m(2m+1) = 0$, so $y_c = c_1 + c_2e^{-x/2}$ and

$$W = \begin{vmatrix} 1 & e^{-x/2} \\ 0 & -\frac{1}{2}e^{-x/2} \end{vmatrix} = -\frac{1}{2}e^{-x/2}$$

Dividing by 2 we identify $f(t) = 3x$ and obtain

$$\begin{aligned} u_1' &= 6x \\ u_2' &= -6xe^{x/2}. \end{aligned}$$

Then

$$\begin{aligned} u_1 &= 3x^2 \\ u_2 &= -12xe^{x/2} + 24e^{x/2} \end{aligned}$$

and

$$\begin{aligned} y &= c_1 + c_2e^{-x/2} + 3x^2 - 12x + 24 \\ &= c_3 + c_2e^{-x/2} + 3x^2 - 12x \end{aligned}$$

**17.** The auxiliary equation is $3m^2 - 6m + 6 = 0$, so $y_c = e^x(c_1 \cos x + c_2 \sin x)$ and

$$W = \begin{vmatrix} e^x \cos x & e^x \sin x \\ e^x \cos x - e^x \sin x & e^x \cos x + e^x \sin x \end{vmatrix} = e^{2x}$$

Identifying $f(x) = \frac{1}{3} e^x \sec x$ we obtain

$$u_1' = -\frac{(e^x \sin x)(e^x \sec x)/3}{e^{2x}} = -\frac{1}{3} \tan x$$

$$u_2' = \frac{(e^x \cos x)(e^x \sec x)/3}{e^{2x}} = \frac{1}{3}.$$

Then $u_1 = \frac{1}{3} \ln |\cos x|$, $u_2 = \frac{1}{3} x$, and

$$y = c_1 e^x \cos x + c_2 e^x \sin x + \frac{1}{3} \ln |\cos x| e^x \cos x + \frac{1}{3} x e^x \sin x.$$

**18.** The auxiliary equation is $4m^2 - 4m + 1 = (2m-1)^2 = 0$, so $y_c = c_1 e^{x/2} + c_2 x e^{x/2}$ and

$$W = \begin{vmatrix} e^{x/2} & x e^{x/2} \\ \frac{1}{2} e^{x/2} & \frac{1}{2} x e^{x/2} + e^{x/2} \end{vmatrix} = e^x$$

Identifying $f(x) = \frac{1}{4} e^{x/2} \sqrt{1 - x^2}$ we obtain

$$u_1' = -\frac{x e^{x/2} e^{x/2} \sqrt{1-x^2}}{4e^x} = -\frac{1}{4} x \sqrt{1-x^2}$$

$$u_2' = \frac{e^{x/2} e^{x/2} \sqrt{1-x^2}}{4e^x} = \frac{1}{4} \sqrt{1-x^2}.$$

To find $u_1$ and $u_2$ we use the substitution $v = 1 - x^2$ and the trig substitution $x = \sin\theta$, respectively:

$$u_1 = \frac{1}{12} \left(1 - x^2\right)^{3/2}$$

$$u_2 = \frac{x}{8} \sqrt{1-x^2} + \frac{1}{8} \sin^{-1} x.$$

Thus

$$y = c_1 e^{x/2} + c_2 x e^{x/2} + \frac{1}{12} e^{x/2} \left(1 - x^2\right)^{3/2} + \frac{1}{8} x^2 e^{x/2} \sqrt{1-x^2} + \frac{1}{8} x e^{x/2} \sin^{-1} x.$$

**19.** The auxiliary equation is $4m^2 - 1 = (2m-1)(2m+1) = 0$, so $y_c = c_1 e^{x/2} + c_2 e^{-x/2}$ and

$$W = \begin{vmatrix} e^{x/2} & e^{-x/2} \\ \frac{1}{2}e^{x/2} & -\frac{1}{2}e^{-x/2} \end{vmatrix} = -1$$

Identifying $f(x) = xe^{x/2}/4$ we obtain $u_1' = x/4$ and $u_2' = -xe^x/4$. Then $u_1 = x^2/8$ and $u_2 = -xe^x/4 + e^x/4$. Thus

$$\begin{aligned} y &= c_1 e^{x/2} + c_2 e^{-x/2} + \frac{1}{8}x^2 e^{x/2} - \frac{1}{4}xe^{x/2} + \frac{1}{4}e^{x/2} \\ &= c_3 e^{x/2} + c_2 e^{-x/2} + \frac{1}{8}x^2 e^{x/2} - \frac{1}{4}xe^{x/2} \end{aligned}$$

and

$$y' = \frac{1}{2}c_3 e^{x/2} - \frac{1}{2}c_2 e^{-x/2} + \frac{1}{16}x^2 e^{x/2} + \frac{1}{8}xe^{x/2} - \frac{1}{4}e^{x/2}.$$

The initial conditions imply

$$\begin{aligned} c_3 + \quad c_2 \qquad &= 0 \\ \frac{1}{2}c_3 - \frac{1}{2}c_2 - \frac{1}{4} &= 0 \end{aligned}$$

Thus $c_3 = 3/4$ and $c_2 = 1/4$, and

$$y = \frac{3}{4}e^{x/2} + \frac{1}{4}e^{-x/2} + \frac{1}{8}x^2 e^{x/2} - \frac{1}{4}xe^{x/2}.$$

**20.** The auxiliary equation is $2m^2 + m - 1 = (2m-1)(m+1) = 0$, so $y_c = c_1 e^{x/2} + c_2 e^{-x}$ and

$$W = \begin{vmatrix} e^{x/2} & e^{-x} \\ \frac{1}{2}e^{x/2} & -e^{-x} \end{vmatrix} = -\frac{3}{2}e^{-x/2}$$

Identifying $f(x) = (x+1)/2$ we obtain

$$\begin{aligned} u_1' &= \frac{1}{3}e^{-x/2}(x+1) \\ u_2' &= -\frac{1}{3}e^{x}(x+1). \end{aligned}$$

Then

$$\begin{aligned} u_1 &= -e^{-x/2}\left(\frac{2}{3}x - 2\right) \\ u_2 &= -\frac{1}{3}xe^{x}. \end{aligned}$$

Thus

$$y = c_1 e^{x/2} + c_2 e^{-x} - x - 2$$

and

$$y' = \frac{1}{2}c_1e^{x/2} - c_2e^{-x} - 1.$$

The initial conditions imply

$$c_1 - c_2 - 2 = 1$$

$$\frac{1}{2}c_1 - c_2 - 1 = 0.$$

Thus $c_1 = 8/3$ and $c_2 = 1/3$, and

$$y = \frac{8}{3}e^{x/2} + \frac{1}{3}e^{-x} - x - 2.$$

**21.** The auxiliary equation is $m^2 + 2m - 8 = (m-2)(m+4) = 0$, so $y_c = c_1e^{2x} + c_2e^{-4x}$ and

$$W = \begin{vmatrix} e^{2x} & e^{-4x} \\ 2e^{2x} & -4e^{-4x} \end{vmatrix} = -6e^{-2x}$$

Identifying $f(x) = 2e^{-2x} - e^{-x}$ we obtain

$$u_1' = \frac{1}{3}e^{-4x} - \frac{1}{6}e^{-3x}$$

$$u_2' = \frac{1}{6}e^{3x} - \frac{1}{3}e^{2x}.$$

Then

$$u_1 = -\frac{1}{12}e^{-4x} + \frac{1}{18}e^{-3x}$$

$$u_2 = \frac{1}{18}e^{3x} - \frac{1}{6}e^{2x}.$$

Thus

$$\begin{aligned} y &= c_1e^{2x} + c_2e^{-4x} - \frac{1}{12}e^{-2x} + \frac{1}{18}e^{-x} + \frac{1}{18}e^{-x} - \frac{1}{6}e^{-2x} \\ &= c_1e^{2x} + c_2e^{-4x} - \frac{1}{4}e^{-2x} + \frac{1}{9}e^{-x} \end{aligned}$$

and

$$y' = 2c_1e^{2x} - 4c_2e^{-4x} + \frac{1}{2}e^{-2x} - \frac{1}{9}e^{-x}.$$

The initial conditions imply

$$c_1 + \ c_2 - \frac{5}{36} = 1$$

$$2c_1 - 4c_2 + \frac{7}{18} = 0$$

Thus $c_1 = 25/36$ and $c_2 = 4/9$, and

$$y = \frac{25}{36}e^{2x} + \frac{4}{9}e^{-4x} - \frac{1}{4}e^{-2x} + \frac{1}{9}e^{-x}.$$

**22.** The auxiliary equation is $m^2 - 4m + 4 = (m-2)^2 = 0$, so $y_c = c_1 e^{2x} + c_2 x e^{2x}$ and

$$W = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & 2xe^{2x} + e^{2x} \end{vmatrix} = e^{4x}$$

Identifying $f(x) = \left(12x^2 - 6x\right) e^{2x}$ we obtain

$$\begin{aligned} u_1' &= 6x^2 - 12x^3 \\ u_2' &= 12x^2 - 6x. \end{aligned}$$

Then

$$\begin{aligned} u_1 &= 2x^3 - 3x^4 \\ u_2 &= 4x^3 - 3x^2. \end{aligned}$$

Thus

$$\begin{aligned} y &= c_1 e^{2x} + c_2 x e^{2x} + \left(2x^3 - 3x^4\right) e^{2x} + \left(4x^3 - 3x^2\right) x e^{2x} \\ &= c_1 e^{2x} + c_2 x e^{2x} + e^{2x}\left(x^4 - x^3\right) \end{aligned}$$

and

$$y' = 2c_1 e^{2x} + c_2\left(2xe^{2x} + e^{2x}\right) + e^{2x}\left(4x^3 - 3x^2\right) + 2e^{2x}\left(x^4 - x^3\right).$$

The initial conditions imply

$$\begin{aligned} c_1 \qquad &= 0 \\ 2c_1 + c_2 &= 0 \end{aligned}$$

Thus $c_1 = 1$ and $c_2 = -2$, and

$$y = e^{2x} - 2xe^{2x} + e^{2x}\left(x^4 - x^3\right) = e^{2x}\left(x^4 - x^3 - 2x + 1\right).$$

**23.** The auxiliary equation is $m^2 + 1 = 0$, so $y_c = c_1 \cos x + c_2 \sin x$ and

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

Identifying $f(x) = e^{x^2}$ we obtain $u_1' = -e^{x^2} \sin x$ and $u_2' = e^{x^2} \cos x$. Then

$$\begin{aligned} u_1 &= -\int_{x_0}^{x} e^{t^2} \sin t \, dt \\ u_2 &= \int_{x_0}^{x} e^{t^2} \cos t \, dt \end{aligned}$$

and

$$y = c_1 \cos x + c_2 \sin x - \cos x \int_{x_0}^{x} e^{t^2} \sin t \, dt + \sin x \int_{x_0}^{x} e^{t^2} \cos t \, dt.$$

**24.** The auxiliary equation is $m^2 - 4 = 0$, so $y_c = c_1 e^{2x} + c_2 e^{-2x}$ and

$$W = \begin{vmatrix} e^{2x} & e^{-2x} \\ 2e^{2x} & -2e^{-2x} \end{vmatrix} = -4$$

Identifying $f(x) = e^{2x}/x$ we obtain $u_1' = 1/4x$ and $u_2' = -e^{4x}/4x$. Then

$$u_1 = \frac{1}{4} \ln |x|,$$

$$u_2 = -\frac{1}{4} \int_{x_0}^{x} \frac{e^{4t}}{t}\, dt$$

and

$$y = c_1 e^{2x} + c_2 e^{-2x} + \frac{1}{4}\left( e^{2x} \ln |x| - e^{-2x} \int_{x_0}^{x} \frac{e^{4t}}{t}\, dt \right), \qquad x_0 > 0.$$

**25.** The auxiliary equation is $m^2 + m - 2 = 0$, so $y_c = c_1 e^{-2x} + c_2 e^{x}$ and

$$W = \begin{vmatrix} e^{-2x} & e^{x} \\ -2e^{-2x} & e^{x} \end{vmatrix} = 3e^{-x}$$

Identifying $f(x) = \ln x$ we obtain $u_1' = -\frac{1}{3} e^{2x} \ln x$ and $u_2' = \frac{1}{3} e^{-x} \ln x$. Then

$$u_1 = -\frac{1}{3} \int_{x_0}^{x} e^{2t} \ln t\, dt,$$

$$u_2 = \frac{1}{3} \int_{x_0}^{x} e^{-t} \ln t\, dt$$

and

$$y = c_1 e^{-2x} + c_2 e^{x} - \frac{1}{3} e^{-2x} \int_{x_0}^{x} e^{2t} \ln t\, dt + \frac{1}{3} e^{x} \int_{x_0}^{x} e^{-t} \ln t\, dt, \qquad x_0 > 0.$$

**26.** The auxiliary equation is $2m^2 + 2m + 1 = 0$, so $y_c = e^{-x/2}[c_1 \cos (x/2) + c_2 \sin (x/2)]$ and

$$W = \begin{vmatrix} e^{-x/2} \cos \dfrac{x}{2} & e^{-x/2} \sin \dfrac{x}{2} \\ \dfrac{1}{2} e^{-x/2} \cos \dfrac{x}{2} - \dfrac{1}{2} e^{-x/2} \sin \dfrac{x}{2} & \dfrac{1}{2} e^{-x/2} \cos \dfrac{x}{2} - \dfrac{1}{2} e^{x/2} \sin \dfrac{x}{2} \end{vmatrix} = \frac{1}{2} e^{-x}$$

Identifying $f(x) = 2\sqrt{x}$ we obtain

$$u_1' = -\frac{e^{-x/2} \sin (x/2) 2\sqrt{x}}{\frac{1}{2} e^{-x/2}} = -4e^{x/2} \sqrt{x} \sin \frac{x}{2}$$

$$u_2' = -\frac{e^{-x/2} \cos (x/2) 2\sqrt{x}}{\frac{1}{2} e^{-x/2}} = 4e^{x/2} \sqrt{x} \cos \frac{x}{2}.$$

Then

$$u_1 = -4\int_{x_0}^{x} e^{t/2}\sqrt{t}\sin\frac{t}{2}\,dt$$

$$u_2 = 4\int_{x_0}^{x} e^{t/2}\sqrt{t}\cos\frac{t}{2}\,dt$$

and

$$\begin{aligned} y = e^{-x/2}\left(c_1\cos\frac{x}{2} + c_2\sin\frac{x}{2}\right) - 4e^{-x/2}\cos\frac{x}{2}\int_{x_0}^{x} e^{t/2}\sqrt{t}\sin\frac{t}{2}\,dt \\ + 4e^{-x/2}\sin\frac{x}{2}\int_{x_0}^{x} e^{t/2}\sqrt{t}\cos\frac{t}{2}\,dt. \end{aligned}$$

**27.** Write the equation in the form

$$y'' + \frac{1}{x}y' + \left(1 - \frac{1}{4x^2}\right)y = x^{-1/2}$$

and identify $f(x) = x^{-1/2}$. From $y_1 = x^{-1/2}\cos x$ and $y_2 = x^{-1/2}\sin x$ we compute

$$W = \begin{vmatrix} x^{-1/2}\cos x & x^{-1/2}\sin x \\ -x^{-1/2}\sin x - \frac{1}{2}x^{-3/2}\cos x & x^{-1/2}\cos x - \frac{1}{2}x^{-3/2}\sin x \end{vmatrix} = \frac{1}{x}$$

Now

$$u_1' = -\sin x \quad \text{so} \quad u_1 = \cos x,$$

and

$$u_2' = \cos x \quad \text{so} \quad u_2 = \sin x.$$

Thus a particular solution is

$$y_p = x^{-1/2}\cos^2 x + x^{-1/2}\sin^2 x,$$

and the general solution is

$$\begin{aligned} y &= c_1x^{-1/2}\cos x + c_2x^{-1/2}\sin x + x^{-1/2}\cos^2 x + x^{-1/2}\sin^2 x \\ &= c_1x^{-1/2}\cos x + c_2x^{-1/2}\sin x + x^{-1/2}. \end{aligned}$$

**28.** Write the equation in the form

$$y'' + \frac{1}{x}y' + \frac{1}{x^2}y = \frac{\sec(\ln x)}{x^2}$$

and identify $f(x) = \sec(\ln x)/x^2$. From $y_1 = \cos(\ln x)$ and $y_2 = \sin(\ln x)$ we compute

$$W = \begin{vmatrix} \cos(\ln x) & \sin(\ln x) \\ -\dfrac{\sin(\ln x)}{x} & \dfrac{\cos(\ln x)}{x} \end{vmatrix} = \frac{1}{x}$$

Now

$$u_1' = -\frac{\tan(\ln x)}{x} \quad \text{so} \quad u_1 = \ln|\cos(\ln x)|,$$

and

$$u_2' = \frac{1}{x} \quad \text{so} \quad u_2 = \ln x.$$

Thus, a particular solution is

$$y_p = \cos(\ln x)\ln|\cos(\ln x)| + (\ln x)\sin(\ln x),$$

and the general solution is

$$y = c_1\cos(\ln x) + c_2\sin(\ln x) + \cos(\ln x)\ln|\cos(\ln x)| + (\ln x)\sin(\ln x).$$

**29.** The auxiliary equation is $m^3 + m = m(m^2+1) = 0$, so $y_c = c_1 + c_2\cos x + c_3\sin x$ and

$$W = \begin{vmatrix} 1 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ 0 & -\cos x & -\sin x \end{vmatrix} = 1$$

Identifying $f(x) = \tan x$ we obtain

$$u_1' = W_1 = \begin{vmatrix} 0 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ \tan x & -\cos x & -\sin x \end{vmatrix} = \tan x$$

$$u_2' = W_2 = \begin{vmatrix} 1 & 0 & \sin x \\ 0 & 0 & \cos x \\ 0 & \tan x & -\sin x \end{vmatrix} = -\sin x$$

$$u_3' = W_3 = \begin{vmatrix} 1 & \cos x & 0 \\ 0 & -\sin x & 0 \\ 0 & -\cos x & \tan x \end{vmatrix} = -\sin x\tan x = \frac{\cos^2 x - 1}{\cos x} = \cos x - \sec x$$

Then

$$\begin{aligned} u_1 &= -\ln|\cos x| \\ u_2 &= \cos x \\ u_3 &= \sin x - \ln|\sec x + \tan x| \end{aligned}$$

and

$$\begin{aligned} y = c_1 + c_2\cos x + c_3\sin x - \ln|\cos x| + \cos^2 x \\ + \sin^2 x - \sin x\ln|\sec x + \tan x| \\ = c_4 + c_2\cos x + c_3\sin x - \ln|\cos x| - \sin x\ln|\sec x + \tan x| \end{aligned}$$

for $-\pi/2 < x < \pi/2$.

**30.** The auxiliary equation is $m^3+4m=m\left(m^2+4\right)=0$, so $y_c=c_1+c_2\cos 2x+c_3\sin 2x$ and

$$W=\begin{vmatrix}1 & \cos 2x & \sin 2x\\ 0 & -2\sin 2x & 2\cos 2x\\ 0 & -4\cos 2x & -4\sin 2x\end{vmatrix}=8$$

Identifying $f(x)=\sec 2x$ we obtain

$$u_1'=\frac{1}{8}W_1=\frac{1}{8}\begin{vmatrix}0 & \cos 2x & \sin 2x\\ 0 & -2\sin 2x & 2\cos 2x\\ \sec 2x & -4\cos 2x & -4\sin 2x\end{vmatrix}=\frac{1}{4}\sec 2x$$

$$u_2'=\frac{1}{8}W_2=\frac{1}{8}\begin{vmatrix}1 & 0 & \sin 2x\\ 0 & 0 & 2\cos 2x\\ 0 & \sec 2x & -4\sin 2x\end{vmatrix}=-\frac{1}{4}$$

$$u_3'=\frac{1}{8}W_3=\frac{1}{8}\begin{vmatrix}1 & \cos 2x & 0\\ 0 & -2\sin 2x & 0\\ 0 & -4\cos 2x & \sec 2x\end{vmatrix}=-\frac{1}{4}\tan 2x$$

Then

$$u_1=\frac{1}{8}\ln|\sec 2x+\tan 2x|$$

$$u_2=-\frac{1}{4}x$$

$$u_3=\frac{1}{8}\ln|\cos 2x|$$

and

$$y=c_1+c_2\cos 2x+c_3\sin 2x+\frac{1}{8}\ln|\sec 2x+\tan 2x|-\frac{1}{4}x\cos 2x+\frac{1}{8}\sin 2x\ln|\cos 2x|$$

for $-\pi/4<x<\pi/4$.

**31.** The auxiliary equation is $m^3-2m^2-m+2=0$ so $y_c=c_1e^x+c_2e^{-x}+c_3e^{2x}$

$$W=\begin{vmatrix}1 & e^x & e^{-x} & e^{2x}\\ e^x & -e^{-x} & 2e^{2x} & \\ e^x & e^{-x} & 4e^{2x} & \end{vmatrix}=-6e^{2x}$$

Identifying $f(x)=e^{4x}$ we obtain

$$u_1'=\frac{1}{-6e^{2x}}W_1=\frac{1}{-6e^{2x}}\begin{vmatrix}0 & e^{-x} & e^{2x}\\ 0 & -e-x & 2e^{2x}\\ e^{4x} & e^{-x} & 4e^{2x}\end{vmatrix}=\frac{1}{-6e^{2x}}\cdot 3e^{5x}=-\frac{1}{2}e^{3x}$$

$$u_2'=\frac{1}{-6e^{2x}}W_2=\frac{1}{-6e^{2x}}\begin{vmatrix}e^x & 0 & e^{2x}\\ e^x & 0 & 2e^{2x}\\ e^x & e^{4x} & 4e^{2x}\end{vmatrix}=\frac{1}{-6e^{2x}}\cdot\left(-e^{7x}\right)=\frac{1}{6}e^{5x}$$

$$u_3'=\frac{1}{-6e^{2x}}W_3=\frac{1}{-6e^{2x}}\begin{vmatrix}e^x & e^{-x} & 0\\ e^x & -e-x & 0\\ e^x & e^{-x} & e^{4x}\end{vmatrix}=\frac{1}{-6e^{2x}}\cdot\left(-2e^{4x}\right)=\frac{1}{3}e^{4x}$$

Then

$$u_1 = -\frac{1}{6}\,e^{3x}$$

$$u_2 = \frac{1}{30}\,e^{5x}$$

$$u_3 = \frac{1}{6}\,e^{2x}$$

Thus $y_p = -\dfrac{1}{6}\,e^{3x}\cdot e^{x} + \dfrac{1}{30}\,e^{5x}\cdot e^{-x} + \dfrac{1}{6}\,e^{2x}\cdot e^{2x} = \dfrac{1}{30}\,e^{4x}$ and $y = c_1e^{x} + c_2e^{-x} + c_3e^{2x} + \dfrac{1}{30}\,e^{4x}$ on $(-\infty, \infty)$.

**32.** The auxiliary equation is $m^3 - 3m^2 + 2m = m\,(m-1)\,(m-2) = 0$ so $y_c = c_1 + c_2e^{x} + c_3e^{2x}$

$$W = \begin{vmatrix} 1 & e^{x} & e^{2x} & e^{2x} \\ 0 & e^{x} & 2e^{2x} & \\ 0 & e^{x} & 4e^{2x} & \end{vmatrix} = 2e^{3x}$$

Identifying $f(x) = \dfrac{e^{2x}}{1+e^{x}}$ we obtain

$$u_1' = \frac{1}{2e^{3x}}W_1 = \frac{1}{2e^{3x}}\begin{vmatrix} 0 & e^{x} & e^{2x} \\ 0 & e^{x} & 2e^{2x} \\ \frac{e^{2x}}{1+e^{x}} & e^{x} & 4e^{2x} \end{vmatrix} = \frac{1}{2e^{3x}}\cdot\frac{e^{5x}}{1+e^{x}} = \frac{1}{2}\,\frac{e^{2x}}{1+e^{x}} = \frac{1}{2}\,e^{x} - \frac{1}{2}\,\frac{e^{x}}{1+e^{x}}$$

$$u_2' = \frac{1}{2e^{3x}}W_2 = \frac{1}{2e^{3x}}\begin{vmatrix} 1 & 0 & e^{2x} \\ 0 & 0 & 2e^{2x} \\ 0 & \frac{e^{2x}}{1+e^{x}} & 4e^{2x} \end{vmatrix} = \frac{1}{2e^{3x}}\cdot\left(-\frac{2e^{4x}}{1+e^{x}}\right) = -\frac{e^{x}}{1+e^{x}}$$

$$u_3' = \frac{1}{2e^{3x}}W_3 = \frac{1}{2e^{3x}}\begin{vmatrix} 1 & e^{x} & 0 \\ 0 & e^{x} & 0 \\ 0 & e^{x} & \frac{e^{2x}}{1+e^{x}} \end{vmatrix} = \frac{1}{2e^{3x}}\cdot\frac{e^{3x}}{1+e^{x}} = \frac{1}{2}\,\frac{1}{1+e^{x}} = \frac{1}{2}\,\frac{e^{-x}}{1+e^{-x}}$$

Then

$$u_1 = \frac{1}{2}\,e^{x} - \frac{1}{2}\,\ln\left(1+e^{x}\right)$$

$$u_2 = -\ln 1 + e^{x}$$

$$u_3 = -\frac{1}{2}\,\ln\left(1+e^{-x}\right)$$

Thus

$$\begin{aligned} y_p &= 1\cdot\left(\frac{1}{2}\,e^{x} - \frac{1}{2}\,\ln\left(1+e^{x}\right)\right) + e^{x}\cdot\left(-\ln\left(1+e^{x}\right)\right) + e^{2x}\cdot\left(-\frac{1}{2}\,\ln\left(1+e^{-x}\right)\right) \\ &= \frac{1}{2}\,e^{x} - \frac{1}{2}\,\ln\left(1+e^{x}\right) - e^{x}\ln\left(1+e^{x}\right) - \frac{1}{2}\,e^{2x}\ln\left(1+e^{-x}\right) \end{aligned}$$

**33.** The auxiliary equation is $3m^2 - 6m + 30 = 0$, which has roots $1 \pm 3i$, so $y_c = e^x(c_1 \cos 3x + c_2 \sin 3x)$. We consider first the differential equation $3y'' - 6y' + 30y = 15 \sin x$, which can be solved using undetermined coefficients. Letting $y_{p_1} = A\cos x + B\sin x$ and substituting into the differential equation we get

$$(27A - 6B)\cos x + (6A + 27B)\sin x = 15\sin x.$$

Then

$$27A - 6B = 0 \quad \text{and} \quad 6A + 27B = 15,$$

so $A = \frac{2}{17}$ and $B = \frac{9}{17}$. Thus, $y_{p_1} = \frac{2}{17}\cos x + \frac{9}{17}\sin x$. Next, we consider the differential equation $3y'' - 6y' + 30y$, for which a particular solution $y_{p_2}$ can be found using variation of parameters. The Wronskian is

$$W = \begin{vmatrix} e^x \cos 3x & e^x \sin 3x \\ e^x \cos 3x - 3e^x \sin 3x & 3e^x \cos 3x + e^x \sin 3x \end{vmatrix} = 3e^{2x}$$

Identifying $f(x) = \frac{1}{3}e^x \tan x$ we obtain

$$u_1' = -\frac{1}{9}\sin 3x \tan 3x = -\frac{1}{9}\left(\frac{\sin^2 3x}{\cos 3x}\right) = -\frac{1}{9}\left(\frac{1 - \cos^2 3x}{\cos 3x}\right) = -\frac{1}{9}(\sec 3x - \cos 3x)$$

so

$$u_1 = -\frac{1}{27}\ln|\sec 3x + \tan 3x| + \frac{1}{27}\sin 3x.$$

Next

$$u_2' = \frac{1}{9}\sin 3x \quad \text{so} \quad u_2 = -\frac{1}{27}\cos 3x.$$

Thus

$$\begin{aligned} y_{p_2} &= -\frac{1}{27}e^x \cos 3x(\ln|\sec 3x + \tan 3x| - \sin 3x) - \frac{1}{27}e^x \sin 3x \cos 3x \\ &= -\frac{1}{27}e^x(\cos 3x)\ln|\sec 3x + \tan 3x| \end{aligned}$$

and the general solution of the original differential equation is

$$y = e^x(c_1 \cos 3x + c_2 \sin 3x) + y_{p_1}(x) + y_{p_2}(x).$$

**34.** The auxiliary equation is $m^2 - 2m + 1 = (m-1)^2 = 0$, which has repeated root 1, so $y_c = c_1 e^x + c_2 x e^x$. We consider first the differential equation $y'' - 2y' + y = 4x^2 - 3$, which can be solved using undetermined coefficients. Letting $y_{p_1} = Ax^2 + Bx + C$ and substituting into the differential equation we get

$$Ax^2 + (-4A + B)x + (2A - 2B + C) = 4x^2 - 3.$$

Then

$$A = 4, \quad -4A + B = 0, \quad \text{and} \quad 2A - 2B + C = -3,$$

so $A = 4$, $B = 16$, and $C = 21$. Thus, $y_{p_1} = 4x^2 + 16x + 21$. Next we consider the differential equation $y'' - 2y' + y = x^{-1}e^x$, for which a particular solution $y_{p_2}$ can be found using variation of parameters. The Wronskian is

$$W = \begin{vmatrix} e^x & xe^x \\ e^x & xe^x + e^x \end{vmatrix} = e^{2x}$$

Identifying $f(x) = e^x/x$ we obtain $u_1' = -1$ and $u_2' = 1/x$. Then $u_1 = -x$ and $u_2 = \ln x$, so that

$$y_{p_2} = -xe^x + xe^x \ln x,$$

and the general solution of the original differential equation is

$$\begin{aligned} y = y_c + y_{p_1} + y_{p_2} &= c_1e^x + c_2xe^x + 4x^2 + 16x + 21 - xe^x + xe^x \ln x \\ &= c_1e^x + c_3xe^x + 4x^2 + 16x + 21 + xe^x \ln x \end{aligned}$$

**35.** The interval of definition for Problem 1 is $(-\pi/2, \pi/2)$, for Problem 7 is $(-\infty, \infty)$, for Problem 15 is $(0, \infty)$, and for Problem 18 is $(-1, 1)$. In Problem 28 the general solution is

$$y = c_1 \cos(\ln x) + c_2 \sin(\ln x) + \cos(\ln x) \ln|\cos(\ln x)| + (\ln x)\sin(\ln x)$$

for $-\pi/2 < \ln x < \pi/2$ or $e^{-\pi/2} < x < e^{\pi/2}$. The bounds on $\ln x$ are due to the presence of $\sec(\ln x)$ in the differential equation.

**36.** We are given that $y_1 = x^2$ is a solution of $x^4y'' + x^3y' - 4x^2y = 0$. To find a second solution we use reduction of order. Let $y = x^2u(x)$. Then the product rule gives

$$y' = x^2u' + 2xu \quad \text{and} \quad y'' = x^2u'' + 4xu' + 2u,$$

so

$$x^4y'' + x^3y' - 4x^2y = x^5(xu'' + 5u') = 0.$$

Letting $w = u'$, this becomes $xw' + 5w = 0$. Separating variables and integrating we have

$$\frac{dw}{w} = -\frac{5}{x}\,dx \quad \text{and} \quad \ln|w| = -5\ln x + c.$$

Thus, $w = x^{-5}$ and $u = -\frac{1}{4}x^{-4}$. A second solution is then $y_2 = x^2x^{-4} = 1/x^2$, and the general solution of the homogeneous differential equation is $y_c = c_1x^2 + c_2/x^2$. To find a particular solution, $y_p$, we use variation of parameters. The Wronskian is

$$W = \begin{vmatrix} x^2 & \dfrac{1}{x^2} \\ 2x & -\dfrac{2}{x^3} \end{vmatrix} = -\frac{4}{x}$$

Identifying $f(x) = 1/x^4$ we obtain $u_1' = \frac{1}{4}x^{-5}$ and $u_2' = -\frac{1}{4}x^{-1}$. Then $u_1 = -\frac{1}{16}x^{-4}$ and $u_2 = -\frac{1}{4}\ln x$, so

$$y_p = -\frac{1}{16}x^{-4}x^2 - \frac{1}{4}(\ln x)x^{-2} = -\frac{1}{16}x^{-2} - \frac{1}{4}x^{-2}\ln x.$$

The general solution is

$$y = c_1x^2 + \frac{c_2}{x^2} - \frac{1}{16x^2} - \frac{1}{4x^2}\ln x.$$

## 4.7 Cauchy–Euler Equation

**1.** The auxiliary equation is $m^2 - m - 2 = (m+1)(m-2) = 0$ so that $y = c_1x^{-1} + c_2x^2$.

**2.** The auxiliary equation is $4m^2 - 4m + 1 = (2m-1)^2 = 0$ so that $y = c_1x^{1/2} + c_2x^{1/2}\ln x$.

**3.** The auxiliary equation is $m^2 = 0$ so that $y = c_1 + c_2\ln x$.

**4.** The auxiliary equation is $m^2 - 4m = m(m-4) = 0$ so that $y = c_1 + c_2x^4$.

**5.** The auxiliary equation is $m^2 + 4 = 0$ so that $y = c_1\cos(2\ln x) + c_2\sin(2\ln x)$.

**6.** The auxiliary equation is $m^2 + 4m + 3 = (m+1)(m+3) = 0$ so that $y = c_1x^{-1} + c_2x^{-3}$.

**7.** The auxiliary equation is $m^2 - 4m - 2 = 0$ so that $y = c_1x^{2-\sqrt{6}} + c_2x^{2+\sqrt{6}}$.

**8.** The auxiliary equation is $m^2 + 2m - 4 = 0$ so that $y = c_1x^{-1+\sqrt{5}} + c_2x^{-1-\sqrt{5}}$.

**9.** The auxiliary equation is $25m^2 + 1 = 0$ so that $y = c_1\cos\left(\frac{1}{5}\ln x\right) + c_2\sin\left(\frac{1}{5}\ln x\right)$.

**10.** The auxiliary equation is $4m^2 - 1 = (2m-1)(2m+1) = 0$ so that $y = c_1x^{1/2} + c_2x^{-1/2}$.

**11.** The auxiliary equation is $m^2 + 4m + 4 = (m+2)^2 = 0$ so that $y = c_1x^{-2} + c_2x^{-2}\ln x$.

**12.** The auxiliary equation is $m^2 + 7m + 6 = (m+1)(m+6) = 0$ so that $y = c_1x^{-1} + c_2x^{-6}$.

**13.** The auxiliary equation is $3m^2 + 3m + 1 = 0$ so that

$$y = x^{-1/2}\left[c_1\cos\left(\frac{\sqrt{3}}{6}\ln x\right) + c_2\sin\left(\frac{\sqrt{3}}{6}\ln x\right)\right].$$

**14.** The auxiliary equation is $m^2 - 8m + 41 = 0$ so that $y = x^4\left[c_1\cos(5\ln x) + c_2\sin(5\ln x)\right]$.

**15.** Assuming that $y = x^m$ and substituting into the differential equation we obtain

$$m(m-1)(m-2) - 6 = m^3 - 3m^2 + 2m - 6 = (m-3)(m^2+2) = 0.$$

Thus

$$y = c_1x^3 + c_2\cos\left(\sqrt{2}\ln x\right) + c_3\sin\left(\sqrt{2}\ln x\right).$$

**16.** Assuming that $y = x^m$ and substituting into the differential equation we obtain

$$m(m-1)(m-2) + m - 1 = m^3 - 3m^2 + 3m - 1 = (m-1)^3 = 0.$$

Thus

$$y = c_1x + c_2x\ln x + c_3x(\ln x)^2.$$

**17.** Assuming that $y = x^m$ and substituting into the differential equation we obtain

$$m(m-1)(m-2)(m-3)+6m(m-1)(m-2) = m^4-7m^2+6m = m(m-1)(m-2)(m+3) = 0.$$

Thus

$$y = c_1 + c_2x + c_3x^2 + c_4x^{-3}.$$

**18.** Assuming that $y = x^m$ and substituting into the differential equation we obtain

$$m(m-1)(m-2)(m-3)+6m(m-1)(m-2)+9m(m-1)+3m+1 = m^4+2m^2+1 = (m^2+1)^2 = 0.$$

Thus

$$y = c_1 \cos(\ln x) + c_2 \sin(\ln x) + c_3(\ln x)\cos(\ln x) + c_4(\ln x)\sin(\ln x).$$

**19.** The auxiliary equation is $m^2 - 5m = m(m-5) = 0$ so that $y_c = c_1 + c_2x^5$ and

$$W\left(1, x^5\right) = \begin{vmatrix} 1 & x^5 \\ 0 & 5x^4 \end{vmatrix} = 5x^4\,.$$

Identifying $f(x) = x^3$ we obtain $u_1' = -\frac{1}{5}x^4$ and $u_2' = 1/5x$. Then $u_1 = -\frac{1}{25}x^5$, $u_2 = \frac{1}{5}\ln x$, and

$$y = c_1 + c_2x^5 - \frac{1}{25}x^5 + \frac{1}{5}x^5 \ln x = c_1 + c_3x^5 + \frac{1}{5}x^5 \ln x.$$

**20.** The auxiliary equation is $2m^2 + 3m + 1 = (2m+1)(m+1) = 0$ so that $y_c = c_1x^{-1} + c_2x^{-1/2}$ and

$$W\left(x^{-1}, x^{-1/2}\right) = \begin{vmatrix} x^{-1} & x^{-1/2} \\ -x^{-2} & -\dfrac{1}{2}x^{-3/2} \end{vmatrix} = \frac{1}{2}x^{-5/2}\,.$$

Identifying $f(x) = \frac{1}{2} - \frac{1}{2x}$ we obtain $u_1' = x - x^2$ and $u_2' = x^{3/2} - x^{1/2}$. Then $u_1 = \frac{1}{2}x^2 - \frac{1}{3}x^3$, $u_2 = \frac{2}{5}x^{5/2} - \frac{2}{3}x^{3/2}$, and

$$y = c_1x^{-1} + c_2x^{-1/2} + \frac{1}{2}x - \frac{1}{3}x^2 + \frac{2}{5}x^2 - \frac{2}{3}x = c_1x^{-1} + c_2x^{-1/2} - \frac{1}{6}x + \frac{1}{15}x^2.$$

**21.** The auxiliary equation is $m^2 - 2m + 1 = (m-1)^2 = 0$ so that $y_c = c_1x + c_2x\ln x$ and

$$W\left(x, x^2\right) = \begin{vmatrix} x & x\ln x \\ 1 & 1+\ln x \end{vmatrix} = x$$

Identifying $f(x) = 2/x$ we obtain $u_1' = -2(\ln x)/x$ and $u_2' = 2/x$. Then $u_1 = -(\ln x)^2$, $u_2 = 2\ln x$, and

$$\begin{aligned} y &= c_1x + c_2x\ln x - x(\ln x)^2 + 2x(\ln x)^2 \\ &= c_1x + c_2x\ln x + x(\ln x)^2, \qquad x > 0. \end{aligned}$$

**22.** The auxiliary equation is $m^2 - 3m + 2 = (m-1)(m-2) = 0$ so that $y_c = c_1x + c_2x^2$ and

$$W\left(x, x^2\right) = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = x^2$$

Identifying $f(x) = x^2e^x$ we obtain $u_1' = -x^2e^x$ and $u_2' = xe^x$. Then $u_1 = -x^2e^x + 2xe^x - 2e^x$, $u_2 = xe^x - e^x$, and

$$\begin{aligned} y &= c_1x + c_2x^2 - x^3e^x + 2x^2e^x - 2xe^x + x^3e^x - x^2e^x \\ &= c_1x + c_2x^2 + x^2e^x - 2xe^x. \end{aligned}$$

**23.** The auxiliary equation $m(m-1) + m - 1 = m^2 - 1 = 0$ has roots $m_1 = -1$, $m_2 = 1$, so $y_c = c_1x^{-1} + c_2x$. With $y_1 = x^{-1}$, $y_2 = x$, and the identification $f(x) = \ln x/x^2$, we get

$$W = 2x^{-1}, \qquad W_1 = -\left(\ln x\right)/x, \qquad \text{and} \qquad W_2 = \left(\ln x\right)/x^3.$$

Then $u_1' = W_1/W = -(\ln x)/2$, $u_2' = W_2/W = (\ln x)/2x^2$, and integration by parts gives

$$\begin{aligned} u_1 &= \frac{1}{2}x - \frac{1}{2}x\ln x \\ u_2 &= -\frac{1}{2}x^{-1}\ln x - \frac{1}{2}x^{-1}, \end{aligned}$$

so

$$y_p = u_1y_1 + u_2y_2 = \left(\frac{1}{2}x - \frac{1}{2}x\ln x\right)x^{-1} + \left(-\frac{1}{2}x^{-1}\ln x - \frac{1}{2}x^{-1}\right)x = -\ln x$$

and

$$y = y_c + y_p = c_1x^{-1} + c_2x - \ln x, \qquad x > 0.$$

**24.** The auxiliary equation $m(m-1) + m - 1 = m^2 - 1 = 0$ has roots $m_1 = -1$, $m_2 = 1$, so $y_c = c_1x^{-1} + c_2x$. With $y_1 = x^{-1}$, $y_2 = x$, and the identification $f(x) = 1/x^2(x+1)$, we get

$$W = 2x^{-1}, \qquad W_1 = -1/x(x+1), \qquad \text{and} \qquad W_2 = 1/x^3(x+1).$$

Then $u_1' = W_1/W = -1/2(x+1)$, $\quad u_2' = W_2/W = 1/2x^2(x+1)$, and integration (by partial fractions for $u_2'$) gives

$$\begin{aligned} u_1 &= -\frac{1}{2}\ln\left(x+1\right) \\ u_2 &= -\frac{1}{2}x^{-1} - \frac{1}{2}\ln x + \frac{1}{2}\ln\left(x+1\right), \end{aligned}$$

so

$$\begin{aligned} y_p = u_1y_1 + u_2y_2 &= \left[-\frac{1}{2}\ln\left(x+1\right)\right]x^{-1} + \left[-\frac{1}{2}x^{-1} - \frac{1}{2}\ln x + \frac{1}{2}\ln\left(x+1\right)\right]x \\ &= -\frac{1}{2} - \frac{1}{2}x\ln x + \frac{1}{2}x\ln\left(x+1\right) - \frac{\ln\left(x+1\right)}{2x} = -\frac{1}{2} + \frac{1}{2}x\ln\left(1 + \frac{1}{x}\right) - \frac{\ln\left(x+1\right)}{2x} \end{aligned}$$

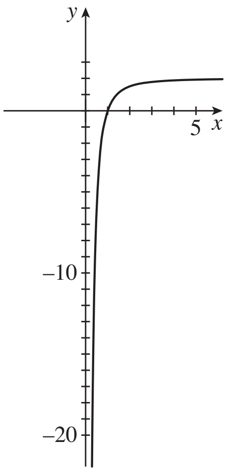
and

$$y = y_c + y_p = c_1x^{-1} + c_2x - \frac{1}{2} + \frac{1}{2}x\ln\left(1 + \frac{1}{x}\right) - \frac{\ln\left(x+1\right)}{2x}, \qquad x > 0.$$

**25.** The auxiliary equation is $m^2 + 2m = m(m+2) = 0$, so that $y = c_1 + c_2x^{-2}$ and $y' = -2c_2x^{-3}$. The initial conditions imply

$$\begin{aligned} c_1 + c_2 &= 0 \\ -2c_2 &= 4. \end{aligned}$$

Thus, $c_1 = 2$, $c_2 = -2$, and $y = 2 - 2x^{-2}$. The graph is given to the right.



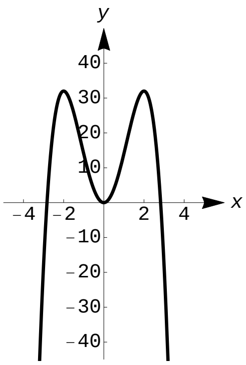
**26.** The auxiliary equation is $m^2 - 6m + 8 = (m-2)(m-4) = 0$, so that

$$y = c_1x^2 + c_2x^4 \quad \text{and} \quad y' = 2c_1x + 4c_2x^3.$$

The initial conditions imply

$$\begin{aligned} 4c_1 + 16c_2 &= 32 \\ 4c_1 + 32c_2v &= 0. \end{aligned}$$

Thus, $c_1 = 16$, $c_2 = -2$, and $y = 16x^2 - 2x^4$. The graph is given to the right.

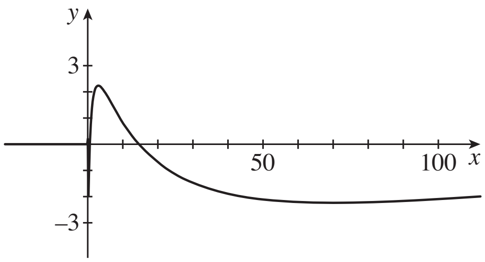


**27.** The auxiliary equation is $m^2 + 1 = 0$, so that

$$y = c_1 \cos(\ln x) + c_2 \sin(\ln x)$$

and

$$y' = -c_1\frac{1}{x}\sin(\ln x) + c_2\frac{1}{x}\cos(\ln x).$$

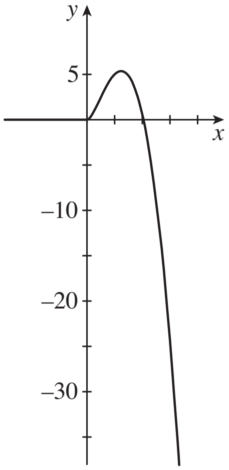


The initial conditions imply $c_1 = 1$ and $c_2 = 2$. Thus $y = \cos(\ln x) + 2\sin(\ln x)$. The graph is given to the right.

**28.** The auxiliary equation is $m^2 - 4m + 4 = (m-2)^2 = 0$, so that

$$y = c_1x^2 + c_2x^2 \ln x \quad \text{and} \quad y' = 2c_1x + c_2(x + 2x\ln x).$$

The initial conditions imply $c_1 = 5$ and $c_2 + 10 = 3$. Thus $y = 5x^2 - 7x^2 \ln x$. The graph is given to the right.



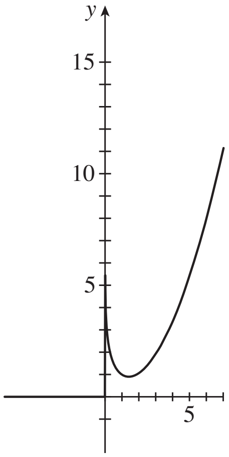
**29.** The auxiliary equation is $m^2 = 0$ so that $y_c = c_1 + c_2 \ln x$ and

$$W(1, \ln x) = \begin{vmatrix} 1 & \ln x \\ 0 & \dfrac{1}{x} \end{vmatrix} = \frac{1}{x}$$

Identifying $f(x) = 1$ we obtain $u_1' = -x\ln x$ and $u_2' = x$. Then $u_1 = \frac{1}{4}x^2 - \frac{1}{2}x^2 \ln x$, $u_2 = \frac{1}{2}x^2$, and

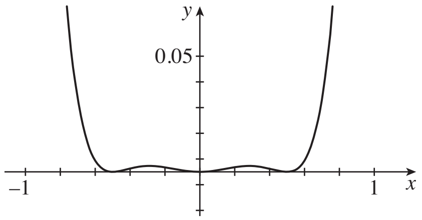
$$y = c_1 + c_2 \ln x + \frac{1}{4}x^2 - \frac{1}{2}x^2 \ln x + \frac{1}{2}x^2 \ln x = c_1 + c_2 \ln x + \frac{1}{4}x^2.$$

The initial conditions imply $c_1 + \frac{1}{4} = 1$ and $c_2 + \frac{1}{2} = -\frac{1}{2}$. Thus, $c_1 = \frac{3}{4}$, $c_2 = -1$, and $y = \frac{3}{4} - \ln x + \frac{1}{4}x^2$. The graph is given to the right.



**30.** The auxiliary equation is $m^2 - 6m + 8 = (m-2)(m-4) = 0$, so that $y_c = c_1x^2 + c_2x^4$ and

$$W\left(x^2, x^4\right) = \begin{vmatrix} x^2 & x^4 \\ 2x & 4x^3 \end{vmatrix} = 2x^5$$



Identifying $f(x) = 8x^4$ we obtain $u_1' = -4x^3$ and $u_2' = 4x$. Then $u_1 = -x^4$, $u_2 = 2x^2$, and $y = c_1x^2 + c_2x^4 + x^6$. The initial conditions imply

$$\frac{1}{4}c_1 + \frac{1}{16}c_2 = -\frac{1}{64}$$
$$c_1 + \frac{1}{2}c_2 = -\frac{3}{16}.$$

Thus $c_1 = \frac{1}{16}$, $c_2 = -\frac{1}{2}$, and $y = \frac{1}{16}x^2 - \frac{1}{2}x^4 + x^6$. The graph is given to the right.

**31.** Substituting $x = e^t$ into the differential equation we obtain

$$\frac{d^2y}{dt^2} + 8\frac{dy}{dt} - 20y = 0.$$

The auxiliary equation is $m^2+8m-20=(m+10)(m-2)=0$ so that

$$y=c_1e^{-10t}+c_2e^{2t} \quad \text{or} \quad y=c_1x^{-10}+c_2x^2.$$

**32.** Substituting $x=e^t$ into the differential equation we obtain

$$\frac{d^2y}{dt^2}-10\frac{dy}{dt}+25y=0.$$

The auxiliary equation is $m^2-10m+25=(m-5)^2=0$ so that

$$y=c_1e^{5t}+c_2te^{5t} \quad \text{or} \quad y=c_1x^5+c_2x^5\ln x.$$

**33.** Substituting $x=e^t$ into the differential equation we obtain

$$\frac{d^2y}{dt^2}+9\frac{dy}{dt}+8y=e^{2t}.$$

The auxiliary equation is $m^2+9m+8=(m+1)(m+8)=0$ so that $y_c=c_1e^{-t}+c_2e^{-8t}$. Using undetermined coefficients we try $y_p=Ae^{2t}$. This leads to $30Ae^{2t}=e^{2t}$, so that $A=1/30$ and

$$y=c_1e^{-t}+c_2e^{-8t}+\frac{1}{30}e^{2t} \quad \text{or} \quad y=c_1x^{-1}+c_2x^{-8}+\frac{1}{30}x^2.$$

**34.** Substituting $x=e^t$ into the differential equation we obtain

$$\frac{d^2y}{dt^2}-5\frac{dy}{dt}+6y=2t.$$

The auxiliary equation is $m^2-5m+6=(m-2)(m-3)=0$ so that $y_c=c_1e^{2t}+c_2e^{3t}$. Using undetermined coefficients we try $y_p=At+B$. This leads to $(-5A+6B)+6At=2t$, so that $A=1/3$, $B=5/18$, and

$$y=c_1e^{2t}+c_2e^{3t}+\frac{1}{3}t+\frac{5}{18} \quad \text{or} \quad y=c_1x^2+c_2x^3+\frac{1}{3}\ln x+\frac{5}{18}.$$

**35.** Substituting $x=e^t$ into the differential equation we obtain

$$\frac{d^2y}{dt^2}-4\frac{dy}{dt}+13y=4+3e^t.$$

The auxiliary equation is $m^2-4m+13=0$ so that $y_c=e^{2t}(c_1\cos 3t+c_2\sin 3t)$. Using undetermined coefficients we try $y_p=A+Be^t$. This leads to $13A+10Be^t=4+3e^t$, so that $A=4/13$, $B=3/10$, and

$$y=e^{2t}(c_1\cos 3t+c_2\sin 3t)+\frac{4}{13}+\frac{3}{10}e^t$$

or

$$y=x^2\left[c_1\cos(3\ln x)+c_2\sin(3\ln x)\right]+\frac{4}{13}+\frac{3}{10}x.$$

**36.** From

$$\frac{d^2y}{dx^2} = \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$$

it follows that

$$\begin{aligned}\frac{d^3y}{dx^3} &= \frac{1}{x^2}\frac{d}{dx}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) - \frac{2}{x^3}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)\\ &= \frac{1}{x^2}\frac{d}{dx}\left(\frac{d^2y}{dt^2}\right) - \frac{1}{x^2}\frac{d}{dx}\left(\frac{dy}{dt}\right) - \frac{2}{x^3}\frac{d^2y}{dt^2} + \frac{2}{x^3}\frac{dy}{dt}\\ &= \frac{1}{x^2}\frac{d^3y}{dt^3}\left(\frac{1}{x}\right) - \frac{1}{x^2}\frac{d^2y}{dt^2}\left(\frac{1}{x}\right) - \frac{2}{x^3}\frac{d^2y}{dt^2} + \frac{2}{x^3}\frac{dy}{dt}\\ &= \frac{1}{x^3}\left(\frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 2\frac{dy}{dt}\right).\end{aligned}$$

Substituting into the differential equation we obtain

$$\frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 2\frac{dy}{dt} - 3\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) + 6\frac{dy}{dt} - 6y = 3 + 3t$$

or

$$\frac{d^3y}{dt^3} - 6\frac{d^2y}{dt^2} + 11\frac{dy}{dt} - 6y = 3 + 3t.$$

The auxiliary equation is $m^3 - 6m^2 + 11m - 6 = (m-1)(m-2)(m-3) = 0$ so that $y_c = c_1e^t + c_2e^{2t} + c_3e^{3t}$. Using undetermined coefficients we try $y_p = A + Bt$. This leads to $(11B - 6A) - 6Bt = 3 + 3t$, so that $A = -17/12$, $B = -1/2$, and

$$y = c_1e^t + c_2e^{2t} + c_3e^{3t} - \frac{17}{12} - \frac{1}{2}t \quad \text{or} \quad y = c_1x + c_2x^2 + c_3x^3 - \frac{17}{12} - \frac{1}{2}\ln x.$$

*In the next two problems we use the substitution $t = -x$ since the initial conditions are on the interval $(-\infty, 0)$. In this case*

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = -\frac{dy}{dx}$$

*and*

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d}{dt}\left(-\frac{dy}{dx}\right) = -\frac{d}{dt}(y') = -\frac{dy'}{dx}\frac{dx}{dt} = -\frac{d^2y}{dx^2}\frac{dx}{dt} = \frac{d^2y}{dx^2}.$$

**37.** The differential equation and initial conditions become

$$4t^2\frac{d^2y}{dt^2} + y = 0; \quad y(t)\Big|_{t=1} = 2, \quad y'(t)\Big|_{t=1} = -4.$$

The auxiliary equation is $4m^2 - 4m + 1 = (2m-1)^2 = 0$, so that

$$y = c_1t^{1/2} + c_2t^{1/2}\ln t \quad \text{and} \quad y' = \frac{1}{2}c_1t^{-1/2} + c_2\left(t^{-1/2} + \frac{1}{2}t^{-1/2}\ln t\right).$$

The initial conditions imply $c_1 = 2$ and $1 + c_2 = -4$. Thus

$$y = 2t^{1/2} - 5t^{1/2}\ln t \quad \text{or} \quad y = 2(-x)^{1/2} - 5(-x)^{1/2}\ln(-x), \quad x < 0.$$

**38.** The differential equation and initial conditions become

$$t^2\frac{d^2y}{dt^2}-4t\frac{dy}{dt}+6y=0; \quad y(t)\Big|_{t=2}=8, \quad y'(t)\Big|_{t=2}=0.$$

The auxiliary equation is $m^2-5m+6=(m-2)(m-3)=0$, so that

$$y=c_1t^2+c_2t^3 \quad \text{and} \quad y'=2c_1t+3c_2t^2.$$

The initial conditions imply

$$\begin{aligned}4c_1+8c_2&=8\\4c_1+12c_2&=0\end{aligned}$$

from which we find $c_1=6$ and $c_2=-2$. Thus

$$y=6t^2-2t^3 \quad \text{or} \quad y=6x^2+2x^3, \quad x<0.$$

**39.** If we force the function $y=(x-(-3))^m=(x+3)^m$ into the equation we get

$$\begin{aligned}(x+3)^2y''-8(x+3)y'+14y&=0\\(x+3)^2m(m-1)(x+3)^{m-2}-8(x+3)m(x+3)^{m-1}+14(x+3)^m&=0\\(x+3)^m[m^2-9m+14]&=0\\(x+3)^m(m-2)(m-7)&=0\end{aligned}$$

The solutions to the auxiliary equation are therefore $m=2$ and $m=7$ from which we get the general solution $y=c_1(x+3)^2+c_2(x+3)^7$.

**40.** If we force the function $y=(x-1)^m$ into the equation we get

$$\begin{aligned}(x-1)^2y''-(x-1)y'+5y&=0\\(x-1)^2m(m-1)(x-1)^{m-2}-(x-1)m(x-1)^{m-1}+5(x-1)^m&=0\\(x-1)^m[m^2-2m+5]&=0\end{aligned}$$

The solutions to the auxiliary equation are therefore the complex roots $m=1\pm2i$ from which we get the general solution $y=c_1(x-1)\cos(2\ln(x-1))+c_2(x-1)\sin(2\ln(x-1))$.

**41.** Letting $t=x+2$ we obtain $dy/dx=dy/dt$ and, using the Chain Rule,

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dt}\right)=\frac{d^2y}{dt^2}\frac{dt}{dx}=\frac{d^2y}{dt^2}(1)=\frac{d^2y}{dt^2}.$$

Substituting into the differential equation we obtain

$$t^2\frac{d^2y}{dt^2}+t\frac{dy}{dt}+y=0.$$

The auxiliary equation is $m^2+1=0$ so that

$$y=c_1\cos(\ln 2t)+c_2\sin(\ln 2t)=c_2\cos[\ln(x+2)]+c_2\sin[\ln(x+2)].$$

**42.** Letting $t = x - 4$ we obtain $dy/dx = dy/dt$ and, using the Chain Rule,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dt}\right) = \frac{d^2y}{dt^2}\frac{dt}{dx} = \frac{d^2y}{dt^2}(1) = \frac{d^2y}{dt^2}.$$

Substituting into the differential equation we obtain

$$t^2\frac{d^2y}{dt^2} - 5t\frac{dy}{dt} + 9y = 0.$$

The auxiliary equation is $m(m-1) - 5m + 9 = m^2 - 6m + 9 = (m-3)^2 = 0$ so that $m_1 = m_2 = 3$. Then

$$y = c_1t^3 + c_2t^3\ln t = c_1(x-4)^3 + c_2(x-4)^3\ln(x-4).$$

**43.** Since the leading coefficient $a_2(x) = (x-4)^2 \neq 0$ for every value of $x$ satisfying $x > 4$ we may take the interval of definition of the general solution to be $(4, \infty)$.

**44.** If $1 - i$ is a root of the auxiliary equation then so is $1 + i$, and the auxiliary equation is

$$(m-2)[m-(1+i)][m-(1-i)] = m^3 - 4m^2 + 6m - 4 = 0.$$

We need $m^3 - 4m^2 + 6m - 4$ to have the form $m(m-1)(m-2) + bm(m-1) + cm + d$. Expanding this last expression and equating coefficients we get $b = -1$, $c = 3$, and $d = -4$. Thus, the differential equation is

$$x^3y''' - x^2y'' + 3xy' - 4y = 0.$$

**45.** For $x^2y'' = 0$ the auxiliary equation is $m(m-1) = 0$ and the general solution is $y = c_1 + c_2x$. The initial conditions imply $c_1 = y_0$ and $c_2 = y_1$, so $y = y_0 + y_1x$. The initial conditions are satisfied for all real values of $y_0$ and $y_1$.

For $x^2y'' - 2xy' + 2y = 0$ the auxiliary equation is $m^2 - 3m + 2 = (m-1)(m-2) = 0$ and the general solution is $y = c_1x + c_2x^2$. The initial condition $y(0) = y_0$ implies $0 = y_0$ and the condition $y'(0) = y_1$ implies $c_1 = y_1$. Thus, the initial conditions are satisfied for $y_0 = 0$ and for all real values of $y_1$.

For $x^2y'' - 4xy' + 6y = 0$ the auxiliary equation is $m^2 - 5m + 6 = (m-2)(m-3) = 0$ and the general solution is $y = c_1x^2 + c_2x^3$. The initial conditions imply $y(0) = 0 = y_0$ and $y'(0) = 0$. Thus, the initial conditions are satisfied only for $y_0 = y_1 = 0$.

**46.** The function $y(x) = -\sqrt{x}\cos(2\ln x)$ is defined for $x > 0$ and has $x$-intercepts where $2\ln x = \pi/2 + k\pi$ for $k$ an integer or where $x = e^{\pi/4 + k\pi/2}$. Solving $\pi/4 + k\pi/2 = 0.5$ we get $k \approx -0.18$, so $e^{\pi/2 + k\pi} < 0.5$ for all negative integers and the graph has infinitely many $x$-intercepts in the interval $(0, 0.5)$.

**47. (a)** Multiplying by $r^3$ we have

$$\frac{d^3w}{dr^3}+\frac{1}{r}\frac{d^2w}{dr^2}-\frac{1}{r^2}\frac{dw}{dr}=\frac{q}{2D}r$$
$$r^3\frac{d^3w}{dr^3}+r^2\frac{d^2w}{dr^2}-r\frac{dw}{dr}=\frac{q}{2D}r^4$$

This is a nonhomogeneous third-order Cauchy-Euler differential equation. We first solve the homogenous equation. The auxiliary equation is

$$\begin{aligned}m(m-1)(m-2)+m(m-1)-m&=0\\ m^3-2m^2&=0\\ m^2(m-2)&=0\end{aligned}$$

Therefore $m_1=m_2=0$ and $m_3=2$. Then the complementary function is $w_c(r)=c_1+c_2\ln r+c_3r^2$. To use variation of parameters we must use the first form of the differential equation and identify $f(r)=qr/2D$. By (10) of Section 3.5 we identify $y_1=1$, $y_2=\ln r$, and $y_3=r^2$, and expand each of the four determinants by the first column:

$$W_1=\begin{vmatrix}0 & \ln r & r^2\\ 0 & \dfrac{1}{r} & 2r\\ \dfrac{q}{2D}r & -\dfrac{1}{r^2} & 2\end{vmatrix}=\frac{q}{2D}r\begin{vmatrix}\ln r & r^2\\ \dfrac{1}{r} & 2r\end{vmatrix}=\frac{q}{2D}\left(2r^2\ln r-r^2\right)$$

$$W_2=\begin{vmatrix}1 & 0 & r^2\\ 0 & 0 & 2r\\ 0 & \dfrac{q}{2D}r & 2\end{vmatrix}=1\cdot\begin{vmatrix}0 & 2r\\ \dfrac{q}{2D}r & 2\end{vmatrix}=-\frac{q}{D}r^2$$

$$W_3=\begin{vmatrix}1 & \ln r & 0\\ 0 & \dfrac{1}{r} & 0\\ 0 & -\dfrac{1}{r^2} & \dfrac{q}{2D}r\end{vmatrix}=1\cdot\begin{vmatrix}\dfrac{1}{r} & 0\\ -\dfrac{1}{r^2} & \dfrac{q}{2D}r\end{vmatrix}=\frac{q}{2D}$$

$$W=\begin{vmatrix}1 & \ln r & r^2\\ 0 & \dfrac{1}{r} & 2r\\ 0 & -\dfrac{1}{r^2} & 2\end{vmatrix}=1\cdot\begin{vmatrix}\dfrac{1}{r} & 2r\\ -\dfrac{1}{r^2} & 2\end{vmatrix}=\frac{2}{r}-\left(-\frac{2}{r}\right)=\frac{4}{r}$$

Then

$$\begin{aligned}
u_1' &= \frac{W_1}{W} = \frac{\dfrac{q}{2D}\,r\,(2r\ln r - r)}{\dfrac{4}{r}} = \frac{q}{4D}\,r^3\ln r - \frac{q}{8D}\,r^3 \\
u_2' &= \frac{W_2}{W} = \frac{-\dfrac{q}{D}\,r^2}{\dfrac{4}{r}} = -\frac{q}{4D}\,r^3 \\
u_3' &= \frac{W_3}{W} = \frac{\dfrac{q}{2D}}{\dfrac{4}{r}} = \frac{q}{8D}\,r
\end{aligned}$$

and

$$\begin{aligned}
u_1 &= \frac{q}{16D}\,r^4\ln r - \frac{3q}{64D}\,r^4 \qquad \text{(integration by parts)} \\
u_2 &= -\frac{q}{16D}\,r^4 \\
u_3 &= \frac{q}{16D}\,r^2
\end{aligned}$$

Thus a particular solution is

$$w_p(r) = \left(\frac{q}{16D}\,r^4\ln r - \frac{3q}{64D}\,r^4\right)\cdot 1 + \left(-\frac{q}{16D}\,r^4\right)\ln r + \left(\frac{4q}{64D}\,r^2\right)r^2 = \frac{q}{64D}\,r^4$$

Therefore the general solution is $w(r) = w_c(r) + w_p(r)$ or

$$w(r) = c_1 + c_2\ln r + c_3r^2 + \frac{q}{64D}\,r^4$$

**(b)** Taking the derivative of $w$ we have

$$w'(r) - \frac{c_2}{r} + 2c_3r + \frac{q}{16D}\,r^3\,.$$

The condition $w'(0) = 0$ requires $c_2 = 0$. Using the condition $w'(a) = 0$ gives

$$\begin{aligned}
0 &= 2c_3a + \frac{q}{16D}\,a^3 \\
c_3 &= -\frac{q}{32D}\,a^2\,.
\end{aligned}$$

So,

$$w(r) = c_1 - \frac{q}{32D}\,a^2r^2 + \frac{q}{64D}\,r^4\,.$$

Using the condition $w(a) = 0$ gives

$$\begin{aligned}
0 &= c_1 - \frac{q}{32D}\,a^4 + \frac{q}{64D}\,a^4 \\
c_1 &= \frac{q}{64D}\,a^4\,.
\end{aligned}$$

Therefore

$$w(r) = \frac{q}{64D}\,a^4 - \frac{q}{32D}\,a^2r^2 + \frac{q}{64D}\,r^4 = \frac{q}{64D}\left(a^2 - r^2\right)^2 .$$

**48. (a)** Using the product rule twice, the left-hand side becomes

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\,\frac{dw}{dr}\right)\right] = \frac{d}{dr}\left[\frac{1}{r}\left(r\frac{d^2w}{dr^2} + \frac{dw}{dr}\right)\right] = \frac{d}{dr}\left[\frac{d^2w}{dr^2} + \frac{1}{r}\frac{dw}{dr}\right]$$
$$= \frac{d^3w}{dr3} + \frac{1}{r}\frac{d^2w}{dr^2} - \frac{1}{r^2}\frac{dw}{dr}$$

This is the left-hand side of equation (8).

**(b)** Integrating both sides with respect to $r$ gives

$$\begin{aligned}
\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\,\frac{dw}{dr}\right)\right] &= \frac{q}{2D}\,r \\
\frac{1}{r}\frac{d}{dr}\left(r\,\frac{dw}{dr}\right) &= \frac{q}{4D}\,r^2 + C_1 \\
\frac{d}{dr}\left(r\,\frac{dw}{dr}\right) &= \frac{1}{4D}\,r^3 + C_1r \\
r\,\frac{dw}{dr} &= \frac{q}{16D}\,r^4 + \frac{C_1}{2}\,r^2 + C_2 \\
\frac{dw}{dr} &= \frac{1}{16D}\,r^3 + \frac{C_1}{2}\,r + \frac{C_2}{r} \\
w(r) &= \frac{q}{64D}\,r^2 + \frac{C_1}{4}\,r^2 + C_2\ln r + C_3\,.
\end{aligned}$$

Relabeling $c_1 = C_3$, $c_2 = C_2$, and $c_3 = C_1/4$ shows this is the same as the general solution obtained in part (a) of Problem 47.

**49.** The auxiliary equation is $2m(m-1)(m-2) - 10.98m(m-1) + 8.5m + 1.3 = 0$, so that $m_1 = -0.053299$, $m_2 = 1.81164$, $m_3 = 6.73166$, and

$$y = c_1x^{-0.053299} + c_2x^{1.81164} + c_3x^{6.73166}.$$

**50.** The auxiliary equation is $m(m-1)(m-2) + 4m(m-1) + 5m - 9 = 0$, so that $m_1 = 1.40819$ and the two complex roots are $-1.20409 \pm 2.22291i$. The general solution of the differential equation is

$$y = c_1x^{1.40819} + x^{-1.20409}\left[c_2\cos\left(2.22291\ln x\right) + c_3\sin\left(2.22291\ln x\right)\right].$$

**51.** The auxiliary equation is

$$m(m-1)(m-2)(m-3) + 6m(m-1)(m-2) + 3m(m-1) - 3m + 4 = 0,$$

so that $m_1 = m_2 = \sqrt{2}$ and $m_3 = m_4 = -\sqrt{2}$. The general solution of the differential equation is

$$y = c_1 x^{\sqrt{2}} + c_2 x^{\sqrt{2}} \ln x + c_3 x^{-\sqrt{2}} + c_4 x^{-\sqrt{2}} \ln x.$$

**52.** The auxiliary equation is $m(m-1)(m-2)(m-3) - 6m(m-1)(m-2) + 33m(m-1) - 105m + 169 = 0$, so that $m_1 = m_2 = 3 + 2i$ and $m_3 = m_4 = 3 - 2i$. The general solution of the differential equation is

$$y = x^3 \left[c_1 \cos(2 \ln x) + c_2 \sin(2 \ln x)\right] + x^3 \ln x \left[c_3 \cos(2 \ln x) + c_4 \sin(2 \ln x)\right].$$

**53.** First solve the associated homogeneous equation. From the auxiliary equation $(m-3)(m-2)(m+1) = 0$ we find $y_c = c_1 x^3 + c_2 x^2 + c_3 x^{-1}$. Before finding the particular solution using variation of parameters, first put the differential equation in standard form by dividing through by $x^3$. We therefore identify $f(x) = x^{-1}$. Now with $y_1 = x^3$, $y_2 = x^2$, and $y_3 = x^{-1}$ we get

$$W = \begin{vmatrix} x^3 & x^2 & x^{-1} \\ 3x^2 & 2x & -x^{-2} \\ 6x & 2 & 2x^{-3} \end{vmatrix} = -12x, \qquad W_1 = \begin{vmatrix} 0 & x^2 & x^{-1} \\ 0 & 2x & -x^{-2} \\ x^{-1} & 2 & 2x^{-3} \end{vmatrix} = -3x,$$

$$W_2 = \begin{vmatrix} x^3 & 0 & x^{-1} \\ 3x^2 & 0 & -x^{-2} \\ 6x & x^{-1} & 2x^{-3} \end{vmatrix} = 4 \qquad W_3 = \begin{vmatrix} x^3 & x^2 & 0 \\ 3x^2 & 2x & 0 \\ 6x & 2 & x^{-1} \end{vmatrix} = -x^3$$

Therefore $u_1' = \dfrac{W_1}{W} = \dfrac{1}{4x^2}$, $u_2' = \dfrac{W_2}{W} = \dfrac{-1}{3x}$, and $u_3' = \dfrac{W_3}{W} = \dfrac{x^2}{12}$. The integrals of these three functions are straight forward: $u_1 = -1/4x$, $u_2 = -(\ln x)/3$, $u_3 = x^3/36$. The particular solution is

$$\begin{aligned} y_p = u_1 y_1 + u_2 y_2 + u_3 y_3 &= -\frac{x^2}{4} - \frac{x^2 \ln x}{3} + \frac{x^2}{36} \\ &= -\frac{2x^2}{9} - \frac{x^2 \ln x}{3} \end{aligned}$$

Finally, because $-2x^2/9$ can be absorbed in the $c_2 x^2$ term the general solutions is

$$y = y_c + y_p = c_1 x^3 + c_2 x^2 + c_3 x^{-1} - \frac{x^2 \ln x}{3}, \quad x > 0.$$

## 4.8 Green's Functions

**1.**

$$\begin{aligned} y''-16y&=f(x)\\ y''-16y&=0\\ y_1=e^{-4x},\quad y_2&=e^{4x}\\ W\left(e^{-4x},e^{4x}\right)&=\begin{vmatrix} e^{-4x} & e^{4x}\\ -4e^{-4x} & 4e^{4x}\end{vmatrix}=8\end{aligned}$$

$$\begin{aligned} G(x,t)&=\frac{e^{-4t}e^{4x}-e^{-4x}e^{4t}}{8}=\frac{1}{8}\left[e^{4(x-t)}-e^{-4(x-t)}\right]\\ y_p(x)&=\frac{1}{8}\int_{x_0}^{x}\left[e^{4(x-t)}-e^{-4(x-t)}\right]f(t)\,dt=\frac{1}{4}\int_{x_0}^{x}\sinh 4(x-t)\,f(t)\,dt\end{aligned}$$

**2.**

$$\begin{aligned} y''+3y'-10y&=f(x)\\ y''+3y'-10y&=0\\ y_1=e^{2x},\quad y_2&=e^{-5x}\\ W\left(e^{2x},e^{-5x}\right)&=\begin{vmatrix} e^{2x} & e^{-5x}\\ 2e^{2x} & -5e^{-5x}\end{vmatrix}=-7e^{-3x}\end{aligned}$$

$$\begin{aligned} G(x,t)&=\frac{e^{2t}e^{-5x}-e^{2x}e^{-5t}}{-7e^{-3t}}=\frac{1}{7}\left[e^{2(x-t)}-e^{-5(x-t)}\right]\\ y_p(x)&=\frac{1}{7}\int_{x_0}^{x}\left[e^{2(x-t)}-e^{-5(x-t)}\right]f(t)\,dt\end{aligned}$$

**3.**

$$\begin{aligned} y''+2y'+y&=f(x)\\ y''+2y'+y&=0\end{aligned}$$

$$\begin{aligned} y_1=e^{-x},\quad y_2&=xe^{-x}\\ W\left(e^{-x},xe^{-x}\right)&=\begin{vmatrix} e^{-x} & xe^{-x}\\ -e^{-x} & -xe^{-x}+e^{-x}\end{vmatrix}=e^{-2x}\end{aligned}$$

$$\begin{aligned} G(x,t)&=\frac{e^{-t}xe^{-x}-e^{-x}te^{-t}}{e^{-2t}}=(x-t)e^{-(x-t)}\\ y_p(x)&=\int_{x_0}^{x}(x-t)e^{-(x-t)}f(t)\,dt\end{aligned}$$

**4.**

$$\begin{aligned}
4y'' - 4y' + y &= f(x) \\
4y'' - 4y' + y &= 0 \\
y_1 = e^{x/2}, \quad y_2 &= xe^{x/2} \\
W\left(e^{x/2}, xe^{x/2}\right) &= \begin{vmatrix} e^{x/2} & xe^{x/2} \\ \frac{1}{2}e^{x/2} & \frac{1}{2}xe^{x/2} + e^{x/2} \end{vmatrix} = e^x
\end{aligned}$$

$$\begin{aligned}
G(x,t) &= \frac{e^{t/2}xe^{x/2} - e^{x/2}te^{t/2}}{e^t} = (x-t)e^{(x-t)/2} \\
y_p(x) &= \int_{x_0}^{x} (x-t)e^{(x-t)/2}\frac{1}{4}f(t)\,dt
\end{aligned}$$

**5.**

$$\begin{aligned}
y'' + 9y &= f(x) \\
y'' + 9y &= 0 \\
y_1 = \cos 3x, \quad y_2 &= \sin 3x \\
W(\cos 3x, \sin 3x) &= \begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix} = 3
\end{aligned}$$

$$\begin{aligned}
G(x,t) &= \frac{\cos 3t \sin 3x - \cos 3x \sin 3t}{3} = \frac{1}{3}\sin 3(x-t) \\
y_p(x) &= \frac{1}{3}\int_{x_0}^{x} \sin 3(x-t)f(t)\,dt
\end{aligned}$$

**6.**

$$\begin{aligned}
y'' - 2y' + 2y &= f(x) \\
y'' - 2y' + 2y &= 0 \\
y_1 = e^x\cos x, \quad y_2 &= e^x \sin x \\
W(e^x\cos x, e^x\sin x) &= \begin{vmatrix} e^x\cos x & e^x\sin x \\ -e^x\sin x + e^x\cos x & e^x\cos x + e^x\sin x \end{vmatrix} = e^{2x}
\end{aligned}$$

$$\begin{aligned}
G(x,t) &= \frac{e^t\cos t e^x\sin x - e^x\cos x e^t\sin t}{e^{2t}} = e^{x-t}\sin(x-t) \\
y_p(x) &= \int_{x_0}^{x} e^{x-t}\sin(x-t)f(t)\,dt
\end{aligned}$$

**7.**

$$\begin{aligned} y'' - 16y &= xe^{-2x} \\ y &= y_c + y_p \\ y &= c_1e^{-4x} + c_2e^{4x} + y_p \\ y &= c_1e^{-4x} + c_2e^{4x} + \frac{1}{4}\int_{x_0}^{x} \sinh 4(x-t)te^{-2t}\,dt \end{aligned}$$

**8.**

$$\begin{aligned} y'' + 3y' - 10y &= x^2 \\ y &= y_c + y_p \\ y &= c_1e^{2x} + c_2e^{-5x} + y_p \\ y &= c_1e^{2x} + c_2e^{-5x} + \frac{1}{7}\int_{x_0}^{x} \left[e^{2(x-t)} - e^{-5(x-t)}\right] t^2\,dt \end{aligned}$$

**9.**

$$\begin{aligned} y'' + 2y' + y &= e^{-x} \\ y &= y_c + y_p \\ y &= c_1e^{-x} + c_2xe^{-x} + y_p \\ y &= c_1e^{-x} + c_2xe^{-x} + \int_{x_0}^{x} (x-t)e^{-(x-t)}e^{-t}\,dt \end{aligned}$$

**10.**

$$\begin{aligned} 4y'' - 4y' + y &= \arctan x \\ y &= y_c + y_p \\ y &= c_1e^{x/2} + c_2xe^{x/2} + y_p \\ y &= c_1e^{x/2} + c_2xe^{x/2} + \int_{x_0}^{x} (x-t)e^{(x-t)/2}\frac{1}{4}\arctan t\,dt \end{aligned}$$

**11.**

$$\begin{aligned} y'' + 9y &= x + \sin x \\ y &= y_c + y_p \\ y &= c_1\cos 3x + c_2\sin 3x + y_p \\ y &= c_1\cos 3x + c_2\sin 3x + \frac{1}{3}\int_{x_0}^{x} \sin 3(x-t)(t + \sin t)\,dt \end{aligned}$$

**12.**

$$\begin{aligned}
y'' - 2y' + 2y &= \cos^2 x \\
y &= y_c + y_p \\
y &= c_1 e^x \cos x + c_2 e^x \sin x + y_p \\
y &= c_1 e^x \cos x + c_2 e^x \sin x = \int_{x_0}^{x} e^{(x-t)} \sin(x-t) \cos^2 t \, dt
\end{aligned}$$

**13.** The initial-value problem is $y'' - 4y = e^{2x}, \quad y(0) = 0,\ y'(0) = 0$. Then we find that

$$\begin{aligned}
y_1 &= e^{-2x}, \quad y_2 = e^{2x} \\
W\left(e^{-2x}, e^{2x}\right) &= \begin{vmatrix} e^{-2x} & e^{2x} \\ -2e^{-2x} & 2e^{2x} \end{vmatrix} = 4.
\end{aligned}$$

Then

$$G(x,t) = \frac{e^{-2t}e^{2x} - e^{-2x}e^{2t}}{4} = \frac{1}{4}\left[e^{2(x-t)} - e^{-2(x-t)}\right]$$

and the solution of the initial-value problem is

$$\begin{aligned}
y_p(x) &= \frac{1}{4}\int_0^x \left[e^{2(x-t)} - e^{-2(x-t)}\right] e^{2t} \, dt \\
&= \frac{1}{4}e^{2x}\int_0^x dt - \frac{1}{4}e^{-2x}\int_0^x e^{4t} \, dt \\
&= \frac{1}{4}xe^{2x} - \frac{1}{16}e^{2x} + \frac{1}{16}e^{-2x}.
\end{aligned}$$

**14.** The initial-value problem is $y'' - y' = 1, \quad y(0) = 0,\ y'(0) = 0$. Then we find that

$$\begin{aligned}
y_1 &= 1, \quad y_2 = e^{x} \\
W\left(1, e^{x}\right) &= \begin{vmatrix} 1 & e^{x} \\ 0 & e^{x} \end{vmatrix} = e^{x}.
\end{aligned}$$

Then

$$G(x,t) = \frac{e^x - e^t}{e^t} = e^{x-t} - 1$$

and the solution of the initial-value problem is

$$\begin{aligned}
y_p(x) &= \int_0^x \left(e^{x-t} - 1\right) dt \\
&= e^x\int_0^x e^{-t} \, dt - \int_0^x dt \\
&= e^x - x - 1.
\end{aligned}$$

**15.** The initial-value problem is $y'' - 10y' + 25y = e^{5x}, \quad y(0) = 0,\ y'(0) = 0$. Then we find that

$$y_1 = e^{5x}, \quad y_2 = xe^{5x}$$

$$W\left(e^{5x}, xe^{5x}\right) = \begin{vmatrix} e^{5x} & xe^{5x} \\ 5e^{5x} & 5xe^{5x} + e^{5x} \end{vmatrix} = e^{10x}.$$

Then

$$G(x,t) = \frac{e^{5t}xe^{5x} - e^{5x}te^{5t}}{e^{10t}} = (x-t)e^{5(x-t)}$$

and the solution of the initial-value problem is

$$\begin{aligned} y_p(x) &= \int_0^x (x-t)e^{5(x-t)}e^{5t}\,dt \\ &= xe^{5x}\int_0^x dt - e^{5x}\int_0^x t\,dt \\ &= x^2e^{5x} - \frac{1}{2}x^2e^{5x} \\ &= \frac{1}{2}x^2e^{5x}. \end{aligned}$$

**16.** The initial-value problem is $y'' + 6y' + 9y = x, \quad y(0) = 0,\ y'(0) = 0$. Then we find that

$$y_1 = e^{-3x}, \quad y_2 = xe^{-3x}$$

$$W\left(e^{-3x}, xe^{-3x}\right) = \begin{vmatrix} e^{-3x} & xe^{-3x} \\ -3e^{-3x} & -3xe^{-3x} + e^{-3x} \end{vmatrix} = e^{-6x}.$$

Then

$$G(x,t) = \frac{e^{-3t}xe^{-3x} - e^{-3x}te^{-3t}}{e^{-6t}} = (x-t)e^{-3(x-t)}$$

and the solution of the initial-value problem is

$$\begin{aligned} y_p(x) &= \int_0^x (x-t)e^{-3(x-t)}t\,dt \\ &= xe^{-3x}\int_0^x te^{3t}\,dt - e^{-3x}\int_0^x t^2e^{3t}\,dt \\ &= \left(-\frac{1}{9}x + \frac{1}{9}xe^{-3x} + \frac{1}{3}x^2\right) - \left(\frac{2}{27} - \frac{2}{27}e^{-3x} - \frac{2}{9}x + \frac{1}{3}x^2\right) \\ &= \frac{2}{27}e^{-3x} + \frac{1}{9}xe^{-3x} - \frac{2}{27} + \frac{1}{9}x. \end{aligned}$$

**17.** The initial-value problem is $y'' + y = \csc x \cot x, \quad y(\pi/2) = 0,\ y'(\pi/2) = 0$. Then we find that

$$y_1 = \cos x, \quad y_2 = \sin x$$

$$W(\cos x, \sin x) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1.$$

Then

$$G(x,t) = \cos t \sin x - \cos x \sin t = \sin(x-t)$$

and the solution of the initial-value problem is

$$\begin{aligned}
y_p(x) &= \int_{\pi/2}^{x} (\cos t \sin x - \cos x \sin t) \csc t \cot t \, dt \\
&= \sin x \int_{\pi/2}^{x} \cot^2 t \, dt - \cos x \int_{\pi/2}^{x} \cot t \, dt \\
&= \sin x \int_{\pi/2}^{x} (\csc^2 t - 1) \, dt - \cos x \int_{\pi/2}^{x} \cot t \, dt \\
&= \sin x \left( -\cot x - x + \frac{\pi}{2} \right) - \cos x \ln|\sin x| \\
&= -\cos x - x \sin x + \frac{\pi}{2} \sin x - \cos x \ln|\sin x| \\
&= -\cos x + \frac{\pi}{2} \sin x - x \sin x - \cos x \ln|\sin x|.
\end{aligned}$$

**18.** The initial-value problem is $y'' + y = \sec^2 x, \quad y(\pi) = 0,\ y'(\pi) = 0$. Then we find that

$$y_1 = \cos x, \quad y_2 = \sin x$$

$$W(\cos x, \sin x) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1.$$

Then

$$G(x,t) = \cos t \sin x - \cos x \sin t = \sin(x-t)$$

and the solution of the initial-value problem is

$$\begin{aligned}
y_p(x) &= \int_{\pi}^{x} (\cos t \sin x - \cos x \sin t) \sec^2 t \, dt \\
&= \sin x \int_{\pi}^{x} \sec t \, dt - \cos x \int_{\pi}^{x} \sec t \tan t \, dt \\
&= -\cos x - 1 + \sin x \ln|\sec x + \tan x|.
\end{aligned}$$

**19.** The initial-value problem is $y'' - 4y = e^{2x}, \quad y(0) = 1,\ y'(0) = -4$, so $y(x) = c_1 e^{-2x} + c_2 e^{2x}$. The initial conditions give

$$\begin{aligned}
c_1 + \phantom{2}c_2 &= \phantom{-}4 \\
-2c_1 + 2c_2 &= -4,
\end{aligned}$$

so $c_1 = \frac{3}{2}$ and $c_2 = -\frac{1}{2}$, which implies that $y_h = \frac{3}{2}e^{-2x} - \frac{1}{2}e^{2x}$. Now, $y_p$ found in the solution of Problem 13 in this section gives

$$\begin{aligned} y = y_h + y_p &= \frac{3}{2}e^{-2x} - \frac{1}{2}e^{2x} + \left(\frac{1}{4}xe^{2x} - \frac{1}{16}e^{2x} + \frac{1}{16}e^{-2x}\right) \\ &= \frac{25}{16}e^{-2x} - \frac{9}{16}e^{2x} + \frac{1}{4}xe^{2x}. \end{aligned}$$

**20.** The initial-value problem is $y'' - y' = 1, \quad y(0) = 10,\ y'(0) = 1$, so $y(x) = c_1 + c_2e^x$. The initial conditions give

$$\begin{aligned} c_1 + c_2 &= 10 \\ c_2 &= \ \ 1, \end{aligned}$$

so $c_1 = 9$ and $c_2 = 1$, which implies that $y_h = 9 + e^x$. Now, $y_p$ found in the solution of Problem 14 in this section gives

$$y = y_h + y_p = 9 + e^x + (e^x - x - 1) = 8 + 2e^x - x.$$

**21.** The initial-value problem is $y'' - 10y' + 25y = e^{5x}, \quad y(0) = -1,\ y'(0) = 1$, so $y(x) = c_1e^{5x} + c_2xe^{5x}$. The initial conditions give

$$\begin{aligned} c_1 &= -1 \\ 5c_1 + c_2 &= \ \ 1, \end{aligned}$$

so $c_1 = -1$ and $c_2 = 6$, which implies that $y_h = -e^{5x} + 6xe^{5x}$. Now, $y_p$ found in the solution of Problem 15 in this section gives

$$y = y_h + y_p = -e^{5x} + 6xe^{5x} + \frac{1}{2}x^2e^{5x}.$$

**22.** The initial-value problem is $y'' + 6y' + 9y = x, \quad y(0) = 1,\ y'(0) = -3$, so $y(x) = c_1e^{-3x} + c_2xe^{-3x}$. The initial conditions give

$$\begin{aligned} c_1 &= \ \ 1 \\ -3c_1 + c_2 &= -3, \end{aligned}$$

so $c_1 = 1$ and $c_2 = 0$, which implies that $y_h = e^{-3x}$. Now, $y_p$ found in the solution of Problem 16 in this section gives

$$y = y_h + y_p = e^{-3x} + \left(\frac{2}{27}e^{-3x} + \frac{1}{9}xe^{-3x} - \frac{2}{27} + \frac{1}{9}\right) = \frac{29}{27}e^{-3x} + \frac{1}{9}xe^{-3x} - \frac{2}{27} + \frac{1}{9}x.$$

**23.** The initial-value problem is $y'' + y = \csc x \cot x, \quad y(\pi/2) = -\frac{\pi}{2},\ y'(\pi/2) = -1$, so $y(x) = c_1 \cos x + c_2 \sin x$. The initial conditions give

$$\begin{aligned} c_2 &= -\frac{\pi}{2} \\ -c_1 &= -1, \end{aligned}$$

so $c_1 = 1$ and $c_2 = -\frac{\pi}{2}$, which implies that $y_h = \cos x - \frac{\pi}{2}\sin x$. Now, $y_p$ found in the solution of Problem 17 in this section gives

$$\begin{aligned} y = y_h + y_p &= \cos x - \frac{\pi}{2}\sin x + \Big(-\cos x + \frac{\pi}{2}\sin x - x\sin x - \cos x\ln|\sin x|\Big) \\ &= -x\sin x - \cos x\ln|\sin x|. \end{aligned}$$

**24.** The initial-value problem is $y'' + y = \sec^2 x, \quad y(\pi) = \frac{1}{2},\ y'(\pi) = -1$, so $y(x) = c_1\cos x + c_2\sin x$. The initial conditions give

$$\begin{aligned} -c_1 &= \ \frac{1}{2} \\ -c_2 &= -1, \end{aligned}$$

so $c_1 = -\frac{1}{2}$ and $c_2 = 1$, which implies that $y_h = -\frac{1}{2}\cos x + \sin x$. Now, $y_p$ found in the solution of Problem 18 in this section gives

$$\begin{aligned} y = y_h + y_p &= -\frac{1}{2}\cos x + \sin x + (-\cos x - 1 + \sin x\ln|\sec x + \tan x|) \\ &= -\frac{3}{2}\cos x + \sin x - 1 + \sin x\ln|\sec x + \tan x|. \end{aligned}$$

**25.** The initial-value problem is $y'' + 3y' + 2y = \sin e^x, \quad y(0) = -1,\ y'(0) = 0$, so $y(x) = c_1e^{-x} + c_2e^{-2x}$. The initial conditions give

$$\begin{aligned} c_1 + \ c_2 &= -1 \\ -c_1 - 2c_2 &= \ \ 0, \end{aligned}$$

so $c_1 = -2$ and $c_2 = 1$, which implies that $y_h = -2e^{-x} + e^{-2x}$. The Wronskian is

$$W\left(e^{-x}, e^{-2x}\right) = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix} = -e^{-3x}.$$

Then $G(x,t) = \dfrac{e^{-t}e^{-2x} - e^{-x}e^{-2t}}{-e^{-3t}} = e^{-x}e^{t} - e^{2t}e^{-2x}$ so

$$\begin{aligned} y_p(x) &= \int_0^x (e^{-x}e^t - e^{2t}e^{-2x})\sin e^t\,dt \\ &= e^{-x}\int_0^x e^t\sin e^t\,dt - e^{-2x}\int_0^x e^{2t}\sin e^t\,dt \\ &= e^{-x}\left(-\cos e^x + \cos 1\right) - e^{-2x}\left(-e^x\cos e^x + \sin e^x + \cos 1 - \sin 1\right) \\ &= e^{-x}\cos 1 + (\sin 1 - \cos 1)e^{-2x} - e^{-2x}\sin e^x. \end{aligned}$$

The solution of the initial-value problem is

$$\begin{aligned} y = y_h + y_p &= -2e^{-x} + e^{-2x} + \left(e^{-x}\cos 1 + e^{-2x}(\sin 1 - \cos 1) - e^{-2x}\sin e^x\right) \\ &= (\cos 1 - 2)e^{-x} + (1 + \sin 1 - \cos 1)e^{-2x} - e^{-2x}\sin e^x. \end{aligned}$$

**26.** The initial-value problem is $y'' + 3y' + 2y = \dfrac{1}{1+e^x}, \quad y(0) = 0,\ y'(0) = -2$, so $y(x) = c_1e^{-x} + c_2e^{-2x}$. The initial conditions give

$$\begin{aligned} c_1 + \ \ c_2 &= 0 \\ -c_1 - 2c_2 &= 1, \end{aligned}$$

so $c_1 = 1$ and $c_2 = -1$, which implies that $y_h = e^{-x} - e^{-2x}$. The Wronskian is

$$W\left(e^{-x}, e^{-2x}\right) = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix} = -e^{-3x}.$$

Then $G(x,t) = \dfrac{e^{-t}e^{-2x} - e^{-x}e^{-2t}}{-e^{-3t}} = e^{-x}e^{t} - e^{2t}e^{-2x}$ so

$$\begin{aligned} y_p(x) &= \int_0^x (e^{-x}e^t - e^{2t}e^{-2x})\frac{1}{1+e^t}\,dt = e^{-x}\int_0^x \frac{e^t}{1+e^t}\,dt - e^{-2x}\int_0^x \frac{e^{2t}}{1+e^t}\,dt \\ &= e^{-x}[\ln(1+e^x) - \ln 2] - e^{-2x}[e^x - \ln(1+e^x) - 1 + \ln 2] \\ &= e^{-x}(-1-\ln 2) + (1-\ln 2)e^{-2x} + e^{-x}\ln(1+e^x) + e^{-2x}\ln(1+e^x). \end{aligned}$$

The solution of the initial-value problem is

$$\begin{aligned} y = y_h + y_p &= e^{-x} - e^{-2x} + \left[e^{-x}(-1-\ln 2) + (1-\ln 2)e^{-2x} + e^{-x}\ln(1+e^x) + e^{-2x}\ln(1+e^x)\right] \\ &= -(\ln 2)e^{-x} - (\ln 2)e^{-2x} + e^{-x}\ln(1+e^x) + e^{-2x}\ln(1+e^x). \end{aligned}$$

**27.** The Cauchy-Euler initial-value problem $x^2y'' - 2xy' + 2y = x, \quad y(1) = 2,\ y'(1) = -1$, has auxiliary equation $m(m-1) - 2m + 2 = m^2 - 3m + 2 = (m-1)(m-2) = 0$ so $m_1 = 1$, $m_2 = 2$, $y(x) = c_1x + c_2x^2$, and $y' = c_1 + 2c_2x$. The initial conditions give

$$\begin{aligned} c_1 + \ \ c_2 &= \ \ 2 \\ c_1 + 2c_2 &= -1, \end{aligned}$$

so $c_1 = 5$ and $c_2 = -3$, which implies that $y_h = 5x - 3x^2$. The Wronskian is

$$W\left(x, x^2\right) = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = x^2.$$

Then $G(x,t) = \dfrac{tx^2 - xt^2}{t^2} = \dfrac{x(x-t)}{t}$. From the standard form of the differential equation we identify the forcing function $f(t) = \dfrac{1}{t}$. Then, for $x > 1$,

$$\begin{aligned} y_p(x) &= \int_1^x \frac{x(x-t)}{t}\,\frac{1}{t}\,dt \\ &= x^2\int_1^x \frac{1}{t^2}\,dt - x\int_1^x \frac{1}{t}\,dt \\ &= x^2\left(-\frac{1}{x} + 1\right) - x(\ln x - \ln 1) \\ &= x^2 - x - x\ln x. \end{aligned}$$

The solution of the initial-value problem is

$$y = y_h + y_p = (5x - 3x^2) + (x^2 - x - x\ln x) = 4x - 2x^2 - x\ln x.$$

**28.** The Cauchy-Euler initial-value problem $x^2y'' - 2xy' + 2y = x\ln x, \quad y(1) = 1,\ y'(1) = 0$, has auxiliary equation $m(m-1) - 2m + 2 = m^2 - 3m + 2 = (m-1)(m-2) = 0$ so $m_1 = 1$, $m_2 = 2$, $y(x) = c_1x + c_2x^2$, and $y' = c_1 + 2c_2x$. The initial conditions give

$$\begin{aligned} c_1 + \ \ c_2 &= 1 \\ c_1 + 2c_2 &= 0, \end{aligned}$$

so $c_1 = 2$ and $c_2 = -1$, which implies that $y_h = 2x - x^2$. The Wronskian is

$$W\left(x, x^2\right) = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = x^2.$$

Then $G(x,t) = \dfrac{tx^2 - xt^2}{t^2} = \dfrac{x(x-t)}{t}$. From the standard form of the differential equation we identify the forcing function $f(t) = \dfrac{\ln t}{t}$. Then, for $x > 1$,

$$\begin{aligned} y_p(x) &= \int_1^x \frac{x(x-t)}{t}\,\frac{\ln t}{t}\,dt = x^2\int_1^x t^{-2}\ln t\,dt - x\int_1^x \frac{\ln t}{t}\,dt \\ &= x^2\left(-\frac{\ln x}{x} - \frac{1}{x} + 1\right) - x\left(\frac{1}{2}(\ln x)^2\right) = x^2 - x - x\ln x - \frac{1}{2}x(\ln x)^2. \end{aligned}$$

The solution of the initial-value problem is

$$y = y_h + y_p = (2x - x^2) + \left(x^2 - x - x\ln x - \frac{1}{2}x(\ln x)^2\right) = x - x\ln x - \frac{1}{2}x(\ln x)^2.$$

**29.** The Cauchy-Euler initial-value problem $x^2y'' - 6y = \ln x, \quad y(1) = 1,\ y'(1) = 3$, has auxiliary equation $m(m-1) - 6 = m^2 - m - 6 = (m-3)(m+2) = 0$ so $m_1 = 3$, $m_2 = -2$, $y(x) = c_1x^3 + c_2x^{-2}$, and $y' = 3c_1x^2 - 2c_2x^{-3}$. The initial conditions give

$$\begin{aligned} c_1 + \ \ c_2 &= 1 \\ 3c_1 - 2c_2 &= 3, \end{aligned}$$

so $c_1 = 1$ and $c_2 = 0$, which implies that $y_h = x^3$. The Wronskian is

$$W\left(x^3, x^{-2}\right) = \begin{vmatrix} x^3 & x^{-2} \\ 3x^2 & -2x^{-3} \end{vmatrix} = -5.$$

Then $G(x,t) = \dfrac{t^3x^{-2} - x^3t^{-2}}{-5} = -\dfrac{1}{5}\left(\dfrac{t^3}{x^2} - \dfrac{x^3}{t^2}\right)$. From the standard form of the differential equation we identify the forcing function $f(t) = \dfrac{\ln t}{t^2}$. Then, for $x > 1$,

$$\begin{aligned} y_p(x) &= \int_1^x \frac{1}{5}\left(\frac{t^3}{x^2} - \frac{x^3}{t^2}\right)\frac{\ln t}{t^2}\,dt \\ &= -\frac{1}{5}x^{-2}\int_1^x t\ln t\,dt + \frac{1}{5}x^3\int_1^x t^{-4}\ln t\,dt \\ &= -\frac{1}{5}x^{-2}\left(-\frac{x^2}{4} + \frac{1}{2}x^2\ln x + \frac{1}{4}\right) + \frac{1}{5}x^3\left(-\frac{1}{9x^3} - \frac{\ln x}{3x^3} + \frac{1}{9}\right) \\ &= \frac{1}{20} - \frac{1}{10}\ln x - \frac{1}{20}x^{-2} - \frac{1}{45} - \frac{1}{15}\ln x + \frac{1}{45}x^3 \\ &= \frac{1}{36} - \frac{1}{6}\ln x + \frac{1}{45}x^3 - \frac{1}{20}x^{-2}. \end{aligned}$$

The solution of the initial-value problem is

$$y = y_h + y_p = x^3 + \left(\frac{1}{36} - \frac{1}{6}\ln x + \frac{1}{45}x^3 - \frac{1}{20}x^{-2}\right) = \frac{46}{45}x^3 - \frac{1}{20}x^{-2} + \frac{1}{36} - \frac{1}{6}\ln x.$$

**30.** The Cauchy-Euler initial-value problem $x^2y'' - xy' + y = x^2$, $y(1) = 4$, $y'(1) = 3$, has auxiliary equation $m(m-1) - m + 1 = m^2 - 2m + 1 = (m-1)^2 = 0$ so $m_1 = m_2 = 1$, $y(x) = c_1x + c_2x\ln x$, and $y' = c_1 + c_2(1 + \ln x)$. The initial conditions give

$$\begin{aligned} c_1 &= 4 \\ c_1 + c_2 &= 3, \end{aligned}$$

so $c_1 = 4$ and $c_2 = -1$, which implies that $y_h = 4x - x\ln x$. The Wronskian is

$$W(x, x\ln x) = \begin{vmatrix} x & x\ln x \\ 1 & 1 + \ln x \end{vmatrix} = x.$$

Then $G(x,t) = \dfrac{tx\ln x - xt\ln t}{t} = x\ln x - x\ln t$. From the standard form of the differential equation we identify the forcing function $f(t) = 1$. Then, for $x > 1$,

$$\begin{aligned} y_p(x) &= \int_1^x (x\ln x - x\ln t)\,dt = x\ln x\int_1^x dt - x\int_1^x \ln t\,dt \\ &= x\ln x(x-1) - x(x\ln x - x + 1) = x^2 - x - x\ln x. \end{aligned}$$

The solution of the initial-value problem is

$$y = y_h + y_p = 4x - x\ln x + \left(x^2 - x - x\ln x\right) = x^2 + 3x - 2x\ln x.$$

**31.** The initial-value problem is $y'' - y = f(x), \quad y(0) = 8,\ y'(0) = 2$, where

$$f(x) = \begin{cases} -1, & x < 0 \\ 1, & x \geq 0. \end{cases}$$

We first find

$$y_h(x) = 5e^x + 3e^{-x} \qquad \text{and} \qquad y_p(x) = \frac{1}{2}\int_0^x \left[e^{x-t} - e^{-(x-t)}\right] f(t)\,dt.$$

Then for $x < 0$,

$$\begin{aligned} y_p(x) &= -\frac{1}{2}\int_0^x [e^{x-t} - e^{-(x-t)}]\,dt = -\frac{1}{2}e^x \int_0^x e^{-t}\,dt + \frac{1}{2}e^{-x}\int_0^x e^t\,dt \\ &= -\frac{1}{2}e^x(1 - e^{-x}) + \frac{1}{2}e^{-x}(e^x - 1) = -\frac{1}{2}e^x + \frac{1}{2} + \frac{1}{2} - \frac{1}{2}e^{-x} \\ &= 1 - \frac{1}{2}e^x - \frac{1}{2}e^{-x}, \end{aligned}$$

and for $x \geq 0$

$$\begin{aligned} y_p(x) &= \frac{1}{2}\int_0^x [e^{x-t} - e^{-(x-t)}]\,dt = \frac{1}{2}e^x \int_0^x e^{-t}\,dt - \frac{1}{2}e^{-x}\int_0^x e^t\,dt \\ &= \frac{1}{2}e^x(1 - e^{-x}) - \frac{1}{2}e^{-x}(e^x - 1) = \frac{1}{2}e^x - \frac{1}{2} - \frac{1}{2} + \frac{1}{2}e^{-x} \\ &= -1 + \frac{1}{2}e^x + \frac{1}{2}e^{-x}. \end{aligned}$$

The solution is

$$y(x) = y_h(x) + y_p(x) = 5e^x + 3e^{-x} + y_p(x),$$

where

$$y_p(x) = \begin{cases} 1 - \dfrac{1}{2}e^x - \dfrac{1}{2}e^{-x}, & x < 0 \\ -1 + \dfrac{1}{2}e^x + \dfrac{1}{2}e^{-x}, & x \geq 0 \end{cases} = \begin{cases} 1 - \cosh x, & x < 0 \\ -1 + \cosh x, & x \geq 0 \end{cases}$$

**32.** The initial-value problem is $y'' - y = f(x), \quad y(0) = 3,\ y'(0) = 2$, where

$$f(x) = \begin{cases} 0, & x < 0 \\ x, & x \geq 0. \end{cases}$$

We first find

$$y_h(x) = \frac{5}{2}e^x + \frac{1}{2}e^{-x} \qquad \text{and} \qquad y_p(x) = \frac{1}{2}\int_0^x [e^{x-t} - e^{-(x-t)} f(t)\,dt.$$

Then for $x < 0$,

$$y_p(x) = \frac{1}{2}\int_0^x [e^{x-t} - e^{-(x-t)}]0\,dt = 0,$$

and for $x \ge 0$

$$\begin{aligned}
y_p(x) &= \frac{1}{2}\int_0^x [e^{x-t} - e^{-(x-t)}]t\,dt \\
&= \frac{1}{2}e^x\int_0^x te^{-t}\,dt - \frac{1}{2}\int_0^x te^t\,dt \\
&= \frac{1}{2}e^x(1 - e^{-x} - xe^{-x}) - \frac{1}{2}e^{-x}(1 - e^x + xe^x) \\
&= \frac{1}{2}e^x - \frac{1}{2} - \frac{1}{2}x - \frac{1}{2}e^{-x} + \frac{1}{2} - \frac{1}{2}x = \frac{1}{2}e^x - \frac{1}{2}e^{-x} - x.
\end{aligned}$$

The solution is

$$y(x) = y_h(x) + y_p(x) = \frac{5}{2}e^x + \frac{1}{2}e^{-x} + y_p(x),$$

where

$$y_p(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{2}e^x - \frac{1}{2}e^{-x} - x, & x \ge 0 \end{cases} = \begin{cases} 0, & x < 0 \\ \sinh x - x, & x \ge 0 \end{cases}$$

**33.** The initial-value problem is $y'' + y = f(x), \quad y(0) = 1,\ y'(0) = -1$, where

$$f(x) = \begin{cases} 0, & x < 0 \\ 10, & 0 \le x \le 3\pi \\ 0, & x > 3\pi. \end{cases}$$

We first find

$$y_h(x) = \cos x - \sin x \qquad \text{and} \qquad y_p(x) = \int_0^x \sin(x - t)f(t)\,dt.$$

Then for $x < 0$

$$y_p(x) = \int_0^x \sin(x - t)0\,dt = 0,$$

for $0 \le x \le 3\pi$

$$y_p(x) = 10\int_0^x \sin(x - t)\,dt = 10 - 10\cos x,$$

and for $x > 3\pi$

$$y_p(x) = 10\int_0^{3\pi} \sin(x - t)\,dt + \int_{3\pi}^x \sin(x - t)0\,dt = -20\cos x.$$

The solution is

$$y(x) = y_h(x) + y_p(x) = \cos x - \sin x + y_p(x),$$

where

$$y_p(x) = \begin{cases} 0, & x < 0 \\ 10 - 10\cos x, & 0 \le x \le 3\pi \\ -20\cos x, & x > 3\pi. \end{cases}$$

**34.** The initial-value problem is $y'' + y = f(x), \quad y(0) = 0, \ y'(0) = 1$, where

$$f(x) = \begin{cases} 0, & x < 0 \\ \cos x, & 0 \le x \le 4\pi \\ 0, & x > 4\pi \end{cases}$$

We first find

$$y_h(x) = \sin x \qquad \text{and} \qquad y_p(x) = \int_0^x \sin(x-t) f(t)\, dt.$$

Then for $x < 0$

$$y_p(x) = \int_0^x \sin(x-t) 0 \, dt = 0,$$

for $0 \le x \le 4\pi$

$$y_p(x) = \int_0^x \sin(x-t)\cos t\, dt = \frac{1}{2} x \sin x,$$

and for $x > 4\pi$

$$y_p(x) = \int_0^{4\pi} \sin(x-t)\cos t\, dt + \int_{4\pi}^x \sin(x-t) 0\, dt = 2\pi \sin x.$$

The solution is

$$y(x) = y_h(x) + y_p(x) = \sin x + y_p(x),$$

where

$$y_p(x) = \begin{cases} 0, & x < 0 \\ \dfrac{1}{2} x \sin x, & 0 \le x \le 4\pi \\ 2\pi \sin x, & x > 4\pi \end{cases}$$

To evaluate the integral of $\int \sin(x-t)\cos t\, dt$ we use the identities

$$\sin(A-B) = \sin A \cos B - \cos A \sin B \quad \text{and} \quad \cos(A-B) = \cos A \cos B + \sin A \sin B.$$

**35.** The boundary-value problem is $y'' = f(x), \quad y(0) = 0, \ y(1) = 0$. The solution of the associated homogeneous equation is $y = c_1 + c_2 x$.

**(a)** To satisfy $y(0) = 0$ we take $y_1(x) = x$ and to satisfy $y(1) = 0$ we take $y_2(x) = x - 1$. The Wronskian of $y_1$ and $y_2$ is

$$W(y_1, y_2) = \begin{vmatrix} x & x-1 \\ 1 & 1 \end{vmatrix} = 1,$$

so

$$G(x,t) = \begin{cases} t(x-1), & 0 \le t \le x \\ x(t-1), & x \le t \le 1. \end{cases}$$

Therefore

$$y_p(x) = \int_0^1 G(x,t) f(t)\, dt = (x-1)\int_0^x t f(t)\, dt + x \int_x^1 (t-1) f(t)\, dt.$$

**(b)** By the Product Rule and the Fundamental Theorem of Calculus, the first two derivative of $y_p(x)$ are

$$\begin{aligned} y_p'(x) &= (x-1)xf(x) + \int_0^x tf(t)\,dt + x[-(x-1)f(x)] + \int_x^1 (t-1)f(t)\,dt \\ y_p''(x) &= (x-1)[xf'(x)+f(x)] + xf(x) + xf(x) \\ &\qquad - [(x^2-x)f'(x) + (2x-1)f(x)] - (x-1)f(x) \\ &= x^2f'(x) + xf(x) - xf'(x) - f(x) + xf(x) + xf(x) - x^2f(x) \\ &\qquad + xf'(x) - 2xf(x) + f(x) - xf(x) + f(x) = f(x) \end{aligned}$$

Thus, $y_p(x)$ satisfies the differential equation. To see that the boundary conditions are satisfied we compute

$$y_p(0) = (0-1)\int_0^0 tf(t)\,dt + 0\cdot\int_0^1 (t-1)f(t)\,dt = 0$$

and

$$y_p(1) = (1-1)\int_0^1 tf(t)\,dt + 1\cdot\int_1^1 (t-1)f(t)\,dt = 0.$$

**36.** The boundary-value problem is $y'' = f(x), \quad y(0)=0,\ y(1)+y'(1)=0$. The solution of the associated homogeneous equation is $y = c_1 + c_2x$.

**(a)** To satisfy $y(0)=0$ we take $y_1(x) = x$ and to satisfy $y(1)+y'(1)=0$ we note that $y(1)+y'(1) = (c_1+c_2)+c_2 = 0$ which implies that $c_1 = -2c_2$. Taking $c_2 = 1$, we find that $c_1 = -2$, so we have $y_2(x) = -2+x$. The Wronskian of $y_1$ and $y_2$ is

$$W(y_1, y_2) = \begin{vmatrix} x & -2+x \\ 1 & 1 \end{vmatrix} = 2,$$

so

$$G(x,t) = \begin{cases} \dfrac{1}{2}t(x-2), & 0 \le t \le x \\[2ex] \dfrac{1}{2}x(t-2), & x \le t \le 1 \end{cases}$$

Therefore

$$y_p(x) = \int_0^1 G(x,t)f(t)\,dt = \frac{1}{2}(x-2)\int_0^x tf(t)\,dt + \frac{1}{2}x\int_x^1 (t-2)f(t)\,dt.$$

**(b)** By the Product Rule and the Fundamental Theorem of Calculus, the first two derivative of $y_p(x)$ are

$$y_p'(x) = \frac{1}{2}(x-2)xf(x) + \frac{1}{2}\int_0^x tf(t)\,dt + \frac{1}{2}x[-(x-2)f(x)] + \frac{1}{2}\int_x^1 (t-2)f(t)\,dt$$

$$\begin{aligned} y_p''(x) &= \frac{1}{2}(x-2)[xf'(x)+f(x)] + \frac{1}{2}xf(x) + \frac{1}{2}xf(x) \\ &\qquad - \frac{1}{2}(x^2-2x)f'(x) - \frac{1}{2}(2x-2)f(x) - \frac{1}{2}(x-2)f(x) \\ &= \frac{1}{2}\big[x^2f'(x) + xf(x) - 2xf'(x) - 2f(x) + xf(x) + xf(x) - x^2f'(x) \\ &\qquad + 2xf'(x) - 2xf(x) + 2f(x) - xf(x) + 2f(x)\big] = f(x) \end{aligned}$$

Thus, $y_p(x)$ satisfies the differential equation. To see that the boundary conditions are satisfied we first compute

$$y_p(0) = \frac{1}{2}(0-2)\int_0^0 tf(t)\,dt + \frac{1}{2}(0)\int_0^1 (t-2)f(t)\,dt = 0.$$

Next we use $y_p'(x)$ found at the beginning of this part of the solution to compute

$$\begin{aligned} y_p(1) + y_p'(1) &= \frac{1}{2}(1-2)\int_0^1 tf(t)\,dt + \frac{1}{2}(1)\int_1^1 (t-2)f(t)\,dt + \frac{1}{2}(1-2)f(x) \\ &\qquad + \frac{1}{2}\int_0^1 tf(t)\,dt + \frac{1}{2}(1)[-(1-2)f(x)] + \frac{1}{2}(1)\int_1^1 (1-2)f(t)\,dt \\ &= \frac{1}{2}(-1)\int_0^1 tf(t)\,dt + \frac{1}{2}(-1)f(1) + \frac{1}{2}\int_0^1 tf(t)\,dt + \frac{1}{2}f(1) = 0. \end{aligned}$$

**37.** If $f(x) = 1$ in Problem 35, then

$$y_p(x) = (x-1)\int_0^x t\,dt + x\int_x^1 (t-1)\,dt = \frac{1}{2}x^2 - \frac{1}{2}x.$$

**38.** If $f(x) = x$ in Problem 36, then

$$y_p(x) = \frac{1}{2}(x-2)\int_0^x t^2\,dt + \frac{1}{2}x\int_x^1 (t-2)t\,dt = \frac{1}{6}x^3 - \frac{1}{3}x.$$

**39.** The boundary-value problem is $y'' + y = 1$, $y(0) = 0$, $y(1) = 0$. The solution of the associated homogeneous equation is $y = c_1\cos x + c_2\sin x$. Since $y(0) = c_1\cos 0 + c_2\sin 0 = c_1 = 0$, we take $y_1(x) = \sin x$. To satisfy $y(1) = 0$ we note that $y(1) = c_1\cos 1 + c_2\sin 1 = 0$ which implies that $c_1 = -c_2\sin 1/\cos 1$ so

$$y(x) = -c_2\frac{\sin 1}{\cos 1}\cos x + c_2\sin x = -\frac{c_2}{\cos 1}(\sin x\cos 1 - \cos x\sin 1).$$

Taking $c_2 = -\cos 1$, we have

$$y_2(x) = \sin x\cos 1 - \cos x\sin 1 = \sin(x-1).$$

The Wronskian of $y_1$ and $y_2$ is

$$W(y_1, y_2) = \begin{vmatrix} \sin x & \sin(x-1) \\ \cos x & \cos(x-1) \end{vmatrix} = \sin x \cos(x-1) - \cos x \sin(x-1) = \sin[x-(x-1)] = \sin 1,$$

so

$$G(x,t) = \begin{cases} \dfrac{\sin t \sin(x-1)}{\sin 1}, & 0 \le t \le x \\ \dfrac{\sin x \sin(t-1)}{\sin 1}, & x \le t \le 1. \end{cases}$$

Therefore, taking $f(t) = 1$,

$$\begin{aligned} y_p(x) &= \int_0^1 G(x,t)\,dt = \frac{\sin(x-1)}{\sin 1}\int_0^x \sin t\,dt + \frac{\sin x}{\sin 1}\int_x^1 \sin(t-1)\,dt \\ &= \frac{\sin(x-1)}{\sin 1}(-\cos x + 1) + \frac{\sin x}{\sin 1}[-1+\cos(x-1)] = \frac{\sin(x-1)}{\sin 1} - \frac{\sin x}{\sin 1} + 1. \end{aligned}$$

**40.** The boundary-value problem is $y'' + 9y = 1, \quad y(0) = 0,\ y'(\pi) = 0$. The solution of the associated homogeneous equation is $y = c_1 \cos 3x + c_2 \sin 3x$. Since $y(0) = c_1 \cos 0 + c_2 \sin 0 = c_1 = 0$, we take $y_1(x) = \sin 3x$. Since $y'(x) = -3c_1 \sin 3x + 3c_2 \cos 3x$ we see that $y'(\pi) = -3c_2$. Then $y'(\pi) = 0$, implies that $-3c_2 = 0$ or $c_2 = 0$. Taking $c_1 = 1$ we have $y_2(x) = \cos 3x$. The Wronskian of $y_1$ and $y_2$ is

$$W(y_1, y_2) = \begin{vmatrix} \sin 3x & \cos 3x \\ 3\cos 3x & -3\sin 3x \end{vmatrix} = -3,$$

so

$$G(x,t) = \begin{cases} -\dfrac{1}{3}\sin 3t \cos 3x, & 0 \le t \le x \\ -\dfrac{1}{3}\sin 3x \cos 3t, & x \le t \le \pi \end{cases}$$

Therefore, taking $f(t) = 1$,

$$\begin{aligned} y_p(x) &= \int_0^\pi G(x,t)\,dt = -\frac{1}{3}\cos 3x \int_0^x \sin 3t\,dt - \frac{1}{3}\sin 3x \int_x^\pi \cos 3t\,dt \\ &= -\frac{1}{3}\cos 3x\left(\frac{1}{3} - \frac{1}{3}\cos 3x\right) - \frac{1}{3}\sin 3x\left(-\frac{1}{3}\sin 3x\right) = \frac{1}{9} - \frac{1}{9}\cos 3x. \end{aligned}$$

**41.** The boundary-value problem is $y'' - 2y' + 2y = e^x, \quad y(0) = 0,\ y(\pi/2) = 0$. The auxiliary equation is

$$m^2 - 2m + 2 = 0 \quad \text{so} \quad m = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i.$$

The solution of the associated homogeneous equation is then $y = e^x(c_1 \cos x + c_2 \sin x)$. It is easily seen that

$$y_1(x) = e^x \sin x \qquad \text{and} \qquad y_2(x) = e^x \cos x$$

satisfy the boundary conditions. The Wronskian of $y_1$ and $y_2$ is

$$\begin{aligned} W(y_1, y_2) &= \begin{vmatrix} e^x \sin x & e^x \cos x \\ e^x \cos x + e^x \sin x & -e^x \sin x + e^x \cos x \end{vmatrix} \\ &= e^{2x}\left[-\sin^2 x + \sin x \cos x - (\cos^2 x + \sin x \cos x)\right] = -e^{2x}, \end{aligned}$$

so

$$G(x,t) = \begin{cases} \dfrac{e^t \sin t e^x \cos x}{-e^{2t}}, & 0 \le t \le x \\[2ex] \dfrac{e^x \sin x e^t \cos t}{-e^{2t}}, & x \le t \le \pi/2 \end{cases}$$

Therefore, taking $f(t) = e^t$,

$$\begin{aligned} y_p(x) &= \int_0^{\pi/2} G(x,t)e^t\,dt = -e^x \cos x \int_0^x \sin t\,dt - e^x \sin x \int_x^{\pi/2} \cos t\,dt \\ &= -e^x \cos x(1-\cos x) - e^x \sin x(1-\sin x) = -e^x \cos x - e^x \sin x + e^x. \end{aligned}$$

**42.** The boundary-value problem is $y'' - y' = e^{2x}$, $y(0) = 0$, $y(1) = 0$. The solution of the associated homogeneous equation is $y = c_1 + c_2 e^x$. Since $y(0) = c_1 + c_2 = 0$, we see that $c_1 = -c_2$ and we take $y_1(x) = 1 - e^x$. To satisfy $y(1) = 0$ we note that $y(1) = c_1 + c_2 e = 0$ which implies that $c_1 = -c_2 e$ so

$$y(x) = -c_2 e + c_2 e^x = -c_2 e(1 - e^{x-1}).$$

Taking $c_2 = -1/e$, we have $y_2(x) = 1 - e^{x-1}$. The Wronskian of $y_1$ and $y_2$ is

$$W(y_1, y_2) = \begin{vmatrix} 1-e^x & 1-e^{x-1} \\ -e^x & -e^{x-1} \end{vmatrix} = -e^{x-1} + e^{2x-1} + e^x - e^{2x-1} = e^x - e^{x-1} = e^x(1-e^{-1}),$$

so

$$G(x,t) = \begin{cases} \dfrac{(1-e^t)(1-e^{x-1})}{e^t(1-e^{-1})}, & 0 \le t \le x \\[2ex] \dfrac{(1-e^x)(1-e^{t-1})}{e^t(1-e^{-1})}, & x \le t \le 1 \end{cases}$$

Therefore, taking $f(t) = e^{2t}$,

$$\begin{aligned} y_p(x) &= \int_0^1 G(x,t)e^{2t}\,dt = \frac{1-e^{x-1}}{1-e^{-1}} \int_0^x (e^t - e^{2t})\,dt + \frac{1-e^x}{1-e^{-1}} \int_x^1 (e^t - e^{2t-1})\,dt \\ &= \frac{1-e^{x-1}}{1-e^{-1}}\left(e^x - \frac{1}{2}e^{2x} - \frac{1}{2}\right) + \frac{1-e^x}{1-e^{-1}}\left(\frac{1}{2}e - e^x + \frac{1}{2}e^{2x-1}\right) \\ &= \frac{1}{2}e^{2x} - \frac{1}{2}e^x - \frac{1}{2}e^{x+1} + \frac{1}{2}e. \end{aligned}$$

**43.** The Cauchy-Euler boundary-value problem $x^2y'' + xy' = 1, \quad y(e^{-1}) = 0,\ y(1) = 0$ has auxiliary equation $m(m-1)+m = m^2 = 0$ so $y(x) = c_1+c_2\ln x$. Since $y(e^{-1}) = c_1+c_2\ln e^{-1} = c_1-c_2 = 0$, $c_1 = c_2$ and $y(x) = c_2+c_2\ln x = c_2(1+\ln x)$. Taking $c_2 = 1$ we have $y_1(x) = 1+\ln x$. To satisfy $y(1) = 0$ we note that $y(1) = c_1 + c_2\ln 1 = c_1 = 0$ which implies that $y(x) = c_2\ln x$. Taking $c_2 = 1$ we find $y_2(x) = \ln x$. The Wronskian of $y_1$ and $y_2$ is

$$W(y_1, y_2) = \begin{vmatrix} 1+\ln x & \ln x \\ 1/x & 1/x \end{vmatrix} = \frac{1}{x},$$

so

$$G(x,t) = \begin{cases} \dfrac{(1+\ln t)(\ln x)}{1/t}, & 0 \le t \le x \\[2ex] \dfrac{(1+\ln x)(\ln t)}{1/t}, & x \le t \le 1. \end{cases}$$

From the standard form of the differential equation we identify the forcing function $f(t) = \dfrac{1}{t^2}$. Then,

$$\begin{aligned} y_p(x) &= \int_{e^{-1}}^{1} G(x,t)\frac{1}{t^2}\,dt = \ln x\int_{e^{-1}}^{x}\left(\frac{1}{t}+\frac{\ln t}{t}\right)dt + (1+\ln x)\int_{x}^{1}\frac{\ln t}{t}\,dt \\ &= (\ln x)\left(\ln x + \frac{1}{2}(\ln x)^2 + \frac{1}{2}\right) + (1+\ln x)\left(-\frac{1}{2}(\ln x)^2\right) = \frac{1}{2}(\ln x)^2 + \frac{1}{2}\ln x. \end{aligned}$$

**44.** The Cauchy-Euler boundary-value problem $x^2y''-4xy'+6y = x^4, \quad y(1)-y'(1) = 0,\ y(3) = 0$ has auxiliary equation $m(m-1) - 4m + 6 = m^2 - 5m + 6 = (m-2)(m-3) = 0$ so $y(x) = c_1x^2 + c_2x^3$. Since $y(1) - y'(1) = (c_1 + c_2) - (2c_1 + 3c_2) = -c_1 - 2c_2 = 0$ we have $c_1 = -2c_2$ and $y(x) = -2c_2x^2 + c_2x^3$. Taking $c_2 = -1$ we have $y_1(x) = 2x^2 - x^3$. From $y(3) = 9c_1 + 27c_2 = 0$ we have $c_1 = -3c_2$, so $y(x) = -3c_2x^2 + c_2x^3 = -c_2(3x^2 - x^3)$. Again, letting $c_2 = -1$ we have $y_2(x) = 3x^2 - x^3$. The Wronskian of $y_1$ and $y_2$ is

$$W(y_1, y_2) = \begin{vmatrix} 2x^2-x^3 & 3x^2-x^3 \\ 4x-3x^2 & 6x-3x^2 \end{vmatrix} = x^4,$$

so

$$G(x,t) = \begin{cases} \dfrac{(2t^2-t^3)(3x^2-x^3)}{t^4}, & 0 \le t \le x \\[2ex] \dfrac{(2x^2-x^3)(3t^2-t^3)}{t^4}, & x \le t \le 3. \end{cases}$$

From the standard form of the differential equation we identify the forcing function $f(t) = t^2$. Then,

$$\begin{aligned} y_p(x) &= \int_{1}^{3} G(x,t)t^2\,dt = (3x^2-x^3)\int_{1}^{x}(2-t)\,dt + (2x^2-x^3)\int_{x}^{3}(3-t)\,dt \\ &= (3x^2-x^3)\left(2x - \frac{1}{2}x^2 - \frac{3}{2}\right) + (2x^2-x^3)\left(\frac{9}{2} - 3x + \frac{1}{2}x^2\right) \\ &= \frac{9}{2}x^2 - 3x^3 + \frac{1}{2}x^4. \end{aligned}$$

## 4.9 Solving Systems of Linear Equations

**1.** From $Dx = 2x - y$ and $Dy = x$ we obtain $y = 2x - Dx$, $Dy = 2Dx - D^2x$, and $(D^2 - 2D + 1)x = 0$. The solution is

$$x = c_1e^t + c_2te^t$$
$$y = (c_1 - c_2)e^t + c_2te^t.$$

**2.** From $Dx = 4x + 7y$ and $Dy = x - 2y$ we obtain $y = \frac{1}{7}Dx - \frac{4}{7}x$, $Dy = \frac{1}{7}D^2x - \frac{4}{7}Dx$, and $(D^2 - 2D - 15)x = 0$. The solution is

$$x = c_1e^{5t} + c_2e^{-3t}$$
$$y = \frac{1}{7}c_1e^{5t} - c_2e^{-3t}.$$

**3.** From $Dx = -y + t$ and $Dy = x - t$ we obtain $y = t - Dx$, $Dy = 1 - D^2x$, and $(D^2 + 1)x = 1 + t$. The solution is

$$x = c_1 \cos t + c_2 \sin t + 1 + t$$
$$y = c_1 \sin t - c_2 \cos t + t - 1.$$

**4.** From $Dx - 4y = 1$ and $x + Dy = 2$ we obtain $y = \frac{1}{4}Dx - \frac{1}{4}$, $Dy = \frac{1}{4}D^2x$, and $(D^2 + 1)x = 2$. The solution is

$$x = c_1 \cos 2t + c_2 \sin 2t + 2$$
$$y = \frac{1}{4}c_2 \cos 2t - \frac{1}{4}c_1 \sin 2t - \frac{1}{4}.$$

**5.** From $(D^2 + 5)x - 2y = 0$ and $-2x + (D^2 + 2)y = 0$ we obtain $y = \frac{1}{2}(D^2 + 5)x$, $D^2y = \frac{1}{2}(D^4 + 5D^2)x$, and $(D^2 + 1)(D^2 + 6)x = 0$. The solution is

$$x = c_1 \cos t + c_2 \sin t + c_3 \cos \sqrt{6}\,t + c_4 \sin \sqrt{6}\,t$$
$$y = 2c_1 \cos t + 2c_2 \sin t - \frac{1}{2}c_3 \cos \sqrt{6}\,t - \frac{1}{2}c_4 \sin \sqrt{6}\,t.$$

**6.** From $(D + 1)x + (D - 1)y = 2$ and $3x + (D + 2)y = -1$ we obtain $x = -\frac{1}{3} - \frac{1}{3}(D + 2)y$, $Dx = -\frac{1}{3}(D^2 + 2D)y$, and $(D^2 + 5)y = -7$. The solution is

$$y = c_1 \cos \sqrt{5}\,t + c_2 \sin \sqrt{5}\,t - \frac{7}{5}$$
$$x = \left(-\frac{2}{3}c_1 - \frac{\sqrt{5}}{3}c_2\right) \cos \sqrt{5}\,t + \left(\frac{\sqrt{5}}{3}c_1 - \frac{2}{3}c_2\right) \sin \sqrt{5}\,t + \frac{3}{5}.$$

**7.** From $D^2x = 4y + e^t$ and $D^2y = 4x - e^t$ we obtain $y = \frac{1}{4}D^2x - \frac{1}{4}e^t$, $D^2y = \frac{1}{4}D^4x - \frac{1}{4}e^t$, and $(D^2+4)(D-2)(D+2)x = -3e^t$. The solution is

$$\begin{aligned} x &= c_1\cos 2t + c_2\sin 2t + c_3e^{2t} + c_4e^{-2t} + \frac{1}{5}e^t \\ y &= -c_1\cos 2t - c_2\sin 2t + c_3e^{2t} + c_4e^{-2t} - \frac{1}{5}e^t. \end{aligned}$$

**8.** From $(D^2+5)x + Dy = 0$ and $(D+1)x + (D-4)y = 0$ we obtain $(D-5)(D^2+4)x = 0$ and $(D-5)(D^2+4)y = 0$. The solution is

$$\begin{aligned} x &= c_1e^{5t} + c_2\cos 2t + c_3\sin 2t \\ y &= c_4e^{5t} + c_5\cos 2t + c_6\sin 2t. \end{aligned}$$

Substituting into $(D+1)x + (D-4)y = 0$ gives

$$(6c_1 + c_4)e^{5t} + (c_2 + 2c_3 - 4c_5 + 2c_6)\cos 2t + (-2c_2 + c_3 - 2c_5 - 4c_6)\sin 2t = 0$$

so that $c_4 = -6c_1$, $c_5 = \frac{1}{2}c_3$, $c_6 = -\frac{1}{2}c_2$, and

$$y = -6c_1e^{5t} + \frac{1}{2}c_3\cos 2t - \frac{1}{2}c_2\sin 2t.$$

**9.** From $Dx + D^2y = e^{3t}$ and $(D+1)x + (D-1)y = 4e^{3t}$ we obtain $D(D^2+1)x = 34e^{3t}$ and $D(D^2+1)y = -8e^{3t}$. The solution is

$$\begin{aligned} y &= c_1 + c_2\sin t + c_3\cos t - \frac{4}{15}e^{3t} \\ x &= c_4 + c_5\sin t + c_6\cos t + \frac{17}{15}e^{3t}. \end{aligned}$$

Substituting into $(D+1)x + (D-1)y = 4e^{3t}$ gives

$$(c_4 - c_1) + (c_5 - c_6 - c_3 - c_2)\sin t + (c_6 + c_5 + c_2 - c_3)\cos t = 0$$

so that $c_4 = c_1$, $c_5 = c_3$, $c_6 = -c_2$, and

$$x = c_1 - c_2\cos t + c_3\sin t + \frac{17}{15}e^{3t}.$$

**10.** From $D^2x - Dy = t$ and $(D+3)x + (D+3)y = 2$ we obtain $D(D+1)(D+3)x = 1 + 3t$ and $D(D+1)(D+3)y = -1 - 3t$. The solution is

$$\begin{aligned} x &= c_1 + c_2e^{-t} + c_3e^{-3t} - t + \frac{1}{2}t^2 \\ y &= c_4 + c_5e^{-t} + c_6e^{-3t} + t - \frac{1}{2}t^2. \end{aligned}$$

Substituting into $(D+3)x+(D+3)y=2$ and $D^2x-Dy=t$ gives

$$3(c_1+c_4)+2(c_2+c_5)e^{-t}=2$$

and

$$(c_2+c_5)e^{-t}+3(3c_3+c_6)e^{-3t}=0$$

so that $c_4=2/3-c_1$, $c_5=-c_2$, $c_6=-3c_3$, and

$$y=\frac{2}{3}-c_1-c_2e^{-t}-3c_3e^{-3t}+t-\frac{1}{2}t^2.$$

**11.** From $(D^2-1)x-y=0$ and $(D-1)x+Dy=0$ we obtain $y=(D^2-1)x$, $Dy=(D^3-D)x$, and $(D-1)(D^2+D+1)x=0$. The solution is

$$x=c_1e^t+e^{-t/2}\left[c_2\cos\frac{\sqrt{3}}{2}t+c_3\sin\frac{\sqrt{3}}{2}t\right]$$

$$y=\left(-\frac{3}{2}c_2-\frac{\sqrt{3}}{2}c_3\right)e^{-t/2}\cos\frac{\sqrt{3}}{2}t+\left(\frac{\sqrt{3}}{2}c_2-\frac{3}{2}c_3\right)e^{-t/2}\sin\frac{\sqrt{3}}{2}t.$$

**12.** From $(2D^2-D-1)x-(2D+1)y=1$ and $(D-1)x+Dy=-1$ we obtain $(2D+1)(D-1)(D+1)x=-1$ and $(2D+1)(D+1)y=-2$. The solution is

$$x=c_1e^{-t/2}+c_2e^{-t}+c_3e^t+1$$
$$y=c_4e^{-t/2}+c_5e^{-t}-2.$$

Substituting into $(D-1)x+Dy=-1$ gives

$$\left(-\frac{3}{2}c_1-\frac{1}{2}c_4\right)e^{-t/2}+(-2c_2-c_5)e^{-t}=0$$

so that $c_4=-3c_1$, $c_5=-2c_2$, and

$$y=-3c_1e^{-t/2}-2c_2e^{-t}-2.$$

**13.** From $(2D-5)x+Dy=e^t$ and $(D-1)x+Dy=5e^t$ we obtain $Dy=(5-2D)x+e^t$ and $(4-D)x=4e^t$. Then

$$x=c_1e^{4t}+\frac{4}{3}e^t$$

and $Dy=-3c_1e^{4t}+5e^t$ so that

$$y=-\frac{3}{4}c_1e^{4t}+c_2+5e^t.$$

**14.** From $Dx+Dy=e^t$ and $(-D^2+D+1)x+y=0$ we obtain $y=(D^2-D-1)x$, $Dy=(D^3-D^2-D)x$, and $D^2(D-1)x=e^t$. The solution is

$$x=c_1+c_2t+c_3e^t+te^t$$
$$y=-c_1-c_2-c_2t-c_3e^t-te^t+e^t.$$

**15.** Multiplying the first equation by $D+1$ and the second equation by $D^2+1$ and subtracting we obtain $(D^4-D^2)x=1$. Then

$$x=c_1+c_2t+c_3e^t+c_4e^{-t}-\frac{1}{2}t^2.$$

Multiplying the first equation by $D+1$ and subtracting we obtain $D^2(D+1)y=1$. Then

$$y=c_5+c_6t+c_7e^{-t}-\frac{1}{2}t^2.$$

Substituting into $(D-1)x+(D^2+1)y=1$ gives

$$(-c_1+c_2+c_5-1)+(-2c_4+2c_7)e^{-t}+(-1-c_2+c_6)t=1$$

so that $c_5=c_1-c_2+2$, $c_6=c_2+1$, and $c_7=c_4$. The solution of the system is

$$\begin{aligned}x&=c_1+c_2t+c_3e^t+c_4e^{-t}-\frac{1}{2}t^2\\ y&=(c_1-c_2+2)+(c_2+1)t+c_4e^{-t}-\frac{1}{2}t^2.\end{aligned}$$

**16.** From $D^2x-2(D^2+D)y=\sin t$ and $x+Dy=0$ we obtain $x=-Dy$, $D^2x=-D^3y$, and $D(D^2+2D+2)y=-\sin t$. The solution is

$$\begin{aligned}y&=c_1+c_2e^{-t}\cos t+c_3e^{-t}\sin t+\frac{1}{5}\cos t+\frac{2}{5}\sin t\\ x&=(c_2+c_3)e^{-t}\sin t+(c_2-c_3)e^{-t}\cos t+\frac{1}{5}\sin t-\frac{2}{5}\cos t.\end{aligned}$$

**17.** From $Dx=y$, $Dy=z.$ and $Dz=x$ we obtain $x=D^2y=D^3x$ so that
$(D-1)(D^2+D+1)x=0$,

$$\begin{aligned}x&=c_1e^t+e^{-t/2}\left[c_2\sin\frac{\sqrt{3}}{2}t+c_3\cos\frac{\sqrt{3}}{2}t\right],\\ y&=c_1e^t+\left(-\frac{1}{2}c_2-\frac{\sqrt{3}}{2}c_3\right)e^{-t/2}\sin\frac{\sqrt{3}}{2}t+\left(\frac{\sqrt{3}}{2}c_2-\frac{1}{2}c_3\right)e^{-t/2}\cos\frac{\sqrt{3}}{2}t,\end{aligned}$$

and

$$z=c_1e^t+\left(-\frac{1}{2}c_2+\frac{\sqrt{3}}{2}c_3\right)e^{-t/2}\sin\frac{\sqrt{3}}{2}t+\left(-\frac{\sqrt{3}}{2}c_2-\frac{1}{2}c_3\right)e^{-t/2}\cos\frac{\sqrt{3}}{2}t.$$

**18.** From $Dx + z = e^t$, $(D-1)x + Dy + Dz = 0$, and $x + 2y + Dz = e^t$ we obtain $z = -Dx + e^t$, $Dz = -D^2x + e^t$, and the system $(-D^2 + D - 1)x + Dy = -e^t$ and $(-D^2 + 1)x + 2y = 0$. Then $y = \frac{1}{2}(D^2 - 1)x$, $Dy = \frac{1}{2}D(D^2 - 1)x$, and $(D-2)(D^2+1)x = -2e^t$ so that the solution is

$$\begin{aligned} x &= c_1e^{2t} + c_2\cos t + c_3\sin t + e^t \\ y &= \frac{3}{2}c_1e^{2t} - c_2\cos t - c_3\sin t \\ z &= -2c_1e^{2t} - c_3\cos t + c_2\sin t. \end{aligned}$$

**19.** Write the system in the form

$$\begin{aligned} Dx - 6y &= 0 \\ x - Dy + z &= 0 \\ x + y - Dz &= 0. \end{aligned}$$

Multiplying the second equation by $D$ and adding to the third equation we obtain $(D+1)x - (D^2 - 1)y = 0$. Eliminating $y$ between this equation and $Dx - 6y = 0$ we find

$$(D^3 - D - 6D - 6)x = (D+1)(D+2)(D-3)x = 0.$$

Thus

$$x = c_1e^{-t} + c_2e^{-2t} + c_3e^{3t},$$

and, successively substituting into the first and second equations, we get

$$\begin{aligned} y &= -\frac{1}{6}c_1e^{-t} - \frac{1}{3}c_2e^{-2t} + \frac{1}{2}c_3e^{3t} \\ z &= -\frac{5}{6}c_1e^{-t} - \frac{1}{3}c_2e^{-2t} + \frac{1}{2}c_3e^{3t}. \end{aligned}$$

**20.** Write the system in the form

$$\begin{aligned} (D+1)x - z &= 0 \\ (D+1)y - z &= 0 \\ x - y + Dz &= 0. \end{aligned}$$

Multiplying the third equation by $D+1$ and adding to the second equation we obtain $(D+1)x + (D^2 + D - 1)z = 0$. Eliminating $z$ between this equation and $(D+1)x - z = 0$ we find $D(D+1)^2x = 0$. Thus

$$x = c_1 + c_2e^{-t} + c_3te^{-t},$$

and, successively substituting into the first and third equations, we get

$$\begin{aligned} y &= c_1 + (c_2 - c_3)e^{-t} + c_3te^{-t} \\ z &= c_1 + c_3e^{-t}. \end{aligned}$$

**21.** From $(D+5)x+y=0$ and $4x-(D+1)y=0$ we obtain $y=-(D+5)x$ so that $Dy=-(D^2+5D)x$. Then $4x+(D^2+5D)x+(D+5)x=0$ and $(D+3)^2x=0$. Thus

$$\begin{aligned} x&=c_1e^{-3t}+c_2te^{-3t}\\ y&=-(2c_1+c_2)e^{-3t}-2c_2te^{-3t}. \end{aligned}$$

Using $x(1)=0$ and $y(1)=1$ we obtain

$$\begin{aligned} c_1e^{-3}+c_2e^{-3}&=0\\ -(2c_1+c_2)e^{-3}-2c_2e^{-3}&=1 \end{aligned}$$

or

$$\begin{aligned} c_1+c_2&=0\\ 2c_1+3c_2&=-e^3. \end{aligned}$$

Thus $c_1=e^3$ and $c_2=-e^3$. The solution of the initial-value problem is

$$\begin{aligned} x&=e^{-3t+3}-te^{-3t+3}\\ y&=-e^{-3t+3}+2te^{-3t+3}. \end{aligned}$$

**22.** From $Dx-y=-1$ and $3x+(D-2)y=0$ we obtain $x=-\frac{1}{3}(D-2)y$ so that $Dx=-\frac{1}{3}(D^2-2D)y$. Then $-\frac{1}{3}(D^2-2D)y=y-1$ and $(D^2-2D+3)y=3$. Thus

$$y=e^t\left(c_1\cos\sqrt{2}\,t+c_2\sin\sqrt{2}\,t\right)+1$$

and

$$x=\frac{1}{3}e^t\left[\left(c_1-\sqrt{2}\,c_2\right)\cos\sqrt{2}\,t+\left(\sqrt{2}\,c_1+c_2\right)\sin\sqrt{2}\,t\right]+\frac{2}{3}.$$

Using $x(0)=y(0)=0$ we obtain

$$\begin{aligned} c_1+1&=0\\ \frac{1}{3}\left(c_1-\sqrt{2}\,c_2\right)+\frac{2}{3}&=0. \end{aligned}$$

Thus $c_1=-1$ and $c_2=\sqrt{2}/2$. The solution of the initial-value problem is

$$x=e^t\left(-\frac{2}{3}\cos\sqrt{2}\,t-\frac{\sqrt{2}}{6}\sin\sqrt{2}\,t\right)+\frac{2}{3}$$

$$y=e^t\left(-\cos\sqrt{2}\,t+\frac{\sqrt{2}}{2}\sin\sqrt{2}\,t\right)+1.$$

**23.** Equating Newton's law with the net forces in the $x$- and $y$-directions gives $m\,d^2x/dt^2=0$ and $m\,d^2y/dt^2=-mg$, respectively. From $mD^2x=0$ we obtain $x(t)=c_1t+c_2$, and from $mD^2y=-mg$ or $D^2y=-g$ we obtain $y(t)=-\frac{1}{2}gt^2+c_3t+c_4$.

**24.** From Newton's second law in the $x$-direction we have

$$m\frac{d^2x}{dt^2} = -k\cos\theta = -k\frac{1}{v}\frac{dx}{dt} = -|c|\frac{dx}{dt}\,.$$

In the $y$-direction we have

$$m\frac{d^2y}{dt^2} = -mg - k\sin\theta = -mg - k\frac{1}{v}\frac{dy}{dt} = -mg - |c|\frac{dy}{dt}\,.$$

From $mD^2x + |c|Dx = 0$ we have $D(mD + |c|)x = 0$ so that $(mD + |c|)x = c_1$ or $(D + |c|/m)x = c_2$. This is a linear first-order differential equation. An integrating factor is $e^{\int |c|dt/m} = e^{|c|t/m}$ so that

$$\frac{d}{dt}\left[e^{|c|t/m}x\right] = c_2e^{|c|t/m}$$

and $e^{|c|t/m}x = (c_2m/|c|)e^{|c|t/m} + c_3$. The general solution of this equation is $x(t) = c_4 + c_3e^{-|c|t/m}$. From $(mD^2 + |c|D)y = -mg$ we have $D(mD + |c|)y = -mg$ so that $(mD + |c|)y = -mgt + c_1$ or $(D + |c|/m)y = -gt + c_2$. This is a linear first-order differential equation with integrating factor $e^{\int |c|\,dt/m} = e^{|c|t/m}$. Thus

$$\frac{d}{dt}\left[e^{|c|t/m}y\right] = (-gt + c_2)e^{|c|t/m}$$

$$e^{|c|t/m}y = -\frac{mg}{|c|}\,te^{|c|t/m} + \frac{m^2g}{c^2}\,e^{|c|t/m} + c_3e^{|c|t/m} + c_4$$

and

$$y(t) = -\frac{mg}{|c|}\,t + \frac{m^2g}{c^2} + c_3 + c_4e^{-|c|t/m}.$$

**25.** Multiplying the first equation by $D + 1$ and the second equation by $D$ we obtain

$$D\,(D+1)\,x - 2D\,(D+1)\,y = 2t + t^2$$

$$D\,(D+1)\,x - 2D\,(D+1)\,y = 0.$$

This leads to $2t + t^2 = 0$, so the system has no solutions.

**26.** The **FindRoot** application of *Mathematica* gives a solution of $x_1(t) = x_2(t)$ as approximately $t = 13.73$ minutes. So tank $B$ contains more salt than tank $A$ for $t > 13.73$ minutes.

**27.** **(a)** Separating variables in the first equation, we have $dx_1/x_1 = -dt/50$, so $x_1 = c_1e^{-t/50}$. From $x_1(0) = 15$ we get $c_1 = 15$. The second differential equation then becomes

$$\frac{dx_2}{dt} = \frac{15}{50}e^{-t/50} - \frac{2}{75}x_2 \qquad \text{or} \qquad \frac{dx_2}{dt} + \frac{2}{75}x_2 = \frac{3}{10}e^{-t/50}.$$

This differential equation is linear and has the integrating factor $e^{\int 2\,dt/75} = e^{2t/75}$. Then

$$\frac{d}{dt}\left[e^{2t/75}x_2\right] = \frac{3}{10}e^{-t/50+2t/75} = \frac{3}{10}e^{t/150}$$

so

$$e^{2t/75}x_2 = 45e^{t/150} + c_2$$

and

$$x_2 = 45e^{-t/50} + c_2e^{-2t/75}.$$

From $x_2(0) = 10$ we get $c_2 = -35$. The third differential equation then becomes

$$\frac{dx_3}{dt} = \frac{90}{75}e^{-t/50} - \frac{70}{75}e^{-2t/75} - \frac{1}{25}x_3$$

or

$$\frac{dx_3}{dt} + \frac{1}{25}x_3 = \frac{6}{5}e^{-t/50} - \frac{14}{15}e^{-2t/75}.$$

This differential equation is linear and has the integrating factor $e^{\int dt/25} = e^{t/25}$. Then

$$\frac{d}{dt}\left[e^{t/25}x_3\right] = \frac{6}{5}e^{-t/50+t/25} - \frac{14}{15}e^{-2t/75+t/25} = \frac{6}{5}e^{t/50} - \frac{14}{15}e^{t/75},$$

so

$$e^{t/25}x_3 = 60e^{t/50} - 70e^{t/75} + c_3$$

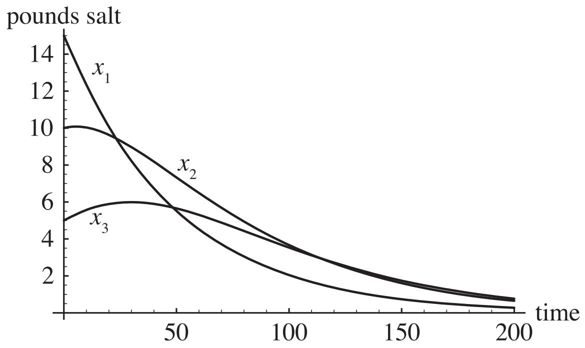
and

$$x_3 = 60e^{-t/50} - 70e^{-2t/75} + c_3e^{-t/25}.$$

From $x_3(0) = 5$ we get $c_3 = 15$. The solution of the initial-value problem is

$$\begin{aligned} x_1(t) &= 15e^{-t/50} \\ x_2(t) &= 45e^{-t/50} - 35e^{-2t/75} \\ x_3(t) &= 60e^{-t/50} - 70e^{-2t/75} + 15e^{-t/25}. \end{aligned}$$

**(b)**



**(c)** Solving $x_1(t) = \frac{1}{2}$, $x_2(t) = \frac{1}{2}$, and $x_3(t) = \frac{1}{2}$, **FindRoot** gives, respectively, $t_1 = 170.06$ min, $t_2 = 214.7$ min, and $t_3 = 224.4$ min. Thus, all three tanks will contain less than or equal to 0.5 pounds of salt after 224.4 minutes.

## 4.10 Nonlinear Differential Equations

**1.** We have $y_1' = y_1'' = e^x$, so

$$(y_1'')^2 = (e^x)^2 = e^{2x} = y_1^2.$$

Also, $y_2' = -\sin x$ and $y_2'' = -\cos x$, so

$$(y_2'')^2 = (-\cos x)^2 = \cos^2 x = y_2^2.$$

However, if $y = c_1y_1 + c_2y_2$, we have $(y'')^2 = (c_1e^x - c_2\cos x)^2$ and $y^2 = (c_1e^x + c_2\cos x)^2$. Thus $(y'')^2 \neq y^2$.

**2.** We have $y_1' = y_1'' = 0$, so

$$y_1y_1'' = 1 \cdot 0 = 0 = \frac{1}{2}(0)^2 = \frac{1}{2}(y_1')^2.$$

Also, $y_2' = 2x$ and $y_2'' = 2$, so

$$y_2y_2'' = x^2(2) = 2x^2 = \frac{1}{2}(2x)^2 = \frac{1}{2}(y_2')^2.$$

However, if $y = c_1y_1 + c_2y_2$, we have $yy'' = (c_1 \cdot 1 + c_2x^2)(c_1 \cdot 0 + 2c_2) = 2c_2(c_1 + c_2x^2)$ and $\frac{1}{2}(y')^2 = \frac{1}{2}[c_1 \cdot 0 + c_2(2x)]^2 = 2c_2^2x^2$. Thus $yy'' \neq \frac{1}{2}(y')^2$.

**3.** Let $u = y'$ so that $u' = y''$. The equation becomes $u' = -u - 1$ which is separable. Thus

$$\begin{aligned}\frac{du}{u^2+1} &= -dx \\ \tan^{-1} u &= -x + c_1 \\ y' &= \tan(c_1 - x) \\ y &= \ln|\cos(c_1 - x)| + c_2.\end{aligned}$$

**4.** Let $u = y'$ so that $u' = y''$. The equation becomes $u' = 1 + u^2$. Separating variables we obtain

$$\begin{aligned}\frac{du}{1+u^2} &= dx \\ \tan^{-1} u &= x + c_1 \\ u &= \tan(x + c_1) \\ y &= -\ln|\cos(x + c_1)| + c_2.\end{aligned}$$

**5.** Let $u = y'$ so that $u' = y''$. The equation becomes $x^2u' + u^2 = 0$. Separating variables we obtain

$$\begin{aligned}
\frac{du}{u^2} &= -\frac{dx}{x^2} \\
-\frac{1}{u} &= \frac{1}{x} + c_1 = \frac{c_1x+1}{x} \\
u &= -\frac{x}{c_1x+1} = \frac{1}{c_1}\left(\frac{1}{c_1x+1} - 1\right) \\
y &= \frac{1}{c_1^2}\ln|c_1x+1| - \frac{1}{c_1}x + c_2.
\end{aligned}$$

**6.** Let $u = y'$ so that $u' = y''$. The equation becomes $e^{-x}u' = u^2$. Separating variables we obtain

$$\begin{aligned}
\frac{du}{u^2} &= e^x\,dx \\
-\frac{1}{u} &= e^x + c_1 \\
u &= -\frac{1}{e^x+c_1} = -\frac{e^{-x}}{1+c_1e^{-x}} = \frac{1}{c_1}\,\frac{-c_1e^{-x}}{1+c_1e^{-x}} \\
y &= \frac{1}{c_1}\ln\left|1+c_1e^{-x}\right| + c_2
\end{aligned}$$

**7.** Let $u = y'$ so that $y'' = u\,du/dy$. The equation becomes $yu\,du/dy + u^2 + 1 = 0$. Separating variables we obtain

$$\begin{aligned}
\frac{u\,du}{u^2+1} + \frac{dy}{y} &= 0 \\
\frac{1}{2}\ln\left|u^2+1\right| + \ln|y| &= \ln c_1 \\
\left(u^2+1\right)y^2 &= c_1^2 \\
u^2 &= \frac{c_1^2-y^2}{y^2} \\
\frac{dy}{dx} &= \pm\frac{\sqrt{c_1^2-y^2}}{y} \\
\pm\frac{y}{\sqrt{c_1^2-y^2}}\,dy &= dx \\
\mp\sqrt{c_1^2-y^2} &= x + c_2 \\
(x+c_2)^2 + y^2 &= c_1^2
\end{aligned}$$

**8.** Let $u = y'$ so that $y'' = u\,du/dy$. The equation becomes $(y+1)u\,du/dy = u^2$. Separating variables we obtain

$$\begin{aligned} \frac{du}{u} &= \frac{dy}{y+1} \\ \ln|u| &= \ln|y+1| + \ln c_1 \\ u &= c_1\,(y+1) \\ \frac{dy}{y+1} &= c_1\,dx \\ \ln|y+1| &= c_1x + c_2 \\ y+1 &= c_3e^{c_1x}. \end{aligned}$$

**9.** Let $u = y'$ so that $y'' = u\,du/dy$. The equation becomes $u\,du/dy + 2yu^3 = 0$. Separating variables we obtain

$$\begin{aligned} \frac{du}{u^2} + 2y\,dy &= 0 \\ -\frac{1}{u} + y^2 &= c \\ u &= \frac{1}{y^2+c_1} \\ \left(y^2+c_1\right)\,dy &= dx \\ \frac{1}{3}y^3 + c_1y &= x + c_2. \end{aligned}$$

**10.** Let $u = y'$ so that $y'' = u\,du/dy$. The equation becomes $y^2u\,du/dy = u$. Separating variables we obtain

$$\begin{aligned} du &= \frac{dy}{y^2} \\ u &= -\frac{1}{y} + c_1 \\ y' &= \frac{c_1y-1}{y} \\ \frac{y}{c_1y-1}\,dy &= dx \\ \frac{1}{c_1}\left(1 + \frac{1}{c_1y-1}\right)\,dy &= dx \quad (\text{for } c_1 \neq 0) \\ \frac{1}{c_1}y + \frac{1}{c_1^2}\ln|y-1| &= x + c_2. \end{aligned}$$

If $c_1 = 0$, then $y\,dy = -dx$ and another solution is $\frac{1}{2}y^2 = -x + c_2$.

**11.** Letting $u = y'$ we have

$$y'' = \frac{du}{dx} = \frac{du}{dy}\frac{dy}{du} = u\frac{du}{dy} \qquad \text{so} \qquad 2y'y'' = 1 \qquad \text{becomes} \qquad 2u^2\frac{du}{dy} = 1.$$

Then separating variables, integrating, and simplifying, we have

$$\begin{aligned} 2u^2\,du &= dy \\ \frac{2}{3}u^3 &= y + c_1 \\ u &= \left(\frac{3}{2}y + c_2\right)^{1/3} = \frac{dy}{dx} \\ \left(\frac{3}{2}y + c_2\right)^{-1/3} dy &= dx \\ \left(\frac{3}{2}y + c_2\right)^{2/3} &= x + c_3 \\ \frac{3}{2}y + c_2 &= (x + c_3)^{3/2}\,. \end{aligned}$$

Now

$$y(0) = 2 \quad \text{implies} \quad 3 + c_2 = c_3^{3/2} \quad \text{and} \quad y'(0) = 1 \quad \text{implies} \quad c_3 = 1.$$

Thus $c_2 = -2$ and $\frac{3}{2}y - 2 = (x+1)^{3/2}$. The solution of the initial-value problem is

$$y = \frac{2}{3}(x+1)^{3/2} + \frac{4}{3}\,.$$

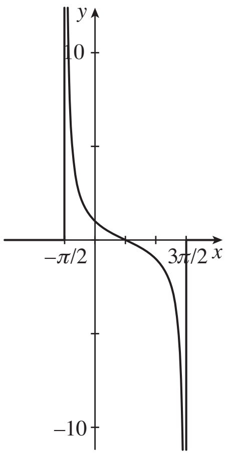
**12.** Letting $u = y'$ the differential equation becomes $u' + xu^2 = 0$. Separating variables, integrating, and simplifying we have

$$\begin{aligned} \frac{du}{dx} &= -xu^2 \\ u^{-2}\,du &= -x\,dx \\ -\frac{1}{u} &= -\frac{1}{2}x^2 + c_2 \\ y' &= \frac{2}{x^2 + c_2}\,. \end{aligned}$$

Using the initial conditions $y'(1) = 2$ we have $2 = 2/(1 + c_2)$, so $c_2 = 0$ and $y' = 2x^{-2}$. INtegrating we find $y = -2x^{-1} + c_3$. Using the other initial condition, $y(1) = 4$, we have $4 = -2 + c_3$ so $c_3 = 6$ and the solution of the intial-value problem is

$$y = 6 - \frac{2}{x}\,.$$

**13. (a)**



**(b)** Let $u = y'$ so that $y'' = u\,du/dy$. The equation becomes $u\,du/dy + yu = 0$. Separating variables we obtain

$$\begin{aligned} du &= -y\,dy \\ u &= -\frac{1}{2}y^2 + c_1 \\ y' &= -\frac{1}{2}y^2 + c_1. \end{aligned}$$

When $x = 0$, $y = 1$ and $y' = -1$ so $-1 = -1/2 + c_1$ and $c_1 = -1/2$. Then

$$\begin{aligned} \frac{dy}{dx} &= -\frac{1}{2}y^2 - \frac{1}{2} \\ \frac{dy}{y^2+1} &= -\frac{1}{2}\,dx \\ \tan^{-1} y &= -\frac{1}{2}x + c_2 \\ y &= \tan\left(-\frac{1}{2}x + c_2\right). \end{aligned}$$

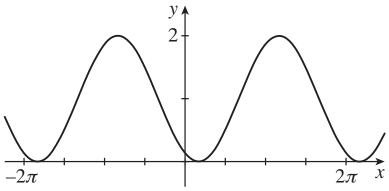
When $x = 0$, $y = 1$ so $1 = \tan c_2$ and $c_2 = \pi/4$. The solution of the initial-value problem is

$$y = \tan\left(\frac{\pi}{4} - \frac{1}{2}x\right).$$

The graph is shown in part (a).

**(c)** The interval of definition is $-\pi/2 < \pi/4 - x/2 < \pi/2$ or $-\pi/2 < x < 3\pi/2$.

**14.** Let $u = y'$ so that $u' = y''$. The equation becomes $(u')^2 + u^2 = 1$ which results in $u' = \pm\sqrt{1-u^2}$. To solve $u' = \sqrt{1-u^2}$ we separate variables:



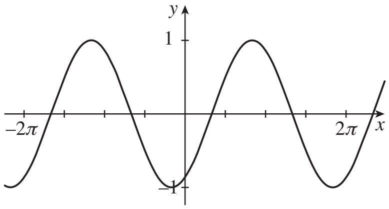
$$\begin{aligned}\frac{du}{\sqrt{1-u^2}} &= dx\\ \sin^{-1} u &= x + c_1\\ u &= \sin(x + c_1)\\ y' &= \sin(x + c_1).\end{aligned}$$

When $x = \pi/2$, $y' = \sqrt{3}/2$, so $\sqrt{3}/2 = \sin(\pi/2 + c_1)$ and $c_1 = -\pi/6$. Thus

$$\begin{aligned}y' &= \sin\left(x - \frac{\pi}{6}\right)\\ y &= -\cos\left(x - \frac{\pi}{6}\right) + c_2.\end{aligned}$$

When $x = \pi/2$, $y = 1/2$, so $1/2 = -\cos(\pi/2 - \pi/6) + c_2 = -1/2 + c_2$ and $c_2 = 1$. The solution of the initial-value problem is $y = 1 - \cos(x - \pi/6)$.

To solve $u' = -\sqrt{1-u^2}$ we separate variables:



$$\begin{aligned}\frac{du}{\sqrt{1-u^2}} &= -dx\\ \cos^{-1} u &= x + c_1\\ u &= \cos(x + c_1)\\ y' &= \cos(x + c_1).\end{aligned}$$

When $x = \pi/2$, $y' = \sqrt{3}/2$, so $\sqrt{3}/2 = \cos(\pi/2 + c_1)$ and $c_1 = -\pi/3$. Thus

$$\begin{aligned}y' &= \cos\left(x - \frac{\pi}{3}\right)\\ y &= \sin\left(x - \frac{\pi}{3}\right) + c_2.\end{aligned}$$

When $x = \pi/2$, $y = 1/2$, so $1/2 = \sin(\pi/2 - \pi/3) + c_2 = 1/2 + c_2$ and $c_2 = 0$. The solution of the initial-value problem is $y = \sin(x - \pi/3)$.

**15.** Let $u = y'$ so that $u' = y''$. The equation becomes $u' - (1/x)u = (1/x)u^3$, which is Bernoulli. Using $w = u^{-2}$ we obtain $dw/dx + (2/x)w = -2/x$. An integrating factor is $x^2$, so

$$\begin{aligned} \frac{d}{dx}[x^2 w] &= -2x \\ x^2 w &= -x^2 + c_1 \\ w &= -1 + \frac{c_1}{x^2} \\ u^{-2} &= -1 + \frac{c_1}{x^2} \\ u &= \frac{x}{\sqrt{c_1 - x^2}} \\ \frac{dy}{dx} &= \frac{x}{\sqrt{c_1 - x^2}} \\ y &= -\sqrt{c_1 - x^2} + c_2 \\ c_1 - x^2 &= (c_2 - y)^2 \\ x^2 + (c_2 - y)^2 &= c_1. \end{aligned}$$

**16.** Let $u = y'$ so that $u' = y''$. The equation becomes $u' - (1/x)u = u^2$, which is a Bernoulli differential equation. Using the substitution $w = u^{-1}$ we obtain $dw/dx + (1/x)w = -1$. An integrating factor is $x$, so

$$\begin{aligned} \frac{d}{dx}[xw] &= -x \\ w &= -\frac{1}{2}x + \frac{1}{x}c \\ \frac{1}{u} &= \frac{c_1 - x^2}{2x} \\ u &= \frac{2x}{c_1 - x^2} \\ y &= -\ln\left|c_1 - x^2\right| + c_2. \end{aligned}$$

*In Problems 17–20 the thinner curve is obtained using a numerical solver, while the thicker curve is the graph of the Taylor polynomial.*

**17.** We look for a solution of the form

$$y(x) = y(0) + y'(0)x + \frac{1}{2!}y''(0)x^2 + \frac{1}{3!}y'''(0)x^3 + \frac{1}{4!}y^{(4)}(0)x^4 + \frac{1}{5!}y^{(5)}(0)x^5.$$

From $y''(x) = x + y^2$ we compute

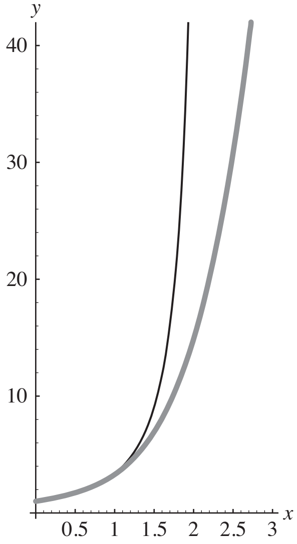
$$\begin{aligned} y'''(x) &= 1 + 2yy' \\ y^{(4)}(x) &= 2yy'' + 2(y')^2 \\ y^{(5)}(x) &= 2yy''' + 6y'y''. \end{aligned}$$

Using $y(0) = 1$ and $y'(0) = 1$ we find

$$y''(0) = 1, \quad y'''(0) = 3, \quad y^{(4)}(0) = 4, \quad y^{(5)}(0) = 12.$$

An approximate solution is

$$y(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4 + \frac{1}{10}x^5.$$



**18.** We look for a solution of the form

$$y(x) = y(0) + y'(0)x + \frac{1}{2!}y''(0)x^2 + \frac{1}{3!}y'''(0)x^3 + \frac{1}{4!}y^{(4)}(0)x^4 + \frac{1}{5!}y^{(5)}(0)x^5.$$

From $y''(x) = 1 - y^2$ we compute

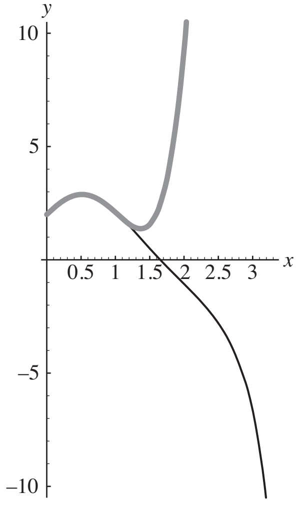
$$\begin{aligned} y'''(x) &= -2yy' \\ y^{(4)}(x) &= -2yy'' - 2(y')^2 \\ y^{(5)}(x) &= -2yy''' - 6y'y''. \end{aligned}$$

Using $y(0) = 2$ and $y'(0) = 3$ we find

$$y''(0) = -3, \quad y'''(0) = -12, \quad y^{(4)}(0) = -6, \quad y^{(5)}(0) = 102.$$

An approximate solution is

$$y(x) = 2 + 3x - \frac{3}{2}x^2 - 2x^3 - \frac{1}{4}x^4 + \frac{17}{20}x^5.$$



**19.** We look for a solution of the form

$$y(x) = y(0) + y'(0)x + \frac{1}{2!}y''(0)x^2 + \frac{1}{3!}y'''(0)x^3 + \frac{1}{4!}y^{(4)}(0)x^4 + \frac{1}{5!}y^{(5)}(0)x^5.$$

From $y''(x) = x^2 + y^2 - 2y'$ we compute

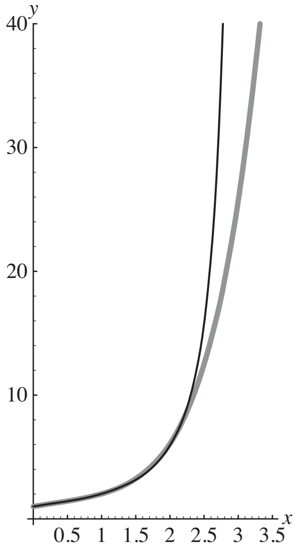
$$\begin{aligned} y'''(x) &= 2x + 2yy' - 2y'' \\ y^{(4)}(x) &= 2 + 2(y')^2 + 2yy'' - 2y''' \\ y^{(5)}(x) &= 6y'y'' + 2yy''' - 2y^{(4)}. \end{aligned}$$

Using $y(0) = 1$ and $y'(0) = 1$ we find

$$y''(0) = -1, \quad y'''(0) = 4, \quad y^{(4)}(0) = -6, \quad y^{(5)}(0) = 14.$$

An approximate solution is

$$y(x) = 1 + x - \frac{1}{2}x^2 + \frac{2}{3}x^3 - \frac{1}{4}x^4 + \frac{7}{60}x^5.$$



**20.** We look for a solution of the form

$$y(x) = y(0) + y'(0)x + \frac{1}{2!}y''(0)x^2 + \frac{1}{3!}y'''(0)x^3 + \frac{1}{4!}y^{(4)}(0)x^4 + \frac{1}{5!}y^{(5)}(0)x^5 + \frac{1}{6!}y^{(6)}(0)x^6.$$

From $y''(x) = e^y$ we compute

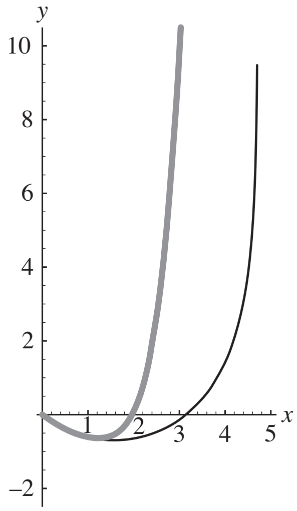
$$\begin{aligned} y'''(x) &= e^y y' \\ y^{(4)}(x) &= e^y(y')^2 + e^y y'' \\ y^{(5)}(x) &= e^y(y')^3 + 3e^y y'y'' + e^y y''' \\ y^{(6)}(x) &= e^y(y')^4 + 6e^y(y')^2y'' + 3e^y(y'')^2 + 4e^y y'y''' + e^y y^{(4)}. \end{aligned}$$

Using $y(0) = 0$ and $y'(0) = -1$ we find

$$y''(0) = 1,\ y'''(0) = -1,\ y^{(4)}(0) = 2,\ y^{(5)}(0) = -5,\ y^{(6)}(0) = 16.$$

An approximate solution is

$$y(x) = -x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{1}{24}x^5 + \frac{1}{45}x^6.$$



**21.** We need to solve $[1+(y')^2]^{3/2} = y''$. Let $u = y'$ so that $u' = y''$. The equation becomes $(1+u^2)^{3/2} = u'$ or $(1+u^2)^{3/2} = du/dx$. Separating variables and using the substitution $u = \tan\theta$ we have

$$\begin{aligned}
\frac{du}{(1+u^2)^{3/2}} &= dx \\
\int \frac{\sec^2\theta}{(1+\tan^2\theta)^{3/2}}\,d\theta &= x \\
\int \frac{\sec^2\theta}{\sec^3\theta}\,d\theta &= x \\
\int \cos\theta\,d\theta &= x \\
\sin\theta &= x \\
\frac{u}{\sqrt{1+u^2}} &= x \\
\frac{y'}{\sqrt{1+(y')^2}} &= x \\
(y')^2 &= x^2\left[1+(y')^2\right] = x^2 + x^2(y')^2 \\
(y')^2 &= \frac{x^2}{1-x^2} \\
y' &= \frac{x}{\sqrt{1-x^2}} \qquad (\text{ for } x > 0) \\
y &= -\sqrt{1-x^2}\,.
\end{aligned}$$

**22.** When $y = \sin x$, $y' = \cos x$, $y'' = -\sin x$, and

$$(y'')^2 - y^2 = \sin^2 x - \sin^2 x = 0.$$

When $y = e^{-x}$, $y' = -e^{-x}$, $y'' = e^{-x}$, and

$$(y'')^2 - y^2 = e^{-2x} - e^{-2x} = 0.$$

From $(y'')^2 - y^2 = 0$ we have $y'' = \pm y$, which can be treated as two linear equations. Since linear combinations of solutions of linear homogeneous differential equations are also solutions, we see that $y = c_1e^x + c_2e^{-x}$ and $y = c_3\cos x + c_4\sin x$ must satisfy the differential equation. However, linear combinations that involve both exponential and trigonometric functions will not be solutions since the differential equation is not linear and each type of function satisfies a different linear differential equation that is part of the original differential equation.

**23.** Letting $u = y''$, separating variables, and integrating we have

$$\frac{du}{dx} = \sqrt{1+u^2}\,, \qquad \frac{du}{\sqrt{1+u^2}} = dx, \quad \text{and} \quad \sinh^{-1} u = x + c_1.$$

Then

$$u = y'' = \sinh{(x+c_1)}, \quad y' = \cosh{(x+c_1)} + c_2, \quad \text{and} \quad y = \sinh{(x+c_1)} + c_2 x + c_3.$$

**24.** If the constant $-c_1^2$ is used instead of $c_1^2$, then, using partial fractions,

$$y = -\int \frac{dx}{x^2 - c_1^2} = -\frac{1}{2c_1}\int \left(\frac{1}{x-c_1} - \frac{1}{x+c_1}\right) dx = \frac{1}{2c_1}\ln\left|\frac{x+c_1}{x-c_1}\right| + c_2.$$

Alternatively, the inverse hyperbolic tangent can be used.

**25.** Let $u = dx/dt$ so that $d^2x/dt^2 = u\,du/dx$. The equation becomes $u\,du/dx = -k^2/x^2$. Separating variables we obtain

$$\begin{aligned} u\,du &= -\frac{k^2}{x^2}\,dx \\ \frac{1}{2}u^2 &= \frac{k^2}{x} + c \\ \frac{1}{2}v^2 &= \frac{k^2}{x} + c. \end{aligned}$$

When $t = 0$, $x = x_0$ and $v = 0$ so $0 = (k^2/x_0) + c$ and $c = -k^2/x_0$. Then

$$\frac{1}{2}v^2 = k^2\left(\frac{1}{x} - \frac{1}{x_0}\right) \quad \text{and} \quad \frac{dx}{dt} = -k\sqrt{2}\sqrt{\frac{x_0 - x}{x x_0}}\,.$$

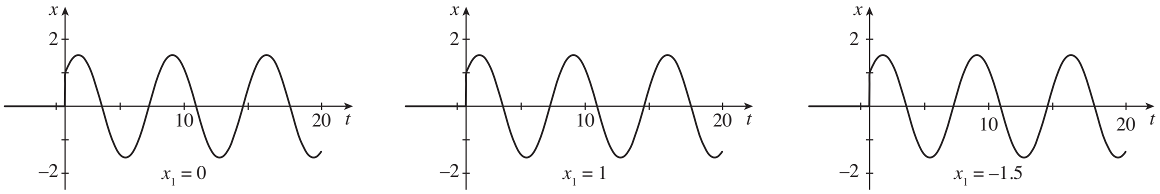
Separating variables we have

$$\begin{aligned} -\sqrt{\frac{x x_0}{x_0 - x}}\,dx &= k\sqrt{2}\,dt \\ t &= -\frac{1}{k}\sqrt{\frac{x_0}{2}}\int\sqrt{\frac{x}{x_0 - x}}\,dx. \end{aligned}$$

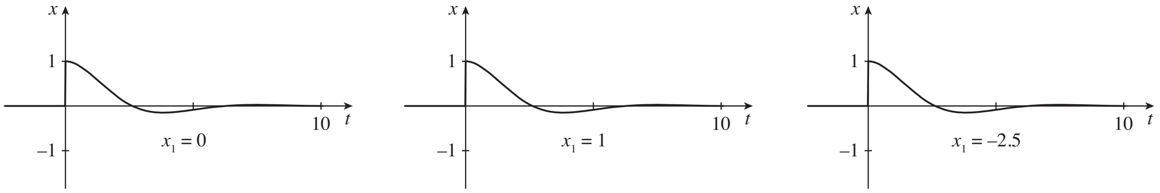
Using *Mathematica* to integrate we obtain

$$\begin{aligned} t &= -\frac{1}{k}\sqrt{\frac{x_0}{2}}\left[-\sqrt{x(x_0-x)} - \frac{x_0}{2}\tan^{-1}\left(\frac{(x_0-2x)}{2x}\sqrt{\frac{x}{x_0-x}}\right)\right] \\ &= \frac{1}{k}\sqrt{\frac{x_0}{2}}\left[\sqrt{x(x_0-x)} + \frac{x_0}{2}\tan^{-1}\frac{x_0-2x}{2\sqrt{x(x_0-x)}}\right]. \end{aligned}$$

**26.**



For $d^2x/dt^2 + \sin x = 0$ the motion appears to be periodic with amplitude 1 when $x_1 = 0$. The amplitude and period are larger for larger magnitudes of $x_1$.



For $d^2x/dt^2 + dx/dt + \sin x = 0$ the motion appears to be periodic with decreasing amplitude. The $dx/dt$ term could be said to have a damping effect.

## Chapter 4 in Review

1. Since, simply by substitution, $y = 0$ is seen to be a solution of the given initial-value problem by Theorem 4.1.1 in the text, it is the only solution

2. Since $y_c = c_1e^x + c_2e^{-x}$, a particular solution for $y'' - y = 1 + e^x$ is $y_p = A + Bxe^x$.

3. False; it is not true unless the differential equation is homogeneous. For example, $y_1 = x$ is a solution of $y'' + y = x$, but $y_2 = 5x$ is not.

4. False; Theorem 4.1.3 in the text requires that $f_1$ and $f_2$ be solutions of a homogeneous linear differential equation. For example, $f_1(x) = x$ and $f_2(x) = |x|$ are defined and linearly independent on $[-1, 1]$, but the Wronskian does not exist at $x = 0$, since $f_2(x)$ is not defined there.

5. The auxiliary equation is second-order and has $5i$ as a root. Thus, the other root of the auxiliary equation must be $-5i$, since complex roots of polynomials with real coefficients must occur in conjugate pairs. Thus, a second solution of the differential equation must be $\cos 5x$ and the general solution is $y = c_1 \cos 5x + c_2 \sin 5x$.

6. The roots are $m_1 = m_2 = m_3 = 0$ and $m_4 = 1$. To see this, note that the general solution of a homogeneous linear fourth-order differential equation with auxiliary equation $m^3(m - 1) = 0$ is $y = c_1 + c_2x + c_3x^2 + c_4e^x$.

**7.** If $y = c_1x^2 + c_2x^2 \ln x$, $x > 0$, is the general solution of a Cauchy-Euler differential equation, then the roots of its auxiliary equation are $m_1 = m_2 = 2$ and the auxiliary equation is

$$m^2 - 4m + 4 = m(m-1) - 3m + 4 = 0.$$

Thus, the Cauchy-Euler differential equation is $x^2y'' - 3xy' + 4y = 0$.

**8.** For $y_p = Ax^2$ we have $y_p' = 2Ax$, $y_p'' = 2A$, and $y_p''' = 0$. Thus, substituting $y_p$ into the differential equation, we have $0 + 2A = 1$ and $A = \frac{1}{2}$.

**9.** By the superposition principle for nonhomogeneous equations a particular solution is

$$y_p = y_{p_1} + y_{p_2} = x + x^2 - 2 = x^2 + x - 2.$$

**10.** True, by the superposition for homogeneous equations.

**11.** The set is linearly independent over $(-\infty, 0)$ and linearly dependent over $(0, \infty)$.

**12.** **(a)** Since $f_2(x) = 2\ln x = 2f_1(x)$, the set of functions is linearly dependent.

**(b)** Since $x^{n+1}$ is not a constant multiple of $x^n$, the set of functions is linearly independent.

**(c)** Since $x + 1$ is not a constant multiple of $x$, the set of functions is linearly independent.

**(d)** Since $f_1(x) = \cos x \cos(\pi/2) - \sin x \sin(\pi/2) = -\sin x = -f_2(x)$, the set of functions is linearly dependent.

**(e)** Since $f_1(x) = 0 \cdot f_2(x)$, the set of functions is linearly dependent.

**(f)** Since $2x$ is not a constant multiple of 2, the set of functions is linearly independent.

**(g)** Since $3(x^2) + 2(1 - x^2) - (2 + x^2) = 0$, the set of functions is linearly dependent.

**(h)** Since $xe^{x+1} + 0(4x - 5)e^x - exe^x = 0$, the set of functions is linearly dependent.

**13.** **(a)** The auxiliary equation is $(m-3)(m+5)(m-1) = m^3 + m^2 - 17m + 15 = 0$, so the differential equation is $y''' + y'' - 17y' + 15y = 0$.

**(b)** The form of the auxiliary equation is

$$m(m-1)(m-2) + bm(m-1) + cm + d = m^3 + (b-3)m^2 + (c-b+2)m + d = 0.$$

Since $(m-3)(m+5)(m-1) = m^3 + m^2 - 17m + 15 = 0$, we have $b - 3 = 1$, $c - b + 2 = -17$, and $d = 15$. Thus, $b = 4$ and $c = -15$, so the differential equation is
$y''' + 4y'' - 15y' + 15y = 0$.

**14.** Variation of parameters will work for all choices of $g(x)$, although the integral involved may not always be able to be expressed in terms of elementary functions. The method of undetermined coefficients will work for the functions in **(b)**, **(c)**, and **(e)**.

**15.** The auxiliary equation is $am(m-1)+bm+c=am^2+(b-a)m+c=0$. If the roots are 3 and $-1$, then we want $(m-3)(m+1)=m^2-2m-3=0$. Thus, let $a=1$, $b=-1$, and $c=-3$, so that the differential equation is $x^2y''-xy'-3y=0$.

**16.** In this case we want the auxiliary equation to be $m^2+1=0$, so let $a=1$, $b=1$, and $c=1$. Then the differential equation is $x^2y''+xy'+y=0$.

**17.** From $m^2-2m-2=0$ we obtain $m=1\pm\sqrt{3}$ so that

$$y=c_1e^{(1+\sqrt{3})x}+c_2e^{(1-\sqrt{3})x}.$$

**18.** From $2m^2+2m+3=0$ we obtain $m=-1/2\pm(\sqrt{5}/2)i$ so that

$$y=e^{-x/2}\left(c_1\cos\frac{\sqrt{5}}{2}x+c_2\sin\frac{\sqrt{5}}{2}x\right).$$

**19.** From $m^3+10m^2+25m=0$ we obtain $m=0$, $m=-5$, and $m=-5$ so that

$$y=c_1+c_2e^{-5x}+c_3xe^{-5x}.$$

**20.** From $2m^3+9m^2+12m+5=0$ we obtain $m=-1$, $m=-1$, and $m=-5/2$ so that

$$y=c_1e^{-5x/2}+c_2e^{-x}+c_3xe^{-x}.$$

**21.** From $3m^3+10m^2+15m+4=0$ we obtain $m=-1/3$ and $m=-3/2\pm(\sqrt{7}/2)i$ so that

$$y=c_1e^{-x/3}+e^{-3x/2}\left(c_2\cos\frac{\sqrt{7}}{2}x+c_3\sin\frac{\sqrt{7}}{2}x\right).$$

**22.** From $2m^4+3m^3+2m^2+6m-4=0$ we obtain $m=1/2$, $m=-2$, and $m=\pm\sqrt{2}\,i$ so that

$$y=c_1e^{x/2}+c_2e^{-2x}+c_3\cos\sqrt{2}\,x+c_4\sin\sqrt{2}\,x.$$

**23.** Applying $D^4$ to the differential equation we obtain $D^4(D^2-3D+5)=0$. Then

$$y=\underbrace{e^{3x/2}\left(c_1\cos\frac{\sqrt{11}}{2}x+c_2\sin\frac{\sqrt{11}}{2}x\right)}_{y_c}+c_3+c_4x+c_5x^2+c_6x^3$$

and $y_p=A+Bx+Cx^2+Dx^3$. Substituting $y_p$ into the differential equation yields

$$(5A-3B+2C)+(5B-6C+6D)x+(5C-9D)x^2+5Dx^3=-2x+4x^3.$$

Equating coefficients gives $A = -222/625$, $B = 46/125$, $C = 36/25$, and $D = 4/5$. The general solution is

$$y = e^{3x/2}\left(c_1 \cos \frac{\sqrt{11}}{2}x + c_2 \sin \frac{\sqrt{11}}{2}x\right) - \frac{222}{625} + \frac{46}{125}x + \frac{36}{25}x^2 + \frac{4}{5}x^3.$$

**24.** Applying $(D-1)^3$ to the differential equation we obtain $(D-1)^3(D-2D+1) = (D-1)^5 = 0$. Then

$$y = \underbrace{c_1e^x + c_2xe^x}_{y_c} + c_3x^2e^x + c_4x^3e^x + c_5x^4e^x$$

and $y_p = Ax^2e^x + Bx^3e^x + Cx^4e^x$. Substituting $y_p$ into the differential equation yields

$$12Cx^2e^x + 6Bxe^x + 2Ae^x = x^2e^x.$$

Equating coefficients gives $A = 0$, $B = 0$, and $C = 1/12$. The general solution is

$$y = c_1e^x + c_2xe^x + \frac{1}{12}x^4e^x.$$

**25.** Applying $D(D^2+1)$ to the differential equation we obtain

$$D(D^2+1)(D^3 - 5D^2 + 6D) = D^2(D^2+1)(D-2)(D-3) = 0.$$

Then

$$y = \underbrace{c_1 + c_2e^{2x} + c_3e^{3x}}_{y_c} + c_4x + c_5 \cos x + c_6 \sin x$$

and $y_p = Ax + B\cos x + C\sin x$. Substituting $y_p$ into the differential equation yields

$$6A + (5B + 5C)\cos x + (-5B + 5C)\sin x = 8 + 2\sin x.$$

Equating coefficients gives $A = 4/3$, $B = -1/5$, and $C = 1/5$. The general solution is

$$y = c_1 + c_2e^{2x} + c_3e^{3x} + \frac{4}{3}x - \frac{1}{5}\cos x + \frac{1}{5}\sin x.$$

**26.** Applying $D$ to the differential equation we obtain $D(D^3 - D^2) = D^3(D-1) = 0$. Then

$$y = \underbrace{c_1 + c_2x + c_3e^x}_{y_c} + c_4x^2$$

and $y_p = Ax^2$. Substituting $y_p$ into the differential equation yields $-2A = 6$. Equating coefficients gives $A = -3$. The general solution is

$$y = c_1 + c_2x + c_3e^x - 3x^2.$$

**27.** The auxiliary equation is $m^2 - 2m + 2 = [m - (1+i)][m - (1-i)] = 0$, so $y_c = c_1 e^x \sin x + c_2 e^x \cos x$ and

$$W = \begin{vmatrix} e^x \sin x & e^x \cos x \\ e^x \cos x + e^x \sin x & -e^x \sin x + e^x \cos x \end{vmatrix} = -e^{2x}$$

Identifying $f(x) = e^x \tan x$ we obtain

$$u_1' = -\frac{(e^x \cos x)(e^x \tan x)}{-e^{2x}} = \sin x$$

$$u_2' = \frac{(e^x \sin x)(e^x \tan x)}{-e^{2x}} = -\frac{\sin^2 x}{\cos x} = \cos x - \sec x.$$

Then $u_1 = -\cos x$, $u_2 = \sin x - \ln|\sec x + \tan x|$, and

$$\begin{aligned} y &= c_1 e^x \sin x + c_2 e^x \cos x - e^x \sin x \cos x + e^x \sin x \cos x - e^x \cos x \ln|\sec x + \tan x| \\ &= c_1 e^x \sin x + c_2 e^x \cos x - e^x \cos x \ln|\sec x + \tan x|. \end{aligned}$$

**28.** The auxiliary equation is $m^2 - 1 = 0$, so $y_c = c_1 e^x + c_2 e^{-x}$ and

$$W = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -2$$

Identifying $f(x) = 2e^x/(e^x + e^{-x})$ we obtain

$$u_1' = \frac{1}{e^x + e^{-x}} = \frac{e^x}{1 + e^{2x}}$$

$$u_2' = -\frac{e^{2x}}{e^x + e^{-x}} = -\frac{e^{3x}}{1 + e^{2x}} = -e^x + \frac{e^x}{1 + e^{2x}}.$$

Then $u_1 = \tan^{-1} e^x$, $u_2 = -e^x + \tan^{-1} e^x$, and

$$y = c_1 e^x + c_2 e^{-x} + e^x \tan^{-1} e^x - 1 + e^{-x} \tan^{-1} e^x.$$

**29.** The auxiliary equation is $6m^2 - m - 1 = 0$ so that

$$y = c_1 x^{1/2} + c_2 x^{-1/3}.$$

**30.** The auxiliary equation is $2m^3 + 13m^2 + 24m + 9 = (m+3)^2(m + 1/2) = 0$ so that

$$y = c_1 x^{-3} + c_2 x^{-3} \ln x + c_3 x^{-1/2}.$$

**31.** The auxiliary equation is $m^2 - 5m + 6 = (m-2)(m-3) = 0$ and a particular solution is $y_p = x^4 - x^2 \ln x$ so that

$$y = c_1 x^2 + c_2 x^3 + x^4 - x^2 \ln x.$$

**32.** The auxiliary equation is $m^2 - 2m + 1 = (m-1)^2 = 0$ and a particular solution is $y_p = \frac{1}{4}x^3$ so that

$$y = c_1 x + c_2 x \ln x + \frac{1}{4}x^3.$$

**33.** The auxiliary equation is $m^2 + \omega^2 = 0$, so $y_c = c_1 \cos\omega t + c_2 \sin\omega t$. When $\omega \neq \alpha$, $y_p = A\cos\alpha t + B\sin\alpha t$ and

$$y = c_1 \cos\omega t + c_2 \sin\omega t + A\cos\alpha t + B\sin\alpha t.$$

When $\omega = \alpha$, $y_p = At\cos\omega t + Bt\sin\omega t$ and

$$y = c_1 \cos\omega t + c_2 \sin\omega t + At\cos\omega t + Bt\sin\omega t.$$

**34.** The auxiliary equation is $m^2 - \omega^2 = 0$, so $y_c = c_1 e^{\omega t} + c_2 e^{-\omega t}$. When $\omega \neq \alpha$, $y_p = Ae^{\alpha t}$ and

$$y = c_1 e^{\omega t} + c_2 e^{-\omega t} + Ae^{\alpha t}.$$

When $\omega = \alpha$, $y_p = Ate^{\omega t}$ and

$$y = c_1 e^{\omega t} + c_2 e^{-\omega t} + Ate^{\omega t}.$$

**35.** If $y = \sin x$ is a solution then so is $y = \cos x$ and $m^2 + 1$ is a factor of the auxiliary equation $m^4 + 2m^3 + 11m^2 + 2m + 10 = 0$. Dividing by $m^2 + 1$ we get $m^2 + 2m + 10$, which has roots $-1 \pm 3i$. The general solution of the differential equation is

$$y = c_1 \cos x + c_2 \sin x + e^{-x}(c_3 \cos 3x + c_4 \sin 3x).$$

**36.** The auxiliary equation is $m(m+1) = m^2 + m = 0$, so the associated homogeneous differential equation is $y'' + y' = 0$. Letting $y = c_1 + c_2 e^{-x} + \frac{1}{2}x^2 - x$ and computing $y'' + y'$ we get $x$. Thus, the differential equation is $y'' + y' = x$.

**37.** **(a)** The auxiliary equation is $m^4 - 2m^2 + 1 = (m^2 - 1)^2 = 0$, so the general solution of the differential equation is

$$y = c_1 \sinh x + c_2 \cosh x + c_3 x \sinh x + c_4 x \cosh x.$$

**(b)** Since both $\sinh x$ and $x \sinh x$ are solutions of the associated homogeneous differential equation , a particular solution of $y^{(4)} - 2y'' + y = \sinh x$ has the form
$y_p = Ax^2 \sinh x + Bx^2 \cosh x$.

**38.** Since $y_1' = 1$ and $y_1'' = 0$, $x^2 y_1'' - (x^2 + 2x)y_1' + (x+2)y_1 = -x^2 - 2x + x^2 + 2x = 0$, and $y_1 = x$ is a solution of the associated homogeneous equation. Using the method of reduction of order, we let $y = ux$. Then $y' = xu' + u$ and $y'' = xu'' + 2u'$, so

$$\begin{aligned} x^2 y'' - (x^2 + 2x)y' + (x+2)y &= x^3 u'' + 2x^2 u' - x^3 u' - 2x^2 u' - x^2 u - 2xu + x^2 u + 2xu \\ &= x^3 u'' - x^3 u' = x^3 (u'' - u'). \end{aligned}$$

To find a second solution of the homogeneous equation we note that $u = e^x$ is a solution of $u'' - u' = 0$. Thus, $y_c = c_1 x + c_2 x e^x$. To find a particular solution we set $x^3(u'' - u') = x^3$ so that $u'' - u' = 1$. This differential equation has a particular solution of the form $Ax$. Substituting, we find $A = -1$, so a particular solution of the original differential equation is $y_p = -x^2$ and the general solution is $y = c_1 x + c_2 x e^x - x^2$.

**39.** The auxiliary equation is $m^2 - 2m + 2 = 0$ so that $m = 1 \pm i$ and $y = e^x(c_1 \cos x + c_2 \sin x)$. Setting $y(\pi/2) = 0$ and $y(\pi) = -1$ we obtain $c_1 = e^{-\pi}$ and $c_2 = 0$. Thus, $y = e^{x-\pi} \cos x$.

**40.** The auxiliary equation is $m^2 + 2m + 1 = (m+1)^2 = 0$, so that $y = c_1 e^{-x} + c_2 x e^{-x}$. Setting $y(-1) = 0$ and $y'(0) = 0$ we get $c_1 e - c_2 e = 0$ and $-c_1 + c_2 = 0$. Thus $c_1 = c_2$ and $y = c_1(e^{-x} + x e^{-x})$ is a solution of the boundary-value problem for any real number $c_1$.

**41.** The auxiliary equation is $m^2 - 1 = (m-1)(m+1) = 0$ so that $m = \pm 1$ and $y = c_1 e^x + c_2 e^{-x}$. Assuming $y_p = Ax + B + C \sin x$ and substituting into the differential equation we find $A = -1$, $B = 0$, and $C = -\frac{1}{2}$. Thus $y_p = -x - \frac{1}{2} \sin x$ and

$$y = c_1 e^x + c_2 e^{-x} - x - \frac{1}{2} \sin x.$$

Setting $y(0) = 2$ and $y'(0) = 3$ we obtain

$$\begin{aligned} c_1 + c_2 &= 2 \\ c_1 - c_2 - \frac{3}{2} &= 3. \end{aligned}$$

Solving this system we find $c_1 = \frac{13}{4}$ and $c_2 = -\frac{5}{4}$. The solution of the initial-value problem is

$$y = \frac{13}{4} e^x - \frac{5}{4} e^{-x} - x - \frac{1}{2} \sin x.$$

**42.** The auxiliary equation is $m^2 + 1 = 0$, so $y_c = c_1 \cos x + c_2 \sin x$ and

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1.$$

Identifying $f(x) = \sec^3 x$ we obtain

$$\begin{aligned} u_1' &= -\sin x \sec^3 x = -\frac{\sin x}{\cos^3 x} \\ u_2' &= \cos x \sec^3 x = \sec^2 x. \end{aligned}$$

Then

$$\begin{aligned} u_1 &= -\frac{1}{2} \frac{1}{\cos^2 x} = -\frac{1}{2} \sec^2 x \\ u_2 &= \tan x. \end{aligned}$$

Thus

$$\begin{aligned} y &= c_1 \cos x + c_2 \sin x - \frac{1}{2}\cos x \sec^2 x + \sin x \tan x \\ &= c_1 \cos x + c_2 \sin x - \frac{1}{2}\sec x + \frac{1-\cos^2 x}{\cos x} \\ &= c_3 \cos x + c_2 \sin x + \frac{1}{2}\sec x. \end{aligned}$$

and

$$y' = -c_3 \sin x + c_2 \cos x + \frac{1}{2}\sec x \tan x.$$

The initial conditions imply

$$\begin{aligned} c_3 + \frac{1}{2} &= 1 \\ c_2 &= \frac{1}{2}. \end{aligned}$$

Thus $c_3 = c_2 = 1/2$ and

$$y = \frac{1}{2}\cos x + \frac{1}{2}\sin x + \frac{1}{2}\sec x.$$

**43.** Let $u = y'$ so that $u' = y''$. The equation becomes $u\,du/dx = 4x$. Separating variables we obtain

$$\begin{aligned} u\,du &= 4x\,dx \\ \frac{1}{2}u^2 &= 2x^2 + c_1 \\ u^2 &= 4x^2 + c_2. \end{aligned}$$

When $x = 1$, $y' = u = 2$, so $4 = 4 + c_2$ and $c_2 = 0$. Then

$$\begin{aligned} u^2 &= 4x^2 \\ \frac{dy}{dx} = 2x \quad &\text{or} \quad \frac{dy}{dx} = -2x \\ y = x^2 + c_3 \quad &\text{or} \quad y = -x^2 + c_4. \end{aligned}$$

When $x = 1$, $y = 5$, so $5 = 1 + c_3$ and $5 = -1 + c_4$. Thus $c_3 = 4$ and $c_4 = 6$. We have $y = x^2 + 4$ and $y = -x^2 + 6$. Note however that when $y = -x^2 + 6$, $y' = -2x$ and $y'(1) = -2 \neq 2$. Thus, the solution of the initial-value problem is $y = x^2 + 4$.

**44.** Let $u = y'$ so that $y'' = u\,du/dy$. The equation becomes $2u\,du/dy = 3y^2$. Separating variables we obtain

$$\begin{aligned} 2u\,du &= 3y^2\,dy \\ u^2 &= y^3 + c_1. \end{aligned}$$

When $x = 0$, $y = 1$ and $y' = u = 1$ so $1 = 1 + c_1$ and $c_1 = 0$. Then

$$\begin{aligned} u^2 &= y^3 \\ \left(\frac{dy}{dx}\right)^2 &= y^3 \\ \frac{dy}{dx} &= y^{3/2} \\ y^{-3/2}\,dy &= dx \\ -2y^{-1/2} &= x + c_2 \\ y &= \frac{4}{(x + c_2)^2}\,. \end{aligned}$$

When $x = 0$, $y = 1$, so $1 = 4/c_2^2$ and $c_2 = \pm 2$. Thus, $y = 4/(x+2)^2$ and $y = 4/(x-2)^2$. Note, however, that when $y = 4/(x+2)^2$, $y' = -8/(x+2)^3$ and $y'(0) = -1 \neq 1$. Thus, the solution of the initial-value problem is $y = 4/(x-2)^2$.

**45.** **(a)** The auxiliary equation is $12m^4 + 64m^3 + 59m^2 - 23m - 12 = 0$ and has roots $-4$, $-\frac{3}{2}$, $-\frac{1}{3}$, and $\frac{1}{2}$. The general solution is

$$y = c_1e^{-4x} + c_2e^{-3x/2} + c_3e^{-x/3} + c_4e^{x/2}.$$

**(b)** The system of equations is

$$\begin{aligned} c_1 + c_2 + c_3 + c_4 &= -1 \\ -4c_1 - \frac{3}{2}c_2 - \frac{1}{3}c_3 + \frac{1}{2}c_4 &= 2 \\ 16c_1 + \frac{9}{4}c_2 + \frac{1}{9}c_3 + \frac{1}{4}c_4 &= 5 \\ -64c_1 - \frac{27}{8}c_2 - \frac{1}{27}c_3 + \frac{1}{8}c_4 &= 0. \end{aligned}$$

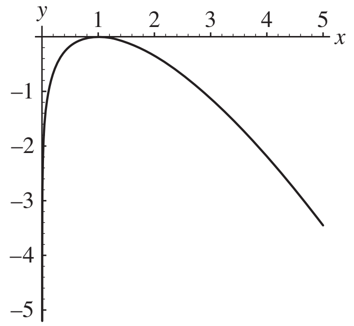
Using a CAS we find $c_1 = -\frac{73}{495}$, $c_2 = \frac{109}{35}$, $c_3 = -\frac{3726}{385}$, and $c_4 = \frac{257}{45}$. The solution of the initial-value problem is

$$y = -\frac{73}{495}e^{-4x} + \frac{109}{35}e^{-3x/2} - \frac{3726}{385}e^{-x/3} + \frac{257}{45}e^{x/2}.$$

**46.** Consider $xy'' + y' = 0$ and look for a solution of the form $y = x^m$. Substituting into the differential equation we have

$$xy'' + y' = m(m-1)x^{m-1} + mx^{m-1} = m^2x^{m-1}.$$

Thus, the general solution of $xy'' + y' = 0$ is $y_c = c_1 + c_2 \ln x$. To find a particular solution of $xy'' + y' = -\sqrt{x}$ we use variation of parameters.



The Wronskian is

$$W = \begin{vmatrix} 1 & \ln x \\ 0 & 1/x \end{vmatrix} = \frac{1}{x}$$

Identifying $f(x) = -x^{-1/2}$ we obtain

$$u_1' = \frac{x^{-1/2}\ln x}{1/x} = \sqrt{x}\ln x \quad\text{and}\quad u_2' = \frac{-x^{-1/2}}{1/x} = -\sqrt{x}\,,$$

so that

$$u_1 = x^{3/2}\left(\frac{2}{3}\ln x - \frac{4}{9}\right) \quad\text{and}\quad u_2 = -\frac{2}{3}x^{3/2}.$$

Then

$$y_p = x^{3/2}\left(\frac{2}{3}\ln x - \frac{4}{9}\right) - \frac{2}{3}x^{3/2}\ln x = -\frac{4}{9}x^{3/2}$$

and the general solution of the differential equation is

$$y = c_1 + c_2\ln x - \frac{4}{9}x^{3/2}.$$

The initial conditions are $y(1) = 0$ and $y'(1) = 0$. These imply that $c_1 = \frac{4}{9}$ and $c_2 = \frac{2}{3}$. The solution of the initial-value problem is

$$y = \frac{4}{9} + \frac{2}{3}\ln x - \frac{4}{9}x^{3/2}.$$

The graph is shown above.

**47.** From $(D-2)x + (D-2)y = 1$ and $Dx + (2D-1)y = 3$ we obtain $(D-1)(D-2)y = -6$ and $Dx = 3 - (2D-1)y$. Then

$$y = c_1e^{2t} + c_2e^t - 3 \quad\text{and}\quad x = -c_2e^t - \frac{3}{2}c_1e^{2t} + c_3.$$

Substituting into $(D-2)x + (D-2)y = 1$ gives $c_3 = \frac{5}{2}$ so that

$$x = -c_2e^t - \frac{3}{2}c_1e^{2t} + \frac{5}{2}\,.$$

**48.** From $(D-2)x - y = t-2$ and $-3x + (D-4)y = -4t$ we obtain $(D-1)(D-5)x = 9 - 8t$. Then

$$x = c_1e^t + c_2e^{5t} - \frac{8}{5}t - \frac{3}{25}$$

and

$$y = (D-2)x - t + 2 = -c_1e^t + 3c_2e^{5t} + \frac{16}{25} + \frac{11}{25}t.$$

**49.** From $(D-2)x - y = -e^t$ and $-3x + (D-4)y = -7e^t$ we obtain $(D-1)(D-5)x = -4e^t$ so that

$$x = c_1e^t + c_2e^{5t} + te^t.$$

Then

$$y = (D-2)x + e^t = -c_1e^t + 3c_2e^{5t} - te^t + 2e^t.$$

**50.** From $(D+2)x+(D+1)y=\sin 2t$ and $5x+(D+3)y=\cos 2t$ we obtain $(D^2+5)y=2\cos 2t-7\sin 2t$. Then

$$y=c_1\cos t+c_2\sin t-\frac{2}{3}\cos 2t+\frac{7}{3}\sin 2t$$

and

$$\begin{aligned} x&=-\frac{1}{5}(D+3)y+\frac{1}{5}\cos 2t \\ &=\left(\frac{1}{5}c_1-\frac{3}{5}c_2\right)\sin t+\left(-\frac{1}{5}c_2-\frac{3}{5}c_1\right)\cos t-\frac{5}{3}\sin 2t-\frac{1}{3}\cos 2t. \end{aligned}$$

# Chapter 5

# Modeling with Higher-Order Differential Equations

## 5.1 Linear Models: Initial-Value Problems

**1.** From $\frac{1}{8}x'' + 16x = 0$ we obtain

$$x = c_1 \cos 8\sqrt{2}\,t + c_2 \sin 8\sqrt{2}\,t$$

so that the period of motion is $2\pi/8\sqrt{2} = \sqrt{2}\,\pi/8$ seconds.

**2.** From $20x'' + kx = 0$ we obtain

$$x = c_1 \cos \frac{1}{2}\sqrt{\frac{k}{5}}\,t + c_2 \sin \frac{1}{2}\sqrt{\frac{k}{5}}\,t$$

so that the frequency $2/\pi = \sqrt{k/20}\,/2\pi$ and $k = 320$ N/m. If $80x'' + 320x = 0$ then $x = c_1 \cos 2t + c_2 \sin 2t$ so that the frequency is $2/2\pi = 1/\pi$ cycles/s.

**3.** From $\frac{3}{4}x'' + 72x = 0$, $x(0) = -1/4$, and $x'(0) = 0$ we obtain $x = -\frac{1}{4}\cos 4\sqrt{6}\,t$.

**4.** From $\frac{3}{4}x'' + 72x = 0$, $x(0) = 0$, and $x'(0) = 2$ we obtain $x = \frac{\sqrt{6}}{12}\sin 4\sqrt{6}\,t$.

**5.** From $\frac{5}{8}x'' + 40x = 0$, $x(0) = 1/2$, and $x'(0) = 0$ we obtain $x = \frac{1}{2}\cos 8t$.

**(a)** $x(\pi/12) = -1/4$, $x(\pi/8) = -1/2$, $x(\pi/6) = -1/4$, $x(\pi/4) = 1/2$, $x(9\pi/32) = \sqrt{2}/4$.

**(b)** $x' = -4\sin 8t$ so that $x'(3\pi/16) = 4$ ft/s directed downward.

**(c)** If $x = \frac{1}{2}\cos 8t = 0$ then $t = (2n+1)\pi/16$ for $n = 0, 1, 2, \ldots$ .

**6.** From $50x'' + 200x = 0$, $x(0) = 0$, and $x'(0) = -10$ we obtain $x = -5\sin 2t$ and $x' = -10\cos 2t$.

**7.** From $20x'' + 20x = 0$, $x(0) = 0$, and $x'(0) = -10$ we obtain $x = -10\sin t$ and $x' = -10\cos t$.

**(a)** The 20 kg mass has the larger amplitude.

**(b)** 20 kg: $x'(\pi/4) = -5\sqrt{2}$ m/s, $x'(\pi/2) = 0$ m/s; 50 kg: $x'(\pi/4) = 0$ m/s, $x'(\pi/2) = 10$ m/s

**(c)** If $-5\sin 2t = -10\sin t$ then $2\sin t(\cos t - 1) = 0$ so that $t = n\pi$ for $n = 0, 1, 2, \ldots,$ placing both masses at the equilibrium position. The 50 kg mass is moving upward; the 20 kg mass is moving upward when $n$ is even and downward when $n$ is odd.

**8.** From $x'' + 16x = 0$, $x(0) = -1$ , and $x'(0) = -2$, we get

$$x(t) = -\cos 4t - \frac{1}{2}\sin 4t = \frac{\sqrt{5}}{2}\cos(4t - 3.605)$$

The period is $T = 2\pi/\omega = \pi/2$ seconds and the amplitude is $\sqrt{5}/2$ feet.

In $4\pi$ seconds it will make 8 complete cycles. To see this, note that the frequency is

$$f = \frac{1}{T} = \frac{\omega}{2\pi} = \frac{4}{2\pi}$$

This means the mass will make 4 complete cycles every $2\pi$ seconds.

**9. (a)** From $x'' + 16x = 0$, $x(0) = 1/2$, and $x'(0) = 3/2$, we get

$$x(t) = \frac{1}{2}\cos 2t + \frac{3}{4}\sin 2t$$

**(b)** We use $x(t) = A\sin(\omega t + \phi)$ where $A = \sqrt{c_1^2 + c_2^2} = \sqrt{(1/2)^2 + (3/4)^2} = \sqrt{13}\,/4$. From this we get

$$\tan\phi = \frac{c_1}{c_2} = \frac{2}{3}$$

$$\phi = \tan^{-1}\left(\frac{2}{3}\right) = 0.588$$

Therefore $x(t) = \frac{\sqrt{13}}{4}\sin(2t + 0.588)$.

**(c)** We use $x(t) = A\cos(\omega t - \phi)$ where $A = \sqrt{c_1^2 + c_2^2} = \sqrt{(1/2)^2 + (3/4)^2} = \sqrt{13}\,/4$. This time we get

$$\tan\phi = \frac{c_2}{c_1} = \frac{3}{2}$$

$$\phi = \tan^{-1}\left(\frac{3}{2}\right) = 0.983$$

Therefore $x(t) = \frac{\sqrt{13}}{4}\cos(2t - 0.983)$.

**10. (a)** From $1.6x'' + 40x = 0$, $x(0) = -1/3$, and $x'(0) = 5/4$, we get

$$x(t) = -\frac{1}{3}\cos 5t + \frac{1}{4}\sin 5t$$

We use $x(t) = A\sin(\omega t + \phi)$ where $A = \sqrt{c_1^2 + c_2^2} = \sqrt{(-1/3)^2 + (1/4)^2} = 5/12$. From this we get

$$\tan\phi = \frac{c_1}{c_2} = -\frac{4}{3}$$
$$\phi = \tan^{-1}\left(-\frac{4}{3}\right) = -0.927$$

Therefore $x(t) = \frac{5}{12}\sin(5t - 0.927)$.

**(b)** We use $x(t) = A\cos(\omega t - \phi)$ where $A = \sqrt{c_1^2 + c_2^2} = 5/12$. Note that $\sin\phi > 0$ and $\cos\phi < 0$ so

$$\tan\phi = \frac{c_2}{c_1} = -\frac{4}{3}$$
$$\phi = \tan^{-1}\left(-\frac{4}{3}\right) = -0.927$$

Therefore $x(t) = \frac{5}{12}\cos(5t - 2.489)$.

**11.** From $2x'' + 200x = 0$, $x(0) = -2/3$, and $x'(0) = 5$ we obtain

**(a)** $x = -\frac{2}{3}\cos 10t + \frac{1}{2}\sin 10t = \frac{5}{6}\sin(10t - 0.927)$.

**(b)** The amplitude is 5/6 ft and the period is $2\pi/10 = \pi/5$

**(c)** $3\pi = \pi k/5$ and $k = 15$ cycles.

**(d)** If $x = 0$ and the weight is moving downward for the second time, then $10t - 0.927 = 2\pi$ or $t = 0.721$ s.

**(e)** If $x' = \frac{25}{3}\cos(10t - 0.927) = 0$ then $10t - 0.927 = \pi/2 + n\pi$ or $t = (2n+1)\pi/20 + 0.0927$ for $n = 0, 1, 2, \ldots$ .

**(f)** $x(3) = -0.597$ ft

**(g)** $x'(3) = -5.814$ ft/s

**(h)** $x''(3) = 59.702$ ft/s$^2$

**(i)** If $x = 0$ then $t = \frac{1}{10}(0.927 + n\pi)$ for $n = 0, 1, 2, \ldots$. The velocity at these times is $x' = \pm 8.33$ ft/s.

**(j)** If $x = 5/12$ then $t = \frac{1}{10}(\pi/6 + 0.927 + 2n\pi)$ and $t = \frac{1}{10}(5\pi/6 + 0.927 + 2n\pi)$ for $n = 0, 1, 2, \ldots$ .

**(k)** If $x = 5/12$ and $x' < 0$ then $t = \frac{1}{10}(5\pi/6 + 0.927 + 2n\pi)$ for $n = 0, 1, 2, \ldots$.

**12.** From $x'' + 9x = 0$, $x(0) = -1$, and $x'(0) = -\sqrt{3}$ we obtain

$$x = -\cos 3t - \frac{\sqrt{3}}{3}\sin 3t = \frac{2}{\sqrt{3}}\sin\left(3t + \frac{4\pi}{3}\right)$$

and $x' = 2\sqrt{3}\cos(3t + 4\pi/3)$. If $x' = 3$ then $t = -7\pi/18 + 2n\pi/3$ and $t = -\pi/2 + 2n\pi/3$ for $n = 1, 2, 3, \ldots$ .

**13.** The springs are parallel as in Figure 3.8.5 so the effective spring constant is $k_{\text{eff}} = k_1 + k_2$. From Hooke's Law we find that a mass weighing 20 lb determines the spring constants $k_1 = 40$ and $k_2 = 120$. Then we have

$$\begin{aligned} k_{\text{eff}} &= k_1 + k_2 = 40 + 120 = 160 \text{ lb/ft} \\ \frac{20}{32}x'' &+ 160x = 0 \\ x'' &+ 256x = 0 \\ x(t) &= c_1\cos 16t + c_2\sin 16t \end{aligned}$$

Using the initial condition $x(0) = 0$ we have $c_1 = 0$ and therefore $x(t) = c_2\sin 16t$. Then $x'(t) = 16c_2\cos 16t$. Using the condition $x'(0) = 2$ we have $c_2 = 1/8$. Thus

$$x(t) = \frac{1}{8}\sin 16t$$

**14.** Hooke's Law applied to the same force $W$ (weight) for the two springs gives

$$\begin{aligned} W &= k_1\frac{1}{3} = k_2\frac{1}{2} \\ 2k_1 &= 3k_2 \\ k_1 &= \frac{3}{2}k_2 \end{aligned}$$

When the mass weighing 8 lb or 1/4 slug is put on the parallel-spring system its period is given by $2\pi/\omega = \pi/15$ or $\omega = 30$. Now

$$\begin{aligned} \omega^2 &= \frac{k_{\text{eff}}}{m} \\ k_{\text{eff}} &= m\omega^2 = \frac{8}{32}\cdot 30^2 = 225 \\ k_1 + k_2 &= 225 \\ \frac{5}{2}k_2 &= 225 \\ k_2 &= \frac{2}{5}\cdot 225 = 90 \\ k_1 &= \frac{3}{2}\cdot 90 = 135 \end{aligned}$$

Therefore $W = mg = \frac{1}{2}k_2 = 45$ lb. Alternatively, $W = mg = \frac{1}{3}k_1 = 45$ lb.

**15.** Using $k_1 = 40$ and $k_2 = 120$ we have

$$\begin{aligned} k_{\text{eff}} &= \frac{40 \cdot 120}{40 + 120} = \frac{40 \cdot 120}{160} = 30 \text{ lb/ft} \\ \frac{20}{32}x'' + 30x &= 0 \\ x'' + 48x &= 0 \\ x(t) &= c_1 \cos 4\sqrt{3}\,t + c_2 \sin 4\sqrt{3}\,t \end{aligned}$$

Using the initial condition $x(0) = 0$ we have $c_1 = 0$ and therefore $x(t) = c_2 \sin 4\sqrt{3}\,t$. Then $x'(t) = 4\sqrt{3}\,c_2 \cos 4\sqrt{3}\,t$. Using the condition $x'(0) = 2$ we have $c_2 = \sqrt{3}\,/6$. Thus

$$x(t) = \frac{\sqrt{3}}{6} \sin 4\sqrt{3}\,t$$

**16.** As is Problem 14,

$$\begin{aligned} W &= k_1\frac{1}{3} = k_2\frac{1}{2} \\ 2k_1 &= 3k_2 \\ k_1 &= \frac{3}{2}k_2 \end{aligned}$$

When the mass weighing 8 lbs or 1/4 slug is put on the series-spring system its period is given as $2\pi/\omega = \pi/15$ or $\omega = 30$. Now

$$\begin{aligned} \omega^2 &= k_{\text{eff}}/m \\ k_{\text{eff}} &= m\omega^2 = \frac{8}{32} \cdot 30^2 = 255 \\ \frac{k_1 k_2}{k_1 + k_2} &= 225 \\ \frac{\frac{3}{2}k_2^2}{\frac{3}{2}k_2 + k_2} &= \frac{3/2}{5/2}k_2 = 225 \\ k_2 &= \frac{5}{3} \cdot 255 = 375 \\ k_1 &= \frac{3}{2} \cdot 365 = 562.5 \end{aligned}$$

Therefore $W = mg = \frac{1}{2}k_2 = 187.5$ lb. Alternatively, $W = mg = \frac{1}{3}k_1 = 187.5$ lb.

**17.** For parallel springs the effective spring constant is $k_{\text{eff}} = k_1 + k_2$. When $k = k_1 = k_2$, the effective spring constant is $k_{\text{eff}} = 2k$. Compared to a single spring with constant $k$, the parallel-spring system is more stiff.

**18.** For springs attached in series the effective spring constant is

$$k_{\text{eff}} = \frac{k_1 k_2}{k_1 + k_2}\,.$$

When $k = k_1 = k_2$, the effective spring constant is

$$k_{\text{eff}} = \frac{k^2}{k+k} = \frac{1}{2}k$$

Compared to a spring with spring constant $k$, the series-spring system is less stiff.

**19.** For large values of $t$ the differential equation is approximated by $x'' = 0$. The solution of this equation is the linear function $x = c_1 t + c_2$. Thus, for large time, the restoring force will have decayed to the point where the spring is incapable of returning the mass, and the spring will simply keep on stretching.

**20.** As $t$ becomes larger the spring constant increases; that is, the spring is stiffening. It would seem that the oscillations would become periodic and the spring would oscillate more rapidly. It is likely that the amplitudes of the oscillations would decrease as $t$ increases.

**21.** **(a)** above **(b)** heading upward

**22.** **(a)** below **(b)** from rest

**23.** **(a)** below **(b)** heading upward

**24.** **(a)** above **(b)** heading downward

**25.** From $\frac{1}{8}x'' + x' + 2x = 0$, $x(0) = -1$, and $x'(0) = 8$ we obtain $x = 4te^{-4t} - e^{-4t}$ and $x' = 8e^{-4t} - 16te^{-4t}$. If $x = 0$ then $t = 1/4$ second. If $x' = 0$ then $t = 1/2$ second and the extreme displacement is $x = e^{-2}$ feet.

**26.** From $\frac{1}{4}x'' + \sqrt{2}\,x' + 2x = 0$, $x(0) = 0$, and $x'(0) = 5$ we obtain $x = 5te^{-2\sqrt{2}\,t}$ and $x' = 5e^{-2\sqrt{2}\,t}\left(1 - 2\sqrt{2}\,t\right)$. If $x' = 0$ then $t = \sqrt{2}/4$ second and the extreme displacement is $x = 5\sqrt{2}\,e^{-1}/4$ feet.

**27.** **(a)** From $x'' + 10x' + 16x = 0$, $x(0) = 1$, and $x'(0) = 0$ we obtain $x = \frac{4}{3}e^{-2t} - \frac{1}{3}e^{-8t}$.

**(b)** From $x'' + 10x' + 16x = 0$, $x(0) = 1$, and $x'(0) = -12$ then $x = -\frac{2}{3}e^{-2t} + \frac{5}{3}e^{-8t}$.

**28.** **(a)** $x = \frac{1}{3}e^{-8t}\left(4e^{6t} - 1\right)$ is not zero for $t \ge 0$; the extreme displacement is $x(0) = 1$ meter.

**(b)** $x = \frac{1}{3}e^{-8t}\left(5 - 2e^{6t}\right) = 0$ when $t = \frac{1}{6}\ln\frac{5}{2} \approx 0.153$ second; if $x' = \frac{4}{3}e^{-8t}\left(e^{6t} - 10\right) = 0$ then $t = \frac{1}{6}\ln 10 \approx 0.384$ second and the extreme displacement is $x = -0.232$ meter.

**29.** **(a)** From $0.1x'' + 0.4x' + 2x = 0$, $x(0) = -1$, and $x'(0) = 0$ we obtain $x = e^{-2t}\left(-\cos 4t - \frac{1}{2}\sin 4t\right)$.

**(b)** $x = \dfrac{\sqrt{5}}{2}e^{-2t}\sin(4t + 4.25)$

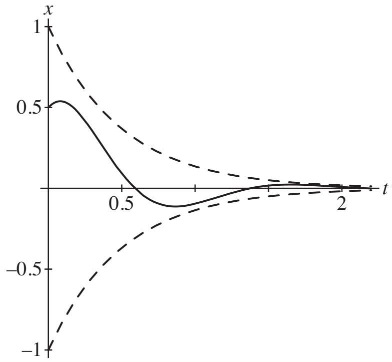
**(c)** If $x = 0$ (and $t > 0$) then $4t + 4.25 = 2\pi,\ 3\pi,\ 4\pi,\ \ldots$ so that the first time heading upward is $t = 1.294$ seconds.

**30.** **(a)** From $\frac{1}{4}x'' + x' + 5x = 0$, $x(0) = 1/2$, and $x'(0) = 1$ we obtain $x = e^{-2t}\left(\frac{1}{2}\cos 4t + \frac{1}{2}\sin 4t\right)$.

**(b)** $x = \dfrac{1}{\sqrt{2}}e^{-2t}\sin\left(4t + \dfrac{\pi}{4}\right)$.

**(c)** If $x = 0$ then $4t + \pi/4 = \pi,\ 2\pi,\ 3\pi,\ \ldots$ so that the times heading downward are $t = (7 + 8n)\pi/16$ for $n = 0,\ 1,\ 2,\ \ldots$ .

**(d)**



**31.** From $\frac{5}{16}x'' + \beta x' + 5x = 0$ we find that the roots of the auxiliary equation are $m = -\frac{8}{5}\beta \pm \frac{4}{5}\sqrt{4\beta^2 - 25}$.

**(a)** If $4\beta^2 - 25 > 0$ then $\beta > 5/2$.

**(b)** If $4\beta^2 - 25 = 0$ then $\beta = 5/2$.

**(c)** If $4\beta^2 - 25 < 0$ then $0 < \beta < 5/2$.

**32.** From $0.75x'' + \beta x' + 6x = 0$ and $\beta > 3\sqrt{2}$ we find that the roots of the auxiliary equation are $m = -\frac{2}{3}\beta \pm \frac{2}{3}\sqrt{\beta^2 - 18}$ and

$$x = e^{-2\beta t/3}\left(c_1\cosh\left(\frac{2}{3}\sqrt{\beta^2 - 18}\,t\right) + c_2\sinh\left(\frac{2}{3}\sqrt{\beta^2 - 18}\,t\right)\right).$$

If $x(0) = 0$ and $x'(0) = -2$ then $c_1 = 0$ and $c_2 = -3/\sqrt{\beta^2 - 18}$.

**33.** If $\frac{1}{2}x'' + \frac{1}{2}x' + 6x = 10\cos 3t$, $x(0) = -2$, and $x'(0) = 0$ then

$$x_c = e^{-t/2}\left(c_1 \cos\left(\frac{\sqrt{47}}{2}t\right) + c_2 \sin\left(\frac{\sqrt{47}}{2}t\right)\right)$$

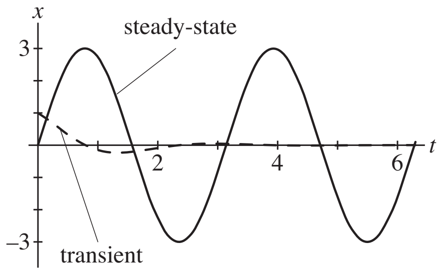
and $x_p = \frac{10}{3}(\cos 3t + \sin 3t)$ so that the equation of motion is

$$x = e^{-t/2}\left(-\frac{4}{3}\cos\left(\frac{\sqrt{47}}{2}t\right) - \frac{64}{3\sqrt{47}}\sin\left(\frac{\sqrt{47}}{2}t\right)\right) + \frac{10}{3}(\cos 3t + \sin 3t).$$

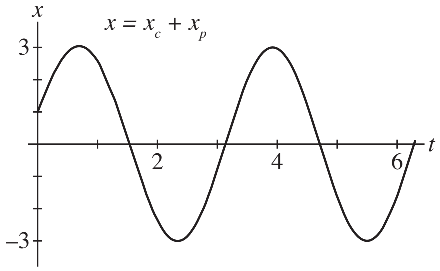
**34.** **(a)** If $x'' + 2x' + 5x = 12\cos 2t + 3\sin 2t$, $x(0) = 1$, and $x'(0) = 5$ then $x_c = e^{-t}(c_1 \cos 2t + c_2 \sin 2t)$ and $x_p = 3\sin 2t$ so that the equation of motion is

$$x = e^{-t}\cos 2t + 3\sin 2t.$$

**(b)**



**(c)**



**35.** From $x'' + 8x' + 16x = 8\sin 4t$, $x(0) = 0$, and $x'(0) = 0$ we obtain $x_c = c_1 e^{-4t} + c_2 te^{-4t}$ and $x_p = -\frac{1}{4}\cos 4t$ so that the equation of motion is

$$x = \frac{1}{4}e^{-4t} + te^{-4t} - \frac{1}{4}\cos 4t.$$

**36.** From $x'' + 8x' + 16x = e^{-t}\sin 4t$, $x(0) = 0$, and $x'(0) = 0$ we obtain $x_c = c_1 e^{-4t} + c_2 te^{-4t}$ and $x_p = -\frac{24}{625}e^{-t}\cos 4t - \frac{7}{625}e^{-t}\sin 4t$ so that

$$x = \frac{1}{625}e^{-4t}(24 + 100t) - \frac{1}{625}e^{-t}(24\cos 4t + 7\sin 4t).$$

As $t \to \infty$ the displacement $x \to 0$.

**37.** From $2x'' + 32x = 68e^{-2t}\cos 4t$, $x(0) = 0$, and $x'(0) = 0$ we obtain $x_c = c_1 \cos 4t + c_2 \sin 4t$ and $x_p = \frac{1}{2}e^{-2t}\cos 4t - 2e^{-2t}\sin 4t$ so that

$$x = -\frac{1}{2}\cos 4t + \frac{9}{4}\sin 4t + \frac{1}{2}e^{-2t}\cos 4t - 2e^{-2t}\sin 4t.$$

**38.** Since $x = \frac{\sqrt{85}}{4}\sin(4t - 0.219) - \frac{\sqrt{17}}{2}e^{-2t}\sin(4t + 2.897)$, the amplitude approaches $\sqrt{85}/4$ as $t \to \infty$.

**39.** **(a)** By Hooke's law the external force is $F(t) = kh(t)$ so that $mx'' + \beta x' + kx = kh(t)$.

**(b)** From $\frac{1}{2}x'' + 2x' + 4x = 20\cos t$, $x(0) = 0$, and $x'(0) = 0$ we obtain $x_c = e^{-2t}(c_1\cos 2t + c_2\sin 2t)$ and $x_p = \frac{56}{13}\cos t + \frac{32}{13}\sin t$ so that

$$x = e^{-2t}\left(-\frac{56}{13}\cos 2t - \frac{72}{13}\sin 2t\right) + \frac{56}{13}\cos t + \frac{32}{13}\sin t.$$

**40.** **(a)** From $100x'' + 1600x = 1600\sin 8t$, $x(0) = 0$, and $x'(0) = 0$ we obtain $x_c = c_1\cos 4t + c_2\sin 4t$ and $x_p = -\frac{1}{3}\sin 8t$ so that by a trigonometric identity

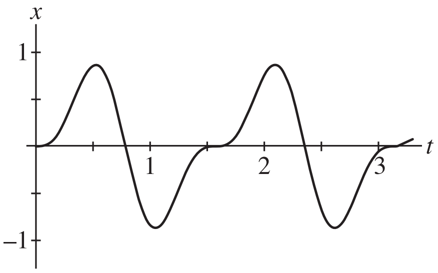
$$x = \frac{2}{3}\sin 4t - \frac{1}{3}\sin 8t = \frac{2}{3}\sin 4t - \frac{2}{3}\sin 4t\cos 4t.$$

**(b)** If $x = \frac{1}{3}\sin 4t(2 - 2\cos 4t) = 0$ then $t = n\pi/4$ for $n = 0, 1, 2, \ldots$ .

**(c)** If $x' = \frac{8}{3}\cos 4t - \frac{8}{3}\cos 8t = \frac{8}{3}(1 - \cos 4t)(1 + 2\cos 4t) = 0$ then $t = \pi/3 + n\pi/2$ and $t = \pi/6 + n\pi/2$ for $n = 0, 1, 2, \ldots$ at the extreme values. *Note*: There are other values of $t$ for which $x' = 0$. However these other values are multiples of $\pi/2$ and the displacement is zero from part (b).

**(d)** $x(\pi/6 + n\pi/2) = \sqrt{3}/2$ cm and $x(\pi/3 + n\pi/2) = -\sqrt{3}/2$ cm.

**(e)**



**41.** From $x''+4x = -5\sin 2t+3\cos 2t$, $x(0) = -1$, and $x'(0) = 1$ we obtain $x_c = c_1\cos 2t+c_2\sin 2t$, $x_p = \frac{3}{4}t\sin 2t + \frac{5}{4}t\cos 2t$, and

$$x = -\cos 2t - \frac{1}{8}\sin 2t + \frac{3}{4}t\sin 2t + \frac{5}{4}t\cos 2t.$$

**42.** From $x'' + 9x = 5\sin 3t$, $x(0) = 2$, and $x'(0) = 0$ we obtain $x_c = c_1\cos 3t + c_2\sin 3t$, $x_p = -\frac{5}{6}t\cos 3t$, and

$$x = 2\cos 3t + \frac{5}{18}\sin 3t - \frac{5}{6}t\cos 3t.$$

**43.** **(a)** From $x'' + \omega^2 x = F_0\cos\gamma t$, $x(0) = 0$, and $x'(0) = 0$ we obtain $x_c = c_1\cos\omega t + c_2\sin\omega t$ and $x_p = (F_0\cos\gamma t)/\left(\omega^2 - \gamma^2\right)$ so that

$$x = -\frac{F_0}{\omega^2 - \gamma^2}\cos\omega t + \frac{F_0}{\omega^2 - \gamma^2}\cos\gamma t.$$

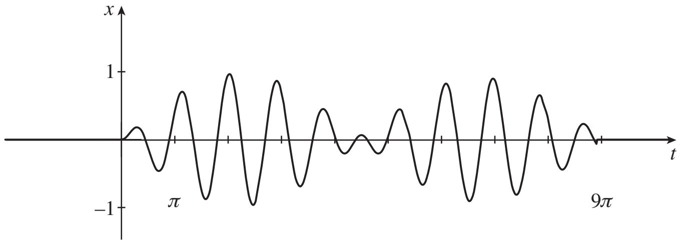
**(b)** $\displaystyle\lim_{\gamma\to\omega}\frac{F_0}{\omega^2-\gamma^2}(\cos\gamma t-\cos\omega t)=\lim_{\gamma\to\omega}\frac{-F_0t\sin\gamma t}{-2\gamma}=\frac{F_0}{2\omega}t\sin\omega t.$

**44.** From $x''+\omega^2x=F_0\cos\omega t$, $x(0)=0$, and $x'(0)=0$ we obtain $x_c=c_1\cos\omega t+c_2\sin\omega t$ and $x_p=(F_0t/2\omega)\sin\omega t$ so that $x=(F_0t/2\omega)\sin\omega t$.

**45. (a)** From $\cos(u-v)=\cos u\cos v+\sin u\sin v$ and $\cos(u+v)=\cos u\cos v-\sin u\sin v$ we obtain $\sin u\sin v=\frac{1}{2}[\cos(u-v)-\cos(u+v)]$. Letting $u=\frac{1}{2}(\gamma-\omega)t$ and $v=\frac{1}{2}(\gamma+\omega)t$, the result follows.

**(b)** If $\epsilon=\frac{1}{2}(\gamma-\omega)$ then $\gamma\approx\omega$ so that $x=(F_0/2\epsilon\gamma)\sin\epsilon t\sin\gamma t$.

**46.** See the article "Distinguished Oscillations of a Forced Harmonic Oscillator" by T.G. Procter in *The College Mathematics Journal*, March, 1995. In this article the author illustrates that for $F_0=1$, $\lambda=0.01$, $\gamma=22/9$, and $\omega=2$ the system exhibits beats oscillations on the interval $[0,9\pi]$, but that this phenomenon is transient as $t\to\infty$.



**47. (a)** The general solution of the homogeneous equation is

$$\begin{aligned}x_c(t)&=c_1e^{-\lambda t}\cos\left(\sqrt{\omega^2-\lambda^2}\,t\right)+c_2e^{-\lambda t}\sin\left(\sqrt{\omega^2-\lambda^2}\,t\right)\\&=Ae^{-\lambda t}\sin\left[\sqrt{\omega^2-\lambda^2}\,t+\phi\right],\end{aligned}$$

where $A=\sqrt{c_1^2+c_2^2}$, $\sin\phi=c_1/A$, and $\cos\phi=c_2/A$. Now

$$x_p(t)=\frac{F_0(\omega^2-\gamma^2)}{(\omega^2-\gamma^2)^2+4\lambda^2\gamma^2}\sin\gamma t+\frac{F_0(-2\lambda\gamma)}{(\omega^2-\gamma^2)^2+4\lambda^2\gamma^2}\cos\gamma t=A\sin(\gamma t+\theta),$$

where

$$\sin\theta=\frac{\dfrac{F_0(-2\lambda\gamma)}{(\omega^2-\gamma^2)^2+4\lambda^2\gamma^2}}{\dfrac{F_0}{\sqrt{\omega^2-\gamma^2+4\lambda^2\gamma^2}}}=\frac{-2\lambda\gamma}{\sqrt{(\omega^2-\gamma^2)^2+4\lambda^2\gamma^2}}$$

and

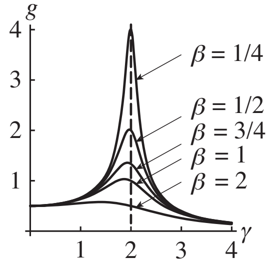
$$\cos\theta=\frac{\dfrac{F_0(\omega^2-\gamma^2)}{(\omega^2-\gamma^2)^2+4\lambda^2\gamma^2}}{\dfrac{F_0}{\sqrt{(\omega^2-\gamma^2)^2+4\lambda^2\gamma^2}}}=\frac{\omega^2-\gamma^2}{\sqrt{(\omega^2-\gamma^2)^2+4\lambda^2\gamma^2}}.$$

**(b)** If $g'(\gamma) = 0$ then $\gamma\left(\gamma^2 + 2\lambda^2 - \omega^2\right) = 0$ so that $\gamma = 0$ or $\gamma = \sqrt{\omega^2 - 2\lambda^2}$. The first derivative test shows that $g$ has a maximum value at $\gamma = \sqrt{\omega^2 - 2\lambda^2}$. The maximum value of $g$ is

$$g\left(\sqrt{\omega^2 - 2\lambda^2}\right) = F_0/2\lambda\sqrt{\omega^2 - \lambda^2}.$$

**(c)** We identify $\omega^2 = k/m = 4$, $\lambda = \beta/2$, and $\gamma_1 = \sqrt{\omega^2 - 2\lambda^2} = \sqrt{4 - \beta^2/2}$. As $\beta \to 0$, $\gamma_1 \to 2$ and the resonance curve grows without bound at $\gamma_1 = 2$. That is, the system approaches pure resonance.

| $\beta$ | $\gamma 1$ | g |
|---|---|---|
| 2.00 | 1.41 | 0.58 |
| 1.00 | 1.87 | 1.03 |
| 0.75 | 1.93 | 1.36 |
| 0.50 | 1.97 | 2.02 |
| 0.25 | 1.99 | 4.01 |



**48. (a)** For $n = 2$, $\sin^2\gamma t = \frac{1}{2}(1 - \cos 2\gamma t)$. The system is in pure resonance when $2\gamma_1/2\pi = \omega/2\pi$, or when $\gamma_1 = \omega/2$.

**(b)** Note that

$$\sin^3\gamma t = \sin\gamma t\sin^2\gamma t = \frac{1}{2}\left[\sin\gamma t - \sin\gamma t\cos 2\gamma t\right].$$

Now

$$\sin(A + B) + \sin(A - B) = 2\sin A\cos B$$

so

$$\sin\gamma t\cos 2\gamma t = \frac{1}{2}\left[\sin 3\gamma t - \sin\gamma t\right]$$

and

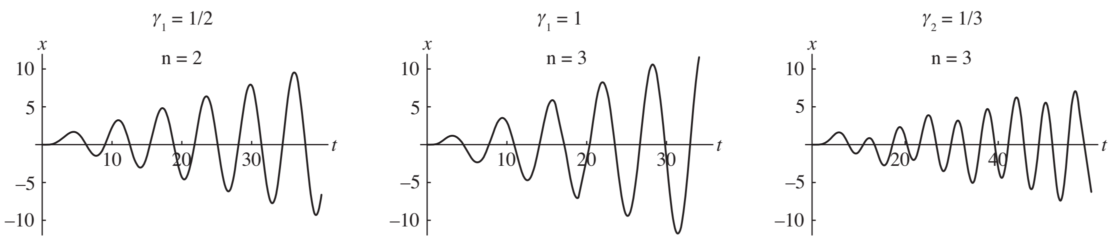
$$\sin^3\gamma t = \frac{3}{4}\sin\gamma t - \frac{1}{4}\sin 3\gamma t.$$

Thus

$$x'' + \omega^2 x = \frac{3}{4}\sin\gamma t - \frac{1}{4}\sin 3\gamma t.$$

The frequency of free vibration is $\omega/2\pi$. Thus, when $\gamma_1/2\pi = \omega/2\pi$ or $\gamma_1 = \omega$, and when $3\gamma_2/2\pi = \omega/2\pi$ or $3\gamma_2 = \omega$ or $\gamma_2 = \omega/3$, the system will be in pure resonance.

**(c)**



**49.** Solving $\frac{1}{20}q'' + 2q' + 100q = 0$ we obtain $q(t) = e^{-20t}(c_1 \cos 40t + c_2 \sin 40t)$. The initial conditions $q(0) = 5$ and $q'(0) = 0$ imply $c_1 = 5$ and $c_2 = 5/2$. Thus

$$q(t) = e^{-20t}\left(5\cos 40t + \frac{5}{2}\sin 40t\right) = \frac{5\sqrt{5}}{2}\, e^{-20t}\sin\left(40t + 1.1071\right)$$

and $q(0.01) \approx 4.5676$ coulombs. The charge is zero for the first time when $40t + 1.1071 = \pi$ or $t \approx 0.0509$ second.

**50.** Solving $\frac{1}{4}q'' + 20q' + 300q = 0$ we obtain $q(t) = c_1 e^{-20t} + c_2 e^{-60t}$. The initial conditions $q(0) = 4$ and $q'(0) = 0$ imply $c_1 = 6$ and $c_2 = -2$. Thus

$$q(t) = 6e^{-20t} - 2e^{-60t}.$$

Setting $q = 0$ we find $e^{40t} = 1/3$ which implies $t < 0$. Therefore the charge is not 0 for $t \geq 0$.

**51.** Solving $\frac{5}{3}q'' + 10q' + 30q = 300$ we obtain $q(t) = e^{-3t}(c_1 \cos 3t + c_2 \sin 3t) + 10$. The initial conditions $q(0) = q'(0) = 0$ imply $c_1 = c_2 = -10$. Thus

$$q(t) = 10 - 10e^{-3t}(\cos 3t + \sin 3t) \quad \text{and} \quad i(t) = 60e^{-3t}\sin 3t.$$

Solving $i(t) = 0$ we see that the maximum charge occurs when $t = \pi/3$ and $q(\pi/3) \approx 10.432$.

**52.** Solving $q'' + 100q' + 2500q = 30$ we obtain $q(t) = c_1 e^{-50t} + c_2 te^{-50t} + 0.012$. The initial conditions $q(0) = 0$ and $q'(0) = 2$ imply $c_1 = -0.012$ and $c_2 = 1.4$. Thus, using $i(t) = q'(t)$ we get

$$q(t) = -0.012e^{-50t} + 1.4te^{-50t} + 0.012 \quad \text{and} \quad i(t) = 2e^{-50t} - 70te^{-50t}.$$

Solving $i(t) = 0$ we see that the maximum charge occurs when $t = 1/35$ second and $q(1/35) \approx 0.01871$ coulomb.

**53.** Solving $q'' + 2q' + 4q = 0$ we obtain $q_c = e^{-t}\left(\cos\sqrt{3}\,t + \sin\sqrt{3}\,t\right)$. The steady-state charge has the form $q_p = A\cos t + B\sin t$. Substituting into the differential equation we find

$$(3A + 2B)\cos t + (3B - 2A)\sin t = 50\cos t.$$

Thus, $A = 150/13$ and $B = 100/13$. The steady-state charge is

$$q_p(t) = \frac{150}{13}\cos t + \frac{100}{13}\sin t$$

and the steady-state current is

$$i_p(t) = -\frac{150}{13}\sin t + \frac{100}{13}\cos t.$$

**54.** From

$$i_p(t) = \frac{E_0}{Z}\left(\frac{R}{Z}\sin\gamma t - \frac{X}{Z}\cos\gamma t\right)$$

and $Z = \sqrt{X^2 + R^2}$ we see that the amplitude of $i_p(t)$ is

$$A = \sqrt{\frac{E_0^2R^2}{Z^4} + \frac{E_0^2X^2}{Z^4}} = \frac{E_0}{Z^2}\sqrt{R^2 + X^2} = \frac{E_0}{Z}.$$

**55.** The differential equation is $\frac{1}{2}q'' + 20q' + 1000q = 100\sin 60t$. To use Example 10 in the text we identify $E_0 = 100$ and $\gamma = 60$. Then

$$X = L\gamma - \frac{1}{c\gamma} = \frac{1}{2}(60) - \frac{1}{0.001(60)} \approx 13.3333,$$

$$Z = \sqrt{X^2 + R^2} = \sqrt{X^2 + 400} \approx 24.0370,$$

and

$$\frac{E_0}{Z} = \frac{100}{Z} \approx 4.1603.$$

From Problem 54, then

$$i_p(t) \approx 4.1603\sin(60t + \phi)$$

where $\sin\phi = -X/Z$ and $\cos\phi = R/Z$. Thus $\tan\phi = -X/R \approx -0.6667$ and $\phi$ is a fourth quadrant angle. Now $\phi \approx -0.5880$ and

$$i_p(t) = 4.1603\sin(60t - 0.5880).$$

**56.** Solving $\frac{1}{2}q'' + 20q' + 1000q = 0$ we obtain $q_c(t) = e^{-20t}(c_1\cos 40t + c_2\sin 40t)$. The steady-state charge has the form $q_p(t) = A\sin 60t + B\cos 60t + C\sin 40t + D\cos 40t$. Substituting into the differential equation we find

$$\begin{aligned}(-1600A - 2400B)\sin 60t &+ (2400A - 1600B)\cos 60t \\ &+ (400C - 1600D)\sin 40t + (1600C + 400D)\cos 40t \\ &= 200\sin 60t + 400\cos 40t.\end{aligned}$$

Equating coefficients we obtain $A = -1/26$, $B = -3/52$, $C = 4/17$, and $D = 1/17$. The steady-state charge is

$$q_p(t) = -\frac{1}{26}\sin 60t - \frac{3}{52}\cos 60t + \frac{4}{17}\sin 40t + \frac{1}{17}\cos 40t$$

and the steady-state current is

$$i_p(t) = -\frac{30}{13}\cos 60t + \frac{45}{13}\sin 60t + \frac{160}{17}\cos 40t - \frac{40}{17}\sin 40t.$$

**57.** Solving $\frac{1}{2}q'' + 10q' + 100q = 150$ we obtain $q(t) = e^{-10t}(c_1 \cos 10t + c_2 \sin 10t) + 3/2$. The initial conditions $q(0) = 1$ and $q'(0) = 0$ imply $c_1 = c_2 = -1/2$. Thus

$$q(t) = -\frac{1}{2}e^{-10t}(\cos 10t + \sin 10t) + \frac{3}{2}.$$

As $t \to \infty$, $q(t) \to 3/2$.

**58.** In Problem 54 it is shown that the amplitude of the steady-state current is $E_0/Z$, where $Z = \sqrt{X^2 + R^2}$ and $X = L\gamma - 1/C\gamma$. Since $E_0$ is constant the amplitude will be a maximum when $Z$ is a minimum. Since $R$ is constant, $Z$ will be a minimum when $X = 0$. Solving $L\gamma - 1/C\gamma = 0$ for $\gamma$ we obtain $\gamma = 1/\sqrt{LC}$. The maximum amplitude will be $E_0/R$.

**59.** By Problem 54 the amplitude of the steady-state current is $E_0/Z$, where $Z = \sqrt{X^2 + R^2}$ and $X = L\gamma - 1/C\gamma$. Since $E_0$ is constant the amplitude will be a maximum when $Z$ is a minimum. Since $R$ is constant, $Z$ will be a minimum when $X = 0$. Solving $L\gamma - 1/C\gamma = 0$ for $C$ we obtain $C = 1/L\gamma^2$.

**60.** Solving $0.1q'' + 10q = 100 \sin \gamma t$ we obtain

$$q(t) = c_1 \cos 10t + c_2 \sin 10t + q_p(t)$$

where $q_p(t) = A \sin \gamma t + B \cos \gamma t$. Substituting $q_p(t)$ into the differential equation we find

$$(100 - \gamma^2)A \sin \gamma t + (100 - \gamma^2)B \cos \gamma t = 100 \sin \gamma t.$$

Equating coefficients we obtain $A = 100/(100 - \gamma^2)$ and $B = 0$. Thus, $q_p(t) = \dfrac{100}{100 - \gamma^2} \sin \gamma t$. The initial conditions $q(0) = q'(0) = 0$ imply $c_1 = 0$ and $c_2 = -10\gamma/(100 - \gamma^2)$. The charge is

$$q(t) = \frac{10}{100 - \gamma^2}(10 \sin \gamma t - \gamma \sin 10t)$$

and the current is

$$i(t) = \frac{100\gamma}{100 - \gamma^2}(\cos \gamma t - \cos 10t).$$

**61.** In an $LC$-series circuit there is no resistor, so the differential equation is

$$L\frac{d^2q}{dt^2} + \frac{1}{C}q = E(t).$$

Then $q(t) = c_1 \cos\left(t/\sqrt{LC}\right) + c_2 \sin\left(t/\sqrt{LC}\right) + q_p(t)$ where $q_p(t) = A \sin \gamma t + B \cos \gamma t$. Substituting $q_p(t)$ into the differential equation we find

$$\left(\frac{1}{C} - L\gamma^2\right) A \sin \gamma t + \left(\frac{1}{C} - L\gamma^2\right) B \cos \gamma t = E_0 \cos \gamma t.$$

Equating coefficients we obtain $A = 0$ and $B = E_0C/(1 - LC\gamma^2)$. Thus, the charge is

$$q(t) = c_1 \cos \frac{1}{\sqrt{LC}} t + c_2 \sin \frac{1}{\sqrt{LC}} t + \frac{E_0C}{1 - LC\gamma^2} \cos \gamma t.$$

The initial conditions $q(0) = q_0$ and $q'(0) = i_0$ imply $c_1 = q_0 - E_0C/(1 - LC\gamma^2)$ and $c_2 = i_0\sqrt{LC}$. The current is $i(t) = q'(t)$ or

$$\begin{aligned} i(t) &= -\frac{c_1}{\sqrt{LC}}\sin\frac{1}{\sqrt{LC}}t + \frac{c_2}{\sqrt{LC}}\cos\frac{1}{\sqrt{LC}}t - \frac{E_0C\gamma}{1 - LC\gamma^2}\sin\gamma t \\ &= i_0\cos\frac{1}{\sqrt{LC}}t - \frac{1}{\sqrt{LC}}\left(q_0 - \frac{E_0C}{1 - LC\gamma^2}\right)\sin\frac{1}{\sqrt{LC}}t - \frac{E_0C\gamma}{1 - LC\gamma^2}\sin\gamma t. \end{aligned}$$

**62.** When the circuit is in resonance the form of $q_p(t)$ is $q_p(t) = At\cos kt + Bt\sin kt$ where $k = 1/\sqrt{LC}$. Substituting $q_p(t)$ into the differential equation we find

$$q_p'' + k^2q_p = -2kA\sin kt + 2kB\cos kt = \frac{E_0}{L}\cos kt.$$

Equating coefficients we obtain $A = 0$ and $B = E_0/2kL$. The charge is

$$q(t) = c_1\cos kt + c_2\sin kt + \frac{E_0}{2kL}t\sin kt.$$

The initial conditions $q(0) = q_0$ and $q'(0) = i_0$ imply $c_1 = q_0$ and $c_2 = i_0/k$. The current is

$$\begin{aligned} i(t) &= -c_1k\sin kt + c_2k\cos kt + \frac{E_0}{2kL}(kt\cos kt + \sin kt) \\ &= \left(\frac{E_0}{2kL} - q_0k\right)\sin kt + i_0\cos kt + \frac{E_0}{2L}t\cos kt. \end{aligned}$$

## 5.2 Linear Models: Boundary-Value Problems

**1. (a)** The general solution is

$$y(x) = c_1 + c_2x + c_3x^2 + c_4x^3 + \frac{w_0}{24EI}x^4.$$

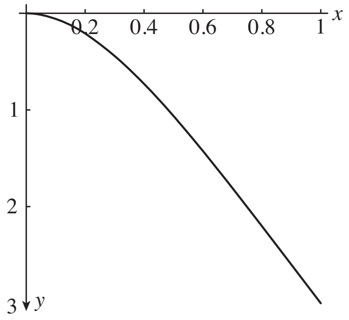
The boundary conditions are $y(0) = 0$, $y'(0) = 0$, $y''(L) = 0$, $y'''(L) = 0$. The first two conditions give $c_1 = 0$ and $c_2 = 0$. The conditions at $x = L$ give the system

$$2c_3 + 6c_4L + \frac{w_0}{2EI}L^2 = 0 6c_4 + \frac{w_0}{EI}L \qquad = 0.$$

Solving, we obtain $c_3 = w_0L^2/4EI$ and $c_4 = -w_0L/6EI$. The deflection is

$$y(x) = \frac{w_0}{24EI}(6L^2x^2 - 4Lx^3 + x^4).$$

**(b)**



**2. (a)** The general solution is

$$y(x) = c_1 + c_2x + c_3x^2 + c_4x^3 + \frac{w_0}{24EI}x^4.$$

The boundary conditions are $y(0) = 0$, $y''(0) = 0$, $y(L) = 0$, $y''(L) = 0$. The first two conditions give $c_1 = 0$ and $c_3 = 0$. The conditions at $x = L$ give the system

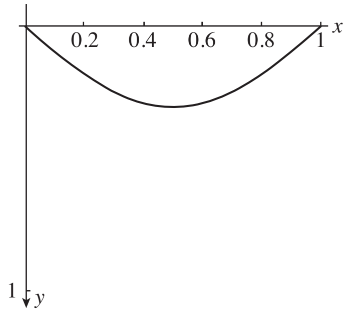
$$c_2L + c_4L^3 + \frac{w_0}{24EI}L^4 = 0$$

$$6c_4L + \frac{w_0}{2EI}L^2 = 0.$$

Solving, we obtain $c_2 = w_0L^3/24EI$ and $c_4 = -w_0L/12EI$. The deflection is

$$y(x) = \frac{w_0}{24EI}(L^3x - 2Lx^3 + x^4).$$

**(b)**



**3. (a)** The general solution is

$$y(x) = c_1 + c_2x + c_3x^2 + c_4x^3 + \frac{w_0}{24EI}x^4.$$

The boundary conditions are $y(0) = 0$, $y'(0) = 0$, $y(L) = 0$, $y''(L) = 0$. The first two conditions give $c_1 = 0$ and $c_2 = 0$. The conditions at $x = L$ give the system

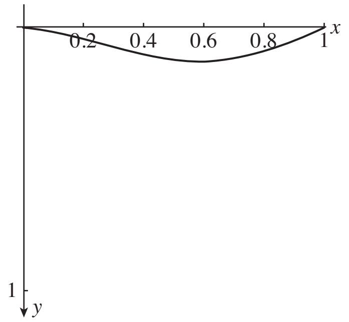
$$c_3L^2 + c_4L^3 + \frac{w_0}{24EI}L^4 = 0$$

$$2c_3 + 6c_4L + \frac{w_0}{2EI}L^2 = 0.$$

Solving, we obtain $c_3 = w_0L^2/16EI$ and $c_4 = -5w_0L/48EI$. The deflection is

$$y(x) = \frac{w_0}{48EI}(3L^2x^2 - 5Lx^3 + 2x^4).$$

**(b)**



**4. (a)** The general solution is

$$y(x) = c_1 + c_2x + c_3x^2 + c_4x^3 + \frac{w_0L^4}{EI\pi^4}\sin\frac{\pi}{L}x.$$

The boundary conditions are $y(0) = 0$, $y'(0) = 0$, $y(L) = 0$, $y''(L) = 0$. The first two conditions give $c_1 = 0$ and $c_2 = -w_0L^3/EI\pi^3$. The conditions at $x = L$ give the system

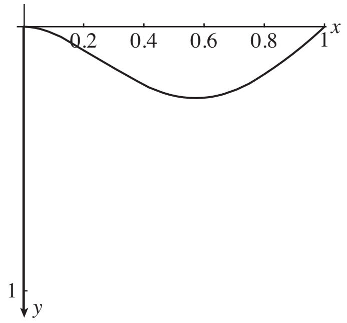
$$c_3L^2 + c_4L^3 + \frac{w_0}{EI\pi^3}L^4 = 0$$

$$2c_3 + 6c_4L = 0.$$

Solving, we obtain $c_3 = 3w_0L^2/2EI\pi^3$ and $c_4 = -w_0L/2EI\pi^3$. The deflection is

$$y(x) = \frac{w_0L}{2EI\pi^3}\left(-2L^2x + 3Lx^2 - x^3 + \frac{2L^3}{\pi}\sin\frac{\pi}{L}x\right).$$

**(b)**



**(c)** Using a CAS we find the maximum deflection to be 0.270806 when $x = 0.572536$.

**5. (a)** The general solution is

$$y(x) = c_1 + c_2x + c_3x^2 + c_4x^3 + \frac{w_0}{120EI}x^5.$$

The boundary conditions are $y(0) = 0$, $y''(0) = 0$, $y(L) = 0$, $y''(L) = 0$. The first two conditions give $c_1 = 0$ and $c_3 = 0$. The conditions at $x = L$ give the system

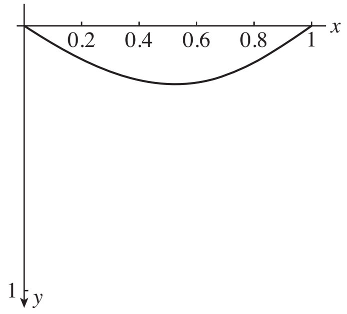
$$c_2L + c_4L^3 + \frac{w_0}{120EI}L^5 = 0$$

$$6c_4L + \frac{w_0}{6EI}L^3 = 0.$$

Solving, we obtain $c_2 = 7w_0L^4/360EI$ and $c_4 = -w_0L^2/36EI$. The deflection is

$$y(x) = \frac{w_0}{360EI}(7L^4x - 10L^2x^3 + 3x^5).$$

**(b)**



**(c)** Using a CAS we find the maximum deflection to be 0.234799 when $x = 0.51933$.

**6.** **(a)** $y_{\max} = y(L) = w_0L^4/8EI$

**(b)** Replacing both $L$ and $x$ by $L/2$ in $y(x)$ we obtain $w_0L^4/128EI$, which is 1/16 of the maximum deflection when the length of the beam is $L$.

**(c)** $y_{\max} = y(L/2) = 5w_0L^4/384EI$

**(d)** The maximum deflection in Example 1 is $y(L/2) = (w_0/24EI)L^4/16 = w_0L^4/384EI$, which is 1/5 of the maximum displacement of the beam in part (c).

**7.** The general solution of the differential equation is

$$y = c_1 \cosh\sqrt{\frac{P}{EI}}\,x + c_2 \sinh\sqrt{\frac{P}{EI}}\,x + \frac{w_0}{2P}\,x^2 + \frac{w_0EI}{P^2}\,.$$

Setting $y(0) = 0$ we obtain $c_1 = -w_0EI/P^2$, so that

$$y = -\frac{w_0EI}{P^2}\cosh\sqrt{\frac{P}{EI}}\,x + c_2 \sinh\sqrt{\frac{P}{EI}}\,x + \frac{w_0}{2P}\,x^2 + \frac{w_0EI}{P^2}\,.$$

Setting $y'(L) = 0$ we find

$$c_2 = \left(\sqrt{\frac{P}{EI}}\,\frac{w_0EI}{P^2}\sinh\sqrt{\frac{P}{EI}}\,L - \frac{w_0L}{P}\right) \Bigg/ \sqrt{\frac{P}{EI}}\cosh\sqrt{\frac{P}{EI}}\,L.$$

**8.** The general solution of the differential equation is

$$y = c_1 \cos\sqrt{\frac{P}{EI}}\,x + c_2 \sin\sqrt{\frac{P}{EI}}\,x - \frac{w_0}{2P}\,x^2 + \frac{w_0EI}{P^2}\,.$$

Setting $y(0) = 0$ we obtain $c_1 = -w_0EI/P^2$, so that

$$y = -\frac{w_0EI}{P^2}\cos\sqrt{\frac{P}{EI}}\,x + c_2 \sin\sqrt{\frac{P}{EI}}\,x - \frac{w_0}{2P}\,x^2 + \frac{w_0EI}{P^2}\,.$$

Setting $y'(L) = 0$ we find

$$c_2 = \left(-\sqrt{\frac{P}{EI}}\,\frac{w_0EI}{P^2}\sin\sqrt{\frac{P}{EI}}\,L + \frac{w_0L}{P}\right) \Bigg/ \sqrt{\frac{P}{EI}}\cos\sqrt{\frac{P}{EI}}\,L.$$

**9.** This is Example 2 in the text with $L = \pi$. The eigenvalues are $\lambda_n = n^2\pi^2/\pi^2 = n^2$, $n = 1, 2, 3, \ldots$ and the corresponding eigenfunctions are $y_n = \sin(n\pi x/\pi) = \sin nx$, $n = 1, 2, 3, \ldots$.

**10.** This is Example 2 in the text with $L = \pi/4$. The eigenvalues are $\lambda_n = n^2\pi^2/(\pi/4)^2 = 16n^2$, $n = 1, 2, 3, \ldots$ and the eigenfunctions are $y_n = \sin\left(n\pi x/(\pi/4)\right) = \sin 4nx$, $n = 1, 2, 3, \ldots$ .

**11.** For $\lambda \le 0$ the only solution of the boundary-value problem is $y = 0$. For $\lambda = \alpha^2 > 0$ we have

$$y = c_1 \cos \alpha x + c_2 \sin \alpha x.$$

Now

$$y'(x) = -c_1 \alpha \sin \alpha x + c_2 \alpha \cos \alpha x$$

and $y'(0) = 0$ implies $c_2 = 0$, so

$$y(L) = c_1 \cos \alpha L = 0$$

gives

$$\alpha L = \frac{(2n-1)\pi}{2} \quad \text{or} \quad \lambda = \alpha^2 = \frac{(2n-1)^2\pi^2}{4L^2}, \ n = 1, 2, 3, \dots .$$

The eigenvalues $(2n-1)^2\pi^2/4L^2$ correspond to the eigenfunctions $\cos \dfrac{(2n-1)\pi}{2L} x$ for $n = 1, 2, 3, \dots$ .

**12.** For $\lambda \le 0$ the only solution of the boundary-value problem is $y = 0$. For $\lambda = \alpha^2 > 0$ we have

$$y = c_1 \cos \alpha x + c_2 \sin \alpha x.$$

Since $y(0) = 0$ implies $c_1 = 0$, $y = c_2 \sin x \, dx$. Now

$$y'\left(\frac{\pi}{2}\right) = c_2 \alpha \cos \alpha \frac{\pi}{2} = 0$$

gives

$$\alpha \frac{\pi}{2} = \frac{(2n-1)\pi}{2} \quad \text{or} \quad \lambda = \alpha^2 = (2n-1)^2, \ n = 1, 2, 3, \dots .$$

The eigenvalues $\lambda_n = (2n-1)^2$ correspond to the eigenfunctions $y_n = \sin(2n-1)x$.

**13.** For $\lambda = -\alpha^2 < 0$ the only solution of the boundary-value problem is $y = 0$. For $\lambda = 0$ we have $y = c_1 x + c_2$. Now $y' = c_1$ and $y'(0) = 0$ implies $c_1 = 0$. Then $y = c_2$ and $y'(\pi) = 0$. Thus, $\lambda = 0$ is an eigenvalue with corresponding eigenfunction $y = 1$. For $\lambda = \alpha^2 > 0$ we have

$$y = c_1 \cos \alpha x + c_2 \sin \alpha x.$$

Now

$$y'(x) = -c_1 \alpha \sin \alpha x + c_2 \alpha \cos \alpha x$$

and $y'(0) = 0$ implies $c_2 = 0$, so

$$y'(\pi) = -c_1 \alpha \sin \alpha \pi = 0$$

gives

$$\alpha \pi = n\pi \quad \text{or} \quad \lambda = \alpha^2 = n^2, \ n = 1, 2, 3, \dots .$$

The eigenvalues $n^2$ correspond to the eigenfunctions $\cos nx$ for $n = 0, 1, 2, \dots$.

**14.** For $\lambda \le 0$ the only solution of the boundary-value problem is $y = 0$. For $\lambda = \alpha^2 > 0$ we have

$$y = c_1 \cos \alpha x + c_2 \sin \alpha x.$$

Now $y(-\pi) = y(\pi) = 0$ implies

$$\begin{aligned} c_1 \cos \alpha\pi - c_2 \sin \alpha\pi &= 0 \\ c_1 \cos \alpha\pi + c_2 \sin \alpha\pi &= 0. \end{aligned} \tag{1}$$

This homogeneous system will have a nontrivial solution when

$$\begin{vmatrix} \cos \alpha\pi & -\sin \alpha\pi \\ \cos \alpha\pi & \sin \alpha\pi \end{vmatrix} = 2 \sin \alpha\pi \cos \alpha\pi = \sin 2\alpha\pi = 0.$$

Then

$$2\alpha\pi = n\pi \qquad \text{or} \qquad \lambda = \alpha^2 = \frac{n^2}{4}; \quad n = 1,\ 2,\ 3,\ \ldots\,.$$

When $n = 2k - 1$ is odd, the eigenvalues are $(2k-1)^2/4$. Since $\cos(2k-1)\pi/2 = 0$ and $\sin(2k-1)\pi/2 \ne 0$, we see from either equation in ((1)) that $c_2 = 0$. Thus, the eigenfunctions corresponding to the eigenvalues $(2k-1)^2/4$ are $y = \cos(2k-1)x/2$ for $k = 1, 2, 3, \ldots$. Similarly, when $n = 2k$ is even, the eigenvalues are $k^2$ with corresponding eigenfunctions $y = \sin kx$ for $k = 1, 2, 3, \ldots$.

**15.** The auxiliary equation has solutions

$$m = \frac{1}{2}\left(-2 \pm \sqrt{4 - 4(\lambda + 1)}\right) = -1 \pm \alpha.$$

For $\lambda = -\alpha^2 < 0$ we have

$$y = e^{-x}\left(c_1 \cosh \alpha x + c_2 \sinh \alpha x\right).$$

The boundary conditions imply

$$y(0) = c_1 = 0y(5) \qquad\qquad = c_2 e^{-5} \sinh 5\alpha = 0$$

so $c_1 = c_2 = 0$ and the only solution of the boundary-value problem is $y = 0$.

For $\lambda = 0$ we have

$$y = c_1 e^{-x} + c_2 x e^{-x}$$

and the only solution of the boundary-value problem is $y = 0$.

For $\lambda = \alpha^2 > 0$ we have

$$y = e^{-x}\left(c_1 \cos \alpha x + c_2 \sin \alpha x\right).$$

Now $y(0) = 0$ implies $c_1 = 0$, so

$$y(5) = c_2 e^{-5} \sin 5\alpha = 0$$

gives

$$5\alpha = n\pi \quad \text{or} \quad \lambda = \alpha^2 = \frac{n^2\pi^2}{25}\,,\ n = 1, 2, 3, \ldots\ .$$

The eigenvalues $\lambda_n = \dfrac{n^2\pi^2}{25}$ correspond to the eigenfunctions $y_n = e^{-x}\sin\dfrac{n\pi}{5}x$ for $n = 1, 2, 3, \ldots$ .

**16.** For $\lambda < -1$ the only solution of the boundary-value problem is $y = 0$. For $\lambda = -1$ we have $y = c_1x + c_2$. Now $y' = c_1$ and $y'(0) = 0$ implies $c_1 = 0$. Then $y = c_2$ and $y'(1) = 0$. Thus, $\lambda = -1$ is an eigenvalue with corresponding eigenfunction $y = 1$. For $\lambda > -1$ or $\lambda + 1 = \alpha^2 > 0$ we have

$$y = c_1\cos\alpha x + c_2\sin\alpha x.$$

Now

$$y' = -c_1\alpha\sin\alpha x + c_2\alpha\cos\alpha x$$

and $y'(0) = 0$ implies $c_2 = 0$, so

$$y'(1) = -c_1\alpha\sin\alpha = 0$$

gives

$$\alpha = n\pi, \quad \lambda + 1 = \alpha^2 = n^2\pi^2, \quad \text{or} \quad \lambda_n = n^2\pi^2 - 1,\ n = 1, 2, 3, \ldots\ .$$

The eigenvalues $n^2\pi^2 - 1$ correspond to the eigenfunctions $y_n = \cos n\pi x$ for $n = 0,\ 1,\ 2,\ \ldots$ .

**17.** For $\lambda = \alpha^2 > 0$ a general solution of the given differential equation is

$$y = c_1\cos(\alpha\ln x) + c_2\sin(\alpha\ln x).$$

Since $\ln 1 = 0$, the boundary condition $y(1) = 0$ implies $c_1 = 0$. Therefore

$$y = c_2\sin(\alpha\ln x).$$

Using $\ln e^{\pi} = \pi$ we find that $y(e^{\pi}) = 0$ implies

$$c_2\sin\alpha\pi = 0$$

or $\alpha\pi = n\pi$, $n = 1, 2, 3, \ldots$ . The eigenvalues and eigenfunctions are, in turn,

$$\lambda_n = \alpha^2 = n^2, \quad n = 1,\ 2,\ 3,\ \ldots \quad \text{and} \quad y_n = \sin(n\ln x).$$

For $\lambda \le 0$ the only solution of the boundary-value problem is $y = 0$.

**18.** For $\lambda = 0$ the general solution is $y = c_1 + c_2\ln x$. Now $y' = c_2/x$, so $y'(e^{-1}) = c_2e = 0$ implies $c_2 = 0$. Then $y = c_1$ and $y(1) = 0$ gives $c_1 = 0$. Thus $y(x) = 0$.

For $\lambda = -\alpha^2 < 0$, $y = c_1x^{-\alpha} + c_2x^{\alpha}$. The boundary conditions give $c_2 = c_1e^{2\alpha}$ and $c_1 = 0$, so that $c_2 = 0$ and $y(x) = 0$.

For $\lambda = \alpha^2 > 0$, $y = c_1 \cos(\alpha \ln x) + c_2 \sin(\alpha \ln x)$. From $y(1) = 0$ we obtain $c_1 = 0$ and $y = c_2 \sin(\alpha \ln x)$. Now $y' = c_2(\alpha/x)\cos(\alpha \ln x)$, so $y'(e^{-1}) = c_2 e\alpha \cos\alpha = 0$ implies $\cos\alpha = 0$ or $\alpha = (2n-1)\pi/2$ and $\lambda_n = \alpha^2 = (2n-1)^2\pi^2/4$ for $n = 1, 2, 3, \ldots$. The corresponding eigenfunctions are

$$y_n = \sin\left(\frac{2n-1}{2}\pi \ln x\right).$$

**19.** For $\lambda = 0$ the general solution is $y = c_1 + c_2 \ln x$. Now $y' = c_2/x$, so $y'(1) = c_2 = 0$ implies $c_2 = 0$. Then $y = c_1$ and $y'(e^2) = 0$ is satisfied for any $c_1$.

For $\lambda = -\alpha^2 < 0$, $y = c_1 x^{-\alpha} + c_2 x^{\alpha}$. The boundary conditions give $c_1 = c_2 e^{4\alpha}$ and $c_1 = 0$, so that $c_2 = 0$ and $y(x) = 0$.

For $\lambda = \alpha^2 > 0$, $y = c_1 \cos(\alpha \ln x) + c_2 \sin(\alpha \ln x)$. From $y'(1) = 0$ we obtain $c_2 = 0$ and $y = c_1 \cos(\alpha \ln x)$. Now $y' = -c_1(\alpha/x)\sin(\alpha \ln x)$, so $y'(e^2) = -c_1(\alpha/e^2)\sin 2\alpha = 0$ implies $\sin 2\alpha = 0$ or $\alpha = n\pi/2$ and $\lambda_n = \alpha^2 = n^2\pi^2/4$ for $n = 0, 1, 2, \ldots$. The corresponding eigenfunctions are

$$y_n = \cos\left(\frac{n\pi}{2} \ln x\right).$$

Note $n = 0$ in $y_n$ gives $y = 1$.

**20.** For $\lambda = 0$ the general solution is $y = c_1 + c_2 \ln x$. Now $y' = c_2/x$, so $y'(e) = c_2/e = 0$ implies $c_2 = 0$. Then $y = c_1$ and $y(1) = 0$ gives $c_1 = 0$. Thus $y(x) = 0$.

For $\lambda = -\alpha^2 < 0$, $y = c_1 x^{-\alpha} + c_2 x^{\alpha}$. The boundary conditions give $c_1 = c_2 e^{2\alpha}$ and $c_2 = 0$, so that $c_1 = 0$ and $y(x) = 0$.

For $\lambda = \alpha^2 > 0$, $y = c_1 \cos(\alpha \ln x) + c_2 \sin(\alpha \ln x)$. From $y(1) = 0$ we obtain $c_1 = 0$ and $y = c_2 \sin(\alpha \ln x)$. Now $y' = c_2(\alpha/x)\cos(\alpha \ln x)$, so $y'(e) = c_2(\alpha/e)\cos\alpha = 0$ implies $\cos\alpha = 0$ or $\alpha = (2n-1)\pi/2$ and $\lambda_n = \alpha^2 = (2n-1)^2\pi^2/4$ for $n = 1, 2, 3, \ldots$. The corresponding eigenfunctions are

$$y_n = \sin\left(\frac{2n-1}{2}\pi \ln x\right).$$

**21.** For $\lambda = \alpha^4$, $\alpha > 0$, the general solution of the boundary-value problem

$$y^{(4)} - \lambda y = 0, \quad y(0) = 0,\ y''(0) = 0,\ y(1) = 0,\ y''(1) = 0$$

is

$$y = c_1 \cos\alpha x + c_2 \sin\alpha x + c_3 \cosh\alpha x + c_4 \sinh\alpha x.$$

The boundary conditions $y(0) = 0$, $y''(0) = 0$ give $c_1 + c_3 = 0$ and $-c_1\alpha^2 + c_3\alpha^2 = 0$, from which we conclude $c_1 = c_3 = 0$. Thus, $y = c_2 \sin\alpha x + c_4 \sinh\alpha x$. The boundary conditions $y(1) = 0$, $y''(1) = 0$ then give

$$\begin{aligned} c_2 \sin\alpha + c_4 \sinh\alpha &= 0 \\ -c_2\alpha^2 \sin\alpha + c_4\alpha^2 \sinh\alpha &= 0. \end{aligned}$$

In order to have nonzero solutions of this system, we must have the determinant of the coefficients equal zero, that is,

$$\begin{vmatrix} \sin\alpha & \sinh\alpha \\ -\alpha^2\sin\alpha & \alpha^2\sinh\alpha \end{vmatrix} = 0 \quad \text{or} \quad 2\alpha^2\sinh\alpha\sin\alpha = 0$$

But since $\alpha > 0$, the only way that this is satisfied is to have $\sin\alpha = 0$ or $\alpha = n\pi$. The system is then satisfied by choosing $c_2 \neq 0$, $c_4 = 0$, and $\alpha = n\pi$. The eigenvalues and corresponding eigenfunctions are then

$$\lambda_n = \alpha^4 = (n\pi)^4,\ n = 1, 2, 3, \ldots \qquad \text{and} \qquad y_n = \sin n\pi x.$$

**22.** For $\lambda = \alpha^4$, $\alpha > 0$, the general solution of the differential equation is

$$y = c_1\cos\alpha x + c_2\sin\alpha x + c_3\cosh\alpha x + c_4\sinh\alpha x.$$

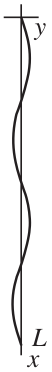
The boundary conditions $y'(0) = 0$, $y'''(0) = 0$ give $c_2\alpha + c_4\alpha = 0$ and $-c_2\alpha^3 + c_4\alpha^3 = 0$ from which we conclude $c_2 = c_4 = 0$. Thus, $y = c_1\cos\alpha x + c_3\cosh\alpha x$. The boundary conditions $y(\pi) = 0$, $y''(\pi) = 0$ then give

$$\begin{aligned} c_1\cos\alpha\pi + c_3\cosh\alpha\pi &= 0 \\ -c_1\alpha^2\cos\alpha\pi + c_3\alpha^2\cosh\alpha\pi &= 0. \end{aligned}$$

The determinant of the coefficients is $2\alpha^2\cosh\alpha\pi\cos\alpha\pi = 0$. But since $\alpha > 0$, the only way that this is satisfied is to have $c_1 \neq 0$, $c_3 = 0$, and $\cos\alpha\pi = 0$ or $\alpha = (2n-1)/2$, $n = 1, 2, 3, \ldots$. The eigenvalues and corresponding eigenfunctions are

$$\lambda_n = \alpha^4 = \left(\frac{2n-1}{2}\right)^4,\ n = 1, 2, 3, \ldots \qquad \text{and} \qquad y_n = \cos\left(\frac{2n-1}{2}\right)x.$$

**23.** If restraints are put on the column at $x = L/4$, $x = L/2$, and $x = 3L/4$, then the critical load will be $P_4$.



**24.** **(a)** The general solution of the differential equation is

$$y = c_1\cos\sqrt{\frac{P}{EI}}\,x + c_2\sin\sqrt{\frac{P}{EI}}\,x + \delta.$$

Since the column is embedded at $x = 0$, the boundary conditions are $y(0) = y'(0) = 0$. If $\delta = 0$ this implies that $c_1 = c_2 = 0$ and $y(x) = 0$. That is, there is no deflection.

**(b)** If $\delta \neq 0$, the boundary conditions give, in turn, $c_1 = -\delta$ and $c_2 = 0$. Then

$$y = \delta\left(1 - \cos\sqrt{\frac{P}{EI}}\,x\right).$$

In order to satisfy the boundary condition $y(L) = \delta$ we must have

$$\delta = \delta\left(1 - \cos\sqrt{\frac{P}{EI}}\,L\right) \quad \text{or} \quad \cos\sqrt{\frac{P}{EI}}\,L = 0.$$

This gives $\sqrt{P/EI}\,L = (2n-1)\pi/2$ for $n = 1, 2, 3, \ldots$. The smallest value of $P_n$, the Euler load, is then

$$\sqrt{\frac{P_1}{EI}}\,L = \frac{\pi}{2} \quad \text{or} \quad P_1 = \frac{1}{4}\left(\frac{\pi^2 EI}{L^2}\right).$$

**25.** If $\lambda = \alpha^2 = P/EI$, then the solution of the differential equation is

$$y = c_1\cos\alpha x + c_2\sin\alpha x + c_3 x + c_4.$$

The conditions $y(0) = 0$, $y''(0) = 0$ yield, in turn, $c_1 + c_4 = 0$ and $c_1 = 0$. With $c_1 = 0$ and $c_4 = 0$ the solution is $y = c_2\sin\alpha x + c_3 x$. The conditions $y(L) = 0$, $y''(L) = 0$, then yield

$$c_2\sin\alpha L + c_3 L = 0 \qquad \text{and} \qquad -c_2\alpha^2\sin\alpha L = 0.$$

Hence, nontrivial solutions of the problem exist only if $\sin\alpha L = 0$. From this point on, the analysis is the same as in Example 3 in the text.

**26.** **(a)** The boundary-value problem is

$$\frac{d^4y}{dx^4} + \lambda\frac{d^2y}{dx^2} = 0, \quad y(0) = 0, y''(0) = 0,\ y(L) = 0, y'(L) = 0,$$

where $\lambda = \alpha^2 = P/EI$. The solution of the differential equation is $y = c_1\cos\alpha x + c_2\sin\alpha x + c_3 x + c_4$ and the conditions $y(0) = 0$, $y''(0) = 0$ yield $c_1 = 0$ and $c_4 = 0$. Next, by applying $y(L) = 0$, $y'(L) = 0$ to $y = c_2\sin\alpha x + c_3 x$ we get the system of equations

$$\begin{aligned} c_2\sin\alpha L + c_3 L &= 0 \\ \alpha c_2\cos\alpha L + c_3 \quad &= 0. \end{aligned}$$

To obtain nontrivial solutions $c_2$, $c_3$, we must have the determinant of the coefficients equal to zero:

$$\begin{vmatrix} \sin\alpha L & L \\ \alpha\cos\alpha L & 1 \end{vmatrix} = 0 \quad \text{or} \quad \tan\beta = \beta$$

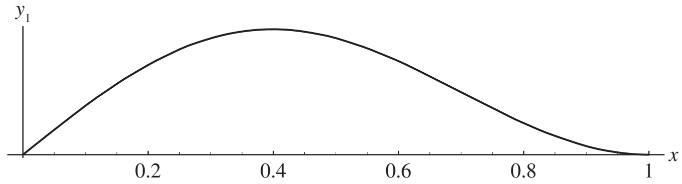
where $\beta = \alpha L$. If $\beta_n$ denotes the positive roots of the last equation, then the eigenvalues are found from $\beta_n = \alpha_n L = \sqrt{\lambda_n}\,L$ or $\lambda_n = (\beta_n/L)^2$. From $\lambda = P/EI$ we see that the critical loads are $P_n = \beta_n^2 EI/L^2$. With the aid of a CAS we find that the first positive root of $\tan\beta = \beta$ is (approximately) $\beta_1 = 4.4934$, and so the Euler load is (approximately) $P_1 = 20.1907EI/L^2$. Finally, if we use $c_3 = -c_2\alpha\cos\alpha L$, then the deflection curves are

$$y_n(x) = c_2\sin\alpha_n x + c_3 x = c_2\left[\sin\left(\frac{\beta_n}{L}x\right) - \left(\frac{\beta_n}{L}\cos\beta_n\right)x\right].$$

**(b)** With $L = 1$ and $c_2$ appropriately chosen, the general shape of the first buckling mode,

$$y_1(x) = c_2\left[\sin\left(\frac{4.4934}{L}x\right) - \left(\frac{4.4934}{L}\cos(4.4934)\right)x\right],$$

is shown below.



**27.** The general solution is

$$y = c_1\cos\sqrt{\frac{\rho}{T}}\,\omega x + c_2\sin\sqrt{\frac{\rho}{T}}\,\omega x.$$

From $y(0) = 0$ we obtain $c_1 = 0$. Setting $y(L) = 0$ we find $\sqrt{\rho/T}\,\omega L = n\pi$, $n = 1, 2, 3, \dots$ . Thus, critical speeds are $\omega_n = n\pi\sqrt{T}/L\sqrt{\rho}$, $n = 1, 2, 3, \dots$ . The corresponding deflection curves are

$$y(x) = c_2\sin\frac{n\pi}{L}x, \quad n = 1, 2, 3, \dots,$$

where $c_2 \neq 0$.

**28. (a)** When $T(x) = x^2$ the given differential equation is the Cauchy-Euler equation

$$x^2y'' + 2xy' + \rho\omega^2 y = 0.$$

The solutions of the auxiliary equation

$$m(m-1) + 2m + \rho\omega^2 = m^2 + m + \rho\omega^2 = 0$$

are

$$m_1 = -\frac{1}{2} - \frac{1}{2}\sqrt{4\rho\omega^2 - 1}\,i, \quad m_2 = -\frac{1}{2} + \frac{1}{2}\sqrt{4\rho\omega^2 - 1}\,i$$

when $\rho\omega^2 > 0.25$. Thus

$$y = c_1x^{-1/2}\cos(\lambda\ln x) + c_2x^{-1/2}\sin(\lambda\ln x)$$

where $\lambda = \frac{1}{2}\sqrt{4\rho\omega^2 - 1}$. Applying $y(1) = 0$ gives $c_1 = 0$ and consequently

$$y = c_2x^{-1/2}\sin(\lambda\ln x).$$

The condition $y(e) = 0$ requires $c_2 e^{-1/2} \sin \lambda = 0$. We obtain a nontrivial solution when $\lambda_n = n\pi$, $n = 1, 2, 3, \ldots$. But

$$\lambda_n = \frac{1}{2}\sqrt{4\rho\omega_n^2 - 1} = n\pi.$$

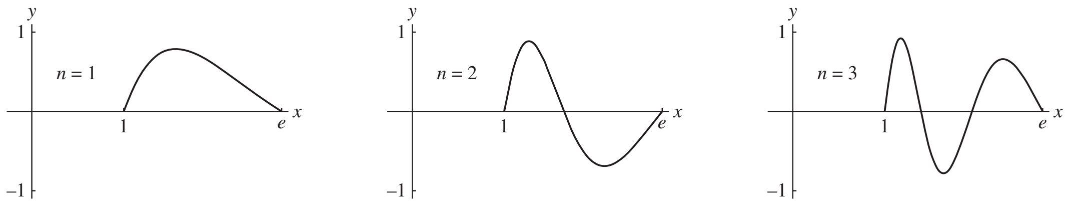
Solving for $\omega_n$ gives

$$\omega_n = \frac{1}{2}\sqrt{(4n^2\pi^2 + 1)/\rho}\,.$$

The corresponding solutions are

$$y_n(x) = c_2 x^{-1/2} \sin(n\pi \ln x).$$

**(b)**



**29.** The auxiliary equation is $m^2 + m = m(m+1) = 0$ so that $u(r) = c_1 r^{-1} + c_2$. The boundary conditions $u(a) = u_0$ and $u(b) = u_1$ yield the system $c_1 a^{-1} + c_2 = u_0$, $c_1 b^{-1} + c_2 = u_1$. Solving gives

$$c_1 = \left(\frac{u_0 - u_1}{b - a}\right) ab \quad \text{and} \quad c_2 = \frac{u_1 b - u_0 a}{b - a}\,.$$

Thus

$$u(r) = \left(\frac{u_0 - u_1}{b - a}\right)\frac{ab}{r} + \frac{u_1 b - u_0 a}{b - a}\,.$$

**30.** The auxiliary equation is $m^2 = 0$ so that $u(r) = c_1 + c_2 \ln r$. The boundary conditions $u(a) = u_0$ and $u(b) = u_1$ yield the system $c_1 + c_2 \ln a = u_0$, $c_1 + c_2 \ln b = u_1$. Solving gives

$$c_1 = \frac{u_1 \ln a - u_0 \ln b}{\ln(a/b)} \quad \text{and} \quad c_2 = \frac{u_0 - u_1}{\ln(a/b)}.$$

Thus

$$u(r) = \frac{u_1 \ln a - u_0 \ln b}{\ln(a/b)} + \frac{u_0 - u_1}{\ln(a/b)} \ln r = \frac{u_0 \ln(r/b) - u_1 \ln(r/a)}{\ln(a/b)}\,.$$

**31. (a)** This is similar to Problem 21 with 1 replaced by $L$. Thus the eigenvalues and eigenfunctions are, in turn,

$$\lambda_n = \alpha_n^4 = \frac{n^4\pi^4}{L^4}, \quad n = 1, 2, 3, \ldots \qquad \text{and} \qquad y_n(x) = \sin\left(\frac{n\pi x}{L}\right)$$

**(b)**

$$\lambda_n = \alpha_n^4 = \rho\omega_n^2/EI = n^4\pi^4/L^4$$

$$\omega_n^2 = \frac{n^4\pi^4}{L^4}\,\frac{EI}{\rho}\,.$$

Therefore the critical speeds are

$$\omega_n = \frac{n^2\pi^2}{L^2}\sqrt{\frac{EI}{\rho}}, \quad n = 1, 2, 3, \dots .$$

Then the fundamental critical speed is

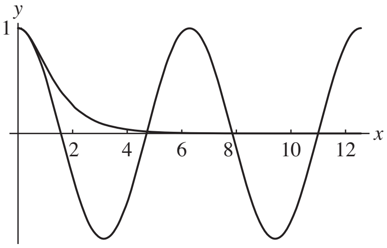
$$\omega_1 = \frac{\pi^2}{L^2}\sqrt{\frac{EI}{\rho}}.$$

Noe that $\omega_n = n^2\omega_1$.

**32.** **(a)** For $\lambda = \alpha^4$, $\alpha > 0$, the solution of the differential equation is

$$y = c_1\cos\alpha x + c_2\sin\alpha x + c_3\cosh\alpha x + c_4\sinh\alpha x.$$

The boundary conditions $y(0) = 0$, $y'(0) = 0$, $y(1) = 0$, $y'(1) = 0$ give, in turn,



$$\begin{aligned} c_1 + c_3 &= 0 \\ \alpha c_2 + \alpha c_4 &= 0, \end{aligned}$$

$$\begin{aligned} c_1\cos\alpha + c_2\sin\alpha + c_3\cosh\alpha + c_4\sinh\alpha &= 0 \\ -c_1\alpha\sin\alpha + c_2\alpha\cos\alpha + c_3\alpha\sinh\alpha + c_4\alpha\cosh\alpha &= 0. \end{aligned}$$

The first two equations enable us to write

$$\begin{aligned} c_1(\cos\alpha - \cosh\alpha) + c_2(\sin\alpha - \sinh\alpha) &= 0 \\ c_1(-\sin\alpha - \sinh\alpha) + c_2(\cos\alpha - \cosh\alpha) &= 0. \end{aligned}$$

The determinant

$$\begin{vmatrix} \cos\alpha - \cosh\alpha & \sin\alpha - \sinh\alpha \\ -\sin\alpha - \sinh\alpha & \cos\alpha - \cosh\alpha \end{vmatrix} = 0$$

simplifies to $\cos\alpha\cosh\alpha = 1$. From the figure showing the graphs of $1/\cosh x$ and $\cos x$, we see that this equation has an infinite number of positive roots.

**(b)** Using the third equation in the system to eliminate $c_2$, we find the eigenfunctions are

$$y_n = (-\sin\alpha_n + \sinh\alpha_n)(\cos\alpha_n x - \cosh\alpha_n x) + (\cos\alpha_n - \cosh\alpha_n)(\sin\alpha_n x - \sinh\alpha_n x).$$

**33.** The solution of the initial-value problem

$$x'' + \omega^2 x = 0, \quad x(0) = 0,\ x'(0) = v_0,\ \omega^2 = 10/m$$

is $x(t) = (v_0/\omega)\sin\omega t$. To satisfy the additional boundary condition $x(1) = 0$ we require that $\omega = n\pi$, $n = 1, 2, 3, \ldots$. The eigenvalues $\lambda = \omega^2 = n^2\pi^2$ and eigenfunctions of the problem are then $x(t) = (v_0/n\pi)\sin n\pi t$. Using $\omega^2 = 10/m$ we find that the *only* masses that can pass through the equilibrium position at $t = 1$ are $m_n = 10/n^2\pi^2$. Note for $n = 1$, the heaviest mass $m_1 = 10/\pi^2$ will *not* pass through the equilibrium position on the interval $0 < t < 1$ (the period of $x(t) = (v_0/\pi)\sin\pi t$ is $T = 2$, so on $0 \le t \le 1$ its graph passes through $x = 0$ only at $t = 0$ and $t = 1$). Whereas for $n > 1$, masses of lighter weight will pass through the equilibrium position $n - 1$ times prior to passing through at $t = 1$. For example, if $n = 2$, the period of $x(t) = (v_0/2\pi)\sin 2\pi t$ is $2\pi/2\pi = 1$, the mass will pass through $x = 0$ only *once* ($t = \frac{1}{2}$) prior to $t = 1$; if $n = 3$, the period of $x(t) = (v_0/3\pi)\sin 3\pi t$ is $\frac{2}{3}$, the mass will pass through $x = 0$ *twice* ($t = \frac{1}{3}$ and $t = \frac{2}{3}$) prior to $t = 1$; and so on.

**34.** The initial-value problem is

$$x'' + \frac{2}{m}x' + \frac{k}{m}x = 0, \quad x(0) = 0,\ x'(0) = v_0.$$

With $k = 10$, the auxiliary equation has roots $\gamma = -1/m \pm \sqrt{1 - 10m}/m$. Consider the three cases:

(*i*) $m = \frac{1}{10}$.

The roots are $\gamma_1 = \gamma_2 = -10$ and the solution of the differential equation is $x(t) = c_1e^{-10t} + c_2te^{-10t}$. The initial conditions imply $c_1 = 0$ and $c_2 = v_0$ and so $x(t) = v_0te^{-10t}$. The condition $x(1) = 0$ implies $v_0e^{-10} = 0$ which is impossible because $v_0 \ne 0$.

(*ii*) $1 - 10m > 0$ or $0 < m < \frac{1}{10}$.

The roots are

$$\gamma_1 = -\frac{1}{m} - \frac{1}{m}\sqrt{1 - 10m} \qquad \text{and} \qquad \gamma_2 = -\frac{1}{m} + \frac{1}{m}\sqrt{1 - 10m}$$

and the solution of the differential equation is $x(t) = c_1e^{\gamma_1 t} + c_2e^{\gamma_2 t}$. The initial conditions imply

$$\begin{aligned} c_1 + c_2 &= 0 \\ \gamma_1c_1 + \gamma_2c_2 &= v_0 \end{aligned}$$

so $c_1 = v_0/(\gamma_1 - \gamma_2)$, $c_2 = -v_0/(\gamma_1 - \gamma_2)$, and

$$x(t) = \frac{v_0}{\gamma_1 - \gamma_2}(e^{\gamma_1 t} - e^{\gamma_2 t}).$$

Again, $x(1) = 0$ is impossible because $v_0 \ne 0$.

(*iii*) $1 - 10m < 0$ or $m > \frac{1}{10}$.

The roots of the auxiliary equation are

$$\gamma_1 = -\frac{1}{m} - \frac{1}{m}\sqrt{10m - 1}\,i \qquad \text{and} \qquad \gamma_2 = -\frac{1}{m} + \frac{1}{m}\sqrt{10m - 1}\,i$$

and the solution of the differential equation is

$$x(t) = c_1e^{-t/m}\cos\frac{1}{m}\sqrt{10m-1}\,t + c_2e^{-t/m}\sin\frac{1}{m}\sqrt{10m-1}\,t.$$

The initial conditions imply $c_1 = 0$ and $c_2 = mv_0/\sqrt{10m-1}$, so that

$$x(t) = \frac{mv_0}{\sqrt{10m-1}}\,e^{-t/m}\sin\left(\frac{1}{m}\sqrt{10m-1}\,t\right),$$

The condition $x(1) = 0$ implies

$$\begin{aligned}
\frac{mv_0}{\sqrt{10m-1}}e^{-1/m}\sin\frac{1}{m}\sqrt{10m-1} &= 0\\
\sin\frac{1}{m}\sqrt{10m-1} &= 0\\
\frac{1}{m}\sqrt{10m-1} &= n\pi\\
\frac{10m-1}{m^2} &= n^2\pi^2,\ \ n = 1, 2, 3, \ldots\\
(n^2\pi^2)m^2 - 10m + 1 &= 0\\
m = \frac{10\sqrt{100-4n^2\pi^2}}{2n^2\pi^2} &= \frac{5\pm\sqrt{25-n^2\pi^2}}{n^2\pi^2}\,.
\end{aligned}$$

Since $m$ is real, $25 - n^2\pi^2 \geq 0$. If $25 - n^2\pi^2 = 0$, then $n^2 = 25/\pi^2$, and $n$ is not an integer. Thus, $25 - n^2\pi^2 = (5 - n\pi)(5 + n\pi) > 0$ and since $n > 0$, $5 + n\pi > 0$, so $5 - n\pi > 0$ also. Then $n < 5/\pi$, and so $n = 1$. Therefore, the mass $m$ will pass through the equilibrium position when $t = 1$ for

$$m_1 = \frac{5+\sqrt{25-\pi^2}}{\pi^2} \qquad \text{and} \qquad m_2 = \frac{5-\sqrt{25-\pi^2}}{\pi^2}\,.$$

**35.** **(a)** The general solution of the differential equation is $y = c_1\cos 4x + c_2\sin 4x$. From $y_0 = y(0) = c_1$ we see that $y = y_0\cos 4x + c_2\sin 4x$. From $y_1 = y(\pi/2) = y_0$ we see that any solution must satisfy $y_0 = y_1$. We also see that when $y_0 = y_1$, $y = y_0\cos 4x + c_2\sin 4x$ is a solution of the boundary-value problem for any choice of $c_2$. Thus, the boundary-value problem does not have a unique solution for any choice of $y_0$ and $y_1$.

**(b)** Whenever $y_0 = y_1$ there are infinitely many solutions.

**(c)** When $y_0 \neq y_1$ there are no solutions.

**(d)** The boundary-value problem has the trivial solution when $y_0 = y_1 = 0$. The solution is not unique.

**36. (a)** The general solution of the differential equation is $y = c_1 \cos 4x + c_2 \sin 4x$. From $1 = y(0) = c_1$ we see that $y = \cos 4x + c_2 \sin 4x$. From $1 = y(L) = \cos 4L + c_2 \sin 4L$ we see that $c_2 = (1 - \cos 4L)/\sin 4L$. Thus,

$$y = \cos 4x + \left(\frac{1 - \cos 4L}{\sin 4L}\right) \sin 4x$$

will be a unique solution when $\sin 4L \neq 0$; that is, when $L \neq k\pi/4$ where $k = 1, 2, 3, \ldots$.

**(b)** There will be infinitely many solutions when $\sin 4L = 0$ and $1 - \cos 4L = 0$; that is, when $L = k\pi/2$ where $k = 1, 2, 3, \ldots$.

**(c)** There will be no solution when $\sin 4L \neq 0$ and $1 - \cos 4L \neq 0$; that is, when $L = k\pi/4$ where $k = 1, 3, 5, \ldots$.

**(d)** There can be no trivial solution since it would fail to satisfy the boundary conditions.

**37. (a)** A solution curve has the same $y$-coordinate at both ends of the interval $[-\pi, \pi]$ and the tangent lines at the endpoints of the interval are parallel.

**(b)** For $\lambda = 0$ the solution of $y'' = 0$ is $y = c_1 x + c_2$. From the first boundary condition we have

$$y(-\pi) = -c_1\pi + c_2 = y(\pi) = c_1\pi + c_2$$

or $2c_1\pi = 0$. Thus, $c_1 = 0$ and $y = c_2$. This constant solution is seen to satisfy the boundary-value problem.

For $\lambda = -\alpha^2 < 0$ we have $y = c_1 \cosh \alpha x + c_2 \sinh \alpha x$. In this case the first boundary condition gives

$$\begin{aligned} y(-\pi) &= c_1 \cosh(-\alpha\pi) + c_2 \sinh(-\alpha\pi) \\ &= c_1 \cosh \alpha\pi - c_2 \sinh \alpha\pi \\ &= y(\pi) = c_1 \cosh \alpha\pi + c_2 \sinh \alpha\pi \end{aligned}$$

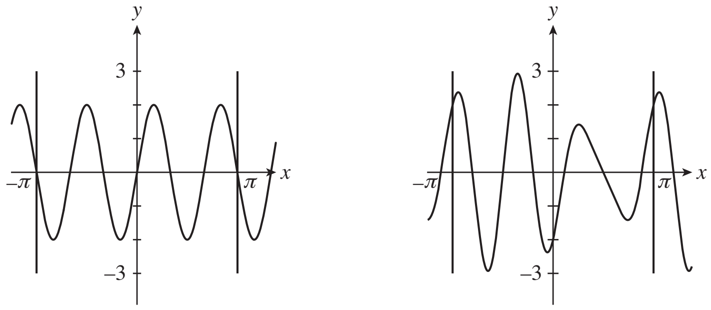
or $2c_2 \sinh \alpha\pi = 0$. Thus $c_2 = 0$ and $y = c_1 \cosh \alpha x$. The second boundary condition implies in a similar fashion that $c_1 = 0$. Thus, for $\lambda < 0$, the only solution of the boundary-value problem is $y = 0$.

For $\lambda = \alpha^2 > 0$ we have $y = c_1 \cos \alpha x + c_2 \sin \alpha x$. The first boundary condition implies

$$\begin{aligned} y(-\pi) &= c_1 \cos(-\alpha\pi) + c_2 \sin(-\alpha\pi) \\ &= c_1 \cos \alpha\pi - c_2 \sin \alpha\pi \\ &= y(\pi) = c_1 \cos \alpha\pi + c_2 \sin \alpha\pi \end{aligned}$$

or $2c_2 \sin \alpha\pi = 0$. Similarly, the second boundary condition implies $2c_1\alpha \sin \alpha\pi = 0$. If $c_1 = c_2 = 0$ the solution is $y = 0$. However, if $c_1 \neq 0$ or $c_2 \neq 0$, then $\sin \alpha\pi = 0$, which implies that $\alpha$ must be an integer, $n$. Therefore, for $c_1$ and $c_2$ not both 0, $y = c_1 \cos nx + c_2 \sin nx$ is a nontrivial solution of the boundary-value problem. Since $\cos(-nx) = \cos nx$ and $\sin(-nx) = -\sin nx$, we may assume without loss of generality that the eigenvalues are $\lambda_n = \alpha^2 = n^2$, for $n$ a positive integer. The corresponding eigenfunctions are $y_n = \cos nx$ and $y_n = \sin nx$.

**(c)**



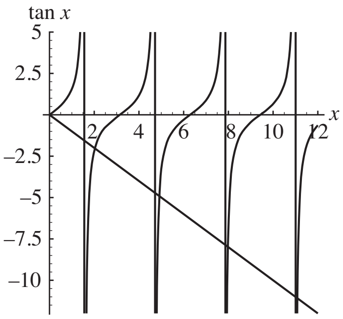
$y = 2\sin 3x$ $\qquad$ $y = \sin 4x - 2\cos 3x$

**38.** For $\lambda = \alpha^2 > 0$ the general solution is $y = c_1 \cos\sqrt{\alpha}\,x + c_2 \sin\sqrt{\alpha}\,x$. Setting $y(0) = 0$ we find $c_1 = 0$, so that $y = c_2 \sin\sqrt{\alpha}\,x$. The boundary condition $y(1) + y'(1) = 0$ implies

$$c_2 \sin\sqrt{\alpha} + c_2\sqrt{\alpha}\cos\sqrt{\alpha} = 0.$$

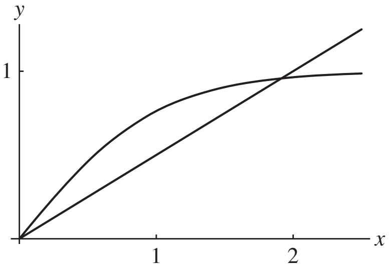
Taking $c_2 \neq 0$, this equation is equivalent to $\tan\sqrt{\alpha} = -\sqrt{\alpha}$. Thus, the eigenvalues are $\lambda_n = \alpha_n^2 = x_n^2$, $n = 1, 2, 3, \dots$ , where the $x_n$ are the consecutive positive roots of $\tan\sqrt{\alpha} = -\sqrt{\alpha}$.

**39.** We see from the graph that $\tan x = -x$ has infinitely many roots. Since $\lambda_n = \alpha_n^2$, there are no new eigenvalues when $\alpha_n < 0$. For $\lambda = 0$, the differential equation $y'' = 0$ has general solution $y = c_1 x + c_2$. The boundary conditions imply $c_1 = c_2 = 0$, so $y = 0$.



**40.** Using a CAS we find that the first four nonnegative roots of $\tan x = -x$ are approximately $2.02876, 4.91318$, $7.97867$, and $11.0855$. The corresponding eigenvalues are $4.11586$, $24.1393$, $63.6591$, and $122.889$, with eigenfunctions $\sin(2.02876x)$, $\sin(4.91318x)$, $\sin(7.97867x)$, and $\sin(11.0855x)$.

**41.** In the case when $\lambda = -\alpha^2 < 0$, the solution of the differential equation is $y = c_1 \cosh\alpha x + c_2 \sinh\alpha x$. The condition $y(0) = 0$ gives $c_1 = 0$. The condition $y(1) - \frac{1}{2}y'(1) = 0$ applied to $y = c_2 \sinh\alpha x$ gives $c_2(\sinh\alpha - \frac{1}{2}\alpha\cosh\alpha) = 0$ or $\tanh\alpha = \frac{1}{2}\alpha$. As can be seen from the figure, the graphs of $y = \tanh x$ and $y = \frac{1}{2}x$ intersect at a single point with approximate $x$-coordinate

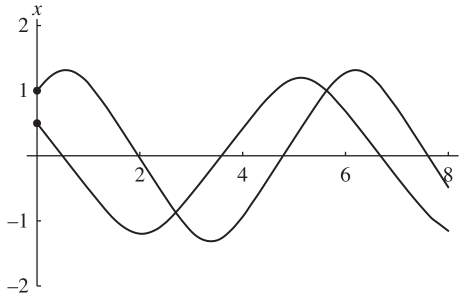


$\alpha_1 = 1.915$. Thus, there is a single negative eigenvalue $\lambda_1 = -\alpha_1^2 \approx -3.667$ and the corresponding eigenfuntion is $y_1 = \sinh 1.915x$. For $\lambda = 0$ the only solution of the boundary-value problem is $y = 0$. For $\lambda = \alpha^2 > 0$ the solution of the differential equation is $y = c_1 \cos \alpha x + c_2 \sin \alpha x$. The condition $y(0) = 0$ gives $c_1 = 0$, so $y = c_2 \sin \alpha x$. The condition $y(1) - \frac{1}{2}y'(1) = 0$ gives $c_2(\sin\alpha - \frac{1}{2}\alpha\cos\alpha) = 0$, so the eigenvalues are $\lambda_n = \alpha_n^2$ when $\alpha_n$, $n = 2, 3, 4, \ldots$, are the positive roots of $\tan\alpha = \frac{1}{2}\alpha$. Using a CAS we find that the first three values of $\alpha$ are $\alpha_2 = 4.27487$, $\alpha_3 = 7.59655$, and $\alpha_4 = 10.8127$. The first three eigenvalues are then $\lambda_2 = \alpha_2^2 = 18.2738$, $\lambda_3 = \alpha_3^2 = 57.7075$, and $\lambda_4 = \alpha_4^2 = 116.9139$ with corresponding eigenfunctions $y_2 = \sin 4.27487x$, $y_3 = \sin 7.59655x$, and $y_4 = \sin 10.8127x$.

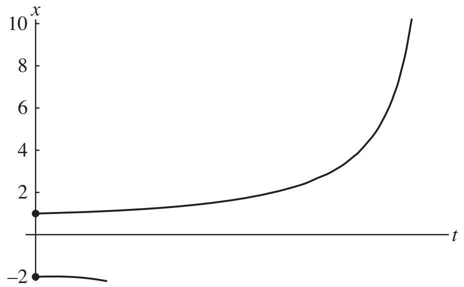
**42.** From the figure in Problem 32 showing the graphs of $1/\cosh x$ and $\cos x$, we see that this equation has an infinite number of positive roots. With the aid of a CAS the first four roots are found to be $\alpha_1 = 4.73004$, $\alpha_2 = 7.8532$, $\alpha_3 = 10.9956$, and $\alpha_4 = 14.1372$, and the corresponding eigenvalues are $\lambda_1 = 500.5636$, $\lambda_2 = 3803.5281$, $\lambda_3 = 14{,}617.5885$, and $\lambda_4 = 39{,}944.1890$.

## 5.3 Nonlinear Models

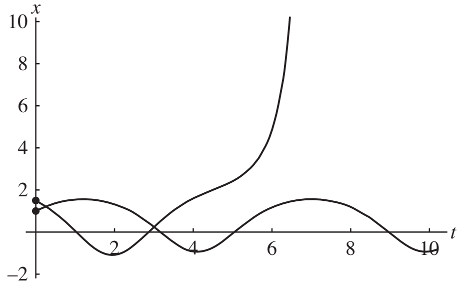
**1.** The period corresponding to $x(0) = 1$, $x'(0) = 1$ is approximately 5.6. The period corresponding to $x(0) = 1/2$, $x'(0) = -1$ is approximately 6.2.



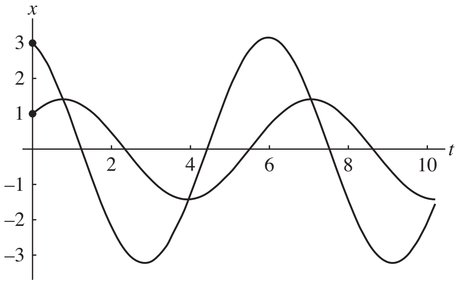
**2.** The solutions are not periodic.



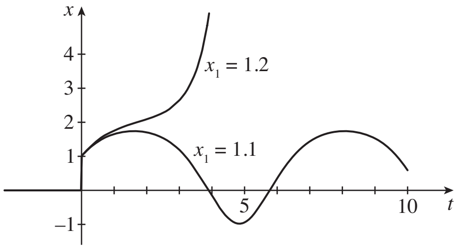
**3.** The period corresponding to $x(0) = 1$, $x'(0) = 1$ is approximately 5.8. The second initial-value problem does not have a periodic solution.



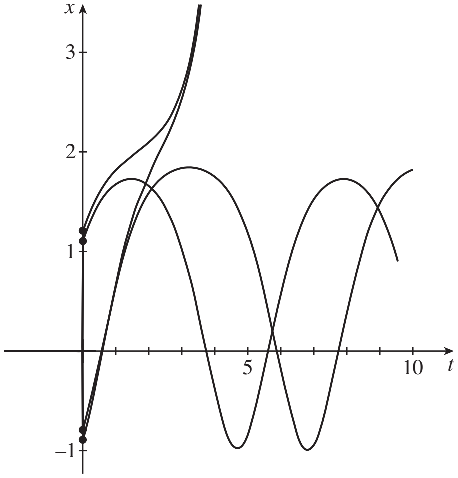
**4.** Both solutions have periods of approximately 6.3.



**5.** From the graph we see that $|x_1| \approx 1.2$.



**6.** From the graphs we see that the interval is approximately $(-0.8, 1.1)$.

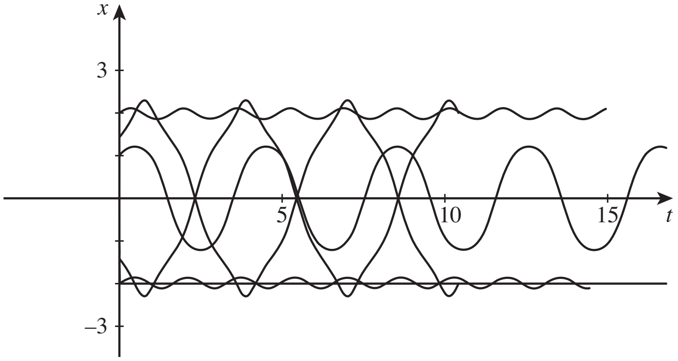


**7.** Since

$$xe^{0.01x} = x[1 + 0.01x + \frac{1}{2!}(0.01x)^2 + \cdots] \approx x$$

for small values of $x$, a linearization is $\dfrac{d^2x}{dt^2} + x = 0$.

**8.**



For $x(0) = 1$ and $x'(0) = 1$ the oscillations are symmetric about the line $x = 0$ with amplitude slightly greater than 1.

For $x(0) = -2$ and $x'(0) = 0.5$ the oscillations are symmetric about the line $x = -2$ with small amplitude.

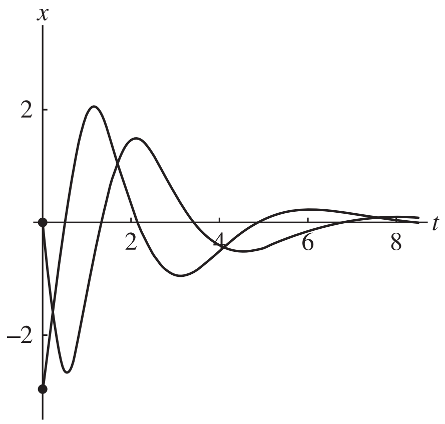
For $x(0) = \sqrt{2}$ and $x'(0) = 1$ the oscillations are symmetric about the line $x = 0$ with amplitude a little greater than 2.

For $x(0) = 2$ and $x'(0) = 0.5$ the oscillations are symmetric about the line $x = 2$ with small amplitude.

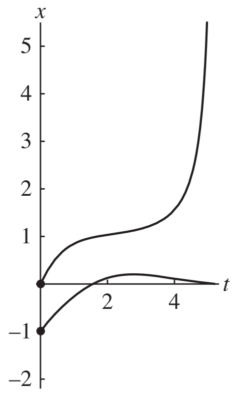
For $x(0) = -2$ and $x'(0) = 0$ there is no oscillation; the solution is constant.

For $x(0) = -\sqrt{2}$ and $x'(0) = -1$ the oscillations are symmetric about the line $x = 0$ with amplitude a little greater than 2.

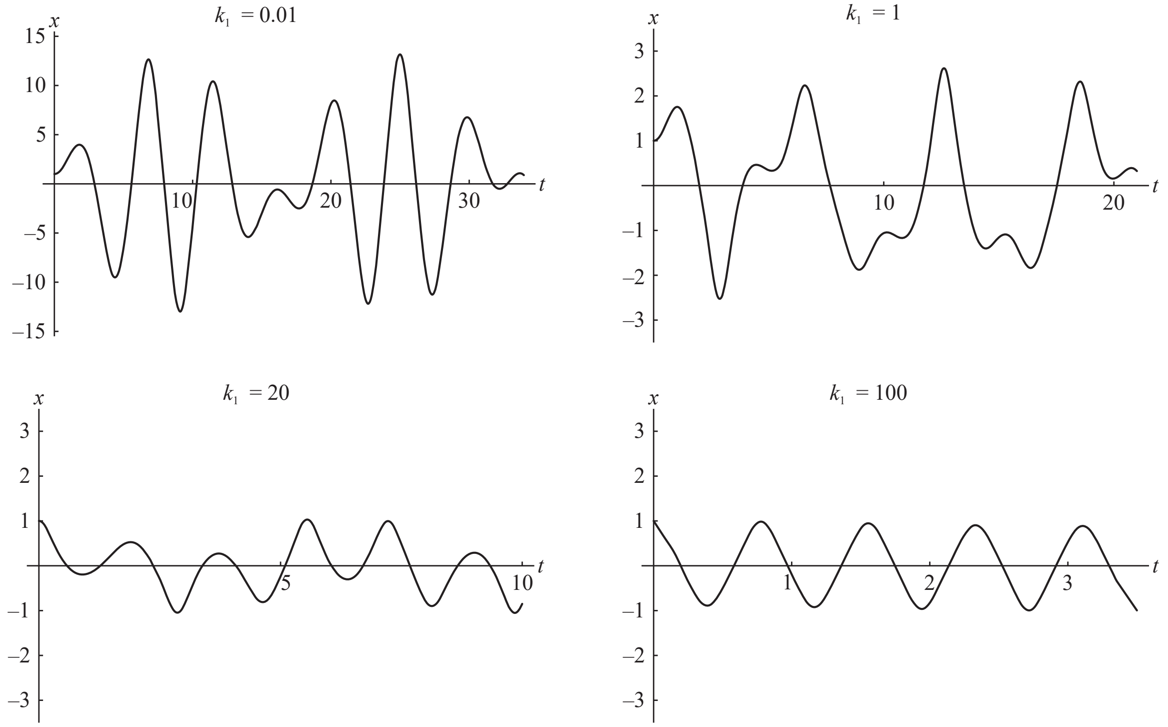
**9.** This is a damped hard spring, so $x$ will approach 0 as $t$ approaches $\infty$.



**10.** This is a damped soft spring, so we might expect no oscillatory solutions. However, if the initial conditions are sufficiently small the spring can oscillate.

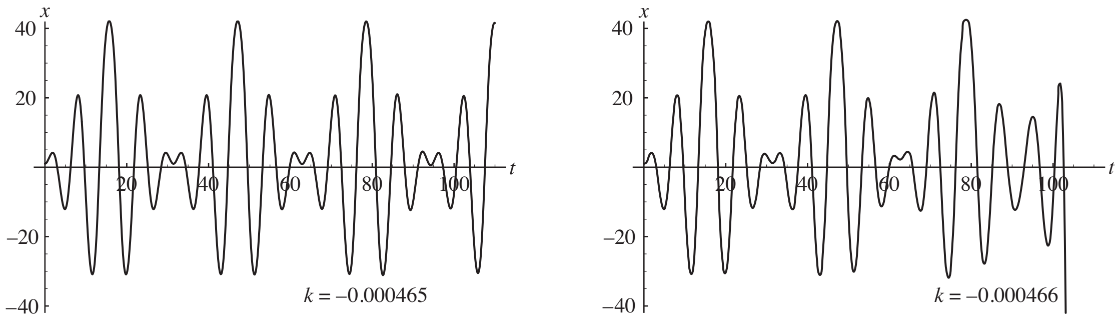


**11.**



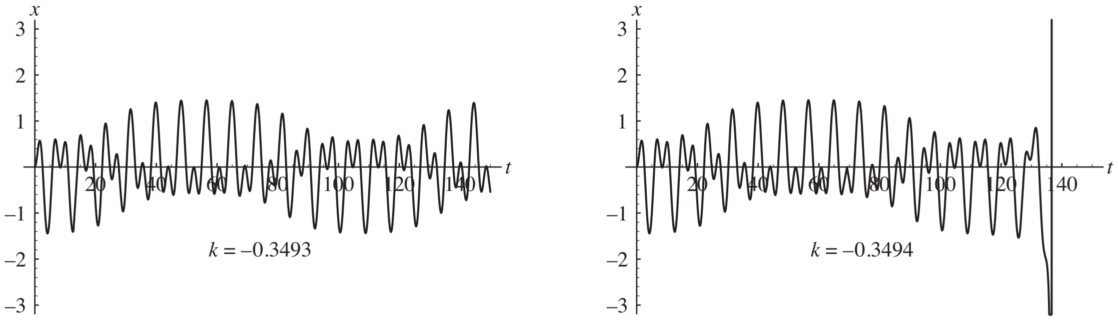
When $k_1$ is very small the effect of the nonlinearity is greatly diminished, and the system is close to pure resonance.

**12. (a)**



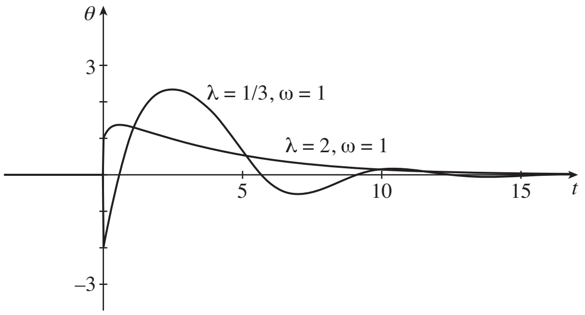
The system appears to be oscillatory for $-0.000465 \le k_1 < 0$ and nonoscillatory for $k_1 \le -0.000466$.

**(b)**



The system appears to be oscillatory for $-0.3493 \le k_1 < 0$ and nonoscillatory for $k_1 \le -0.3494$.

**13.** For $\lambda^2 - \omega^2 > 0$ we choose $\lambda = 2$ and $\omega = 1$ with $\theta(0) = 1$ and $\theta'(0) = 2$. For $\lambda^2 - \omega^2 < 0$ we choose $\lambda = 1/3$ and $\omega = 1$ with $\theta(0) = -2$ and $\theta'(0) = 4$. In both cases the motion corresponds to the overdamped and underdamped cases for spring/mass systems.



**14. (a)** Setting $dy/dt = v$, the differential equation in (13) becomes $dv/dt = -gR^2/y^2$. But, by the chain rule, $dv/dt = (dv/dy)(dy/dt) = v\,dv/dy$, so $v\,dv/dy = -gR^2/y^2$. Separating variables and integrating we obtain

$$v\,dv = -gR^2\frac{dy}{y^2} \quad \text{and} \quad \frac{1}{2}v^2 = \frac{gR^2}{y} + c.$$

Setting $v = v_0$ and $y = R$ we find $c = -gR + \frac{1}{2}v_0^2$ and

$$v^2 = 2g\frac{R^2}{y} - 2gR + v_0^2.$$

**(b)** As $y \to \infty$ we assume that $v \to 0^+$. Then $v_0^2 = 2gR$ and $v_0 = \sqrt{2gR}$.

**(c)** Using $g = 32$ ft/s and $R = 4000(5280)$ ft we find

$$v_0 = \sqrt{2(32)(4000)(5280)} \approx 36765.2\text{ft/s} \approx 25067 \text{ mi/hr}.$$

**(d)** $v_0 = \sqrt{2(0.165)(32)(1080)} \approx 7760\text{ft/s} \approx 5291$ mi/hr

**15. (a)** Intuitively, one might expect that only half of a 10-pound chain could be lifted by a 5-pound vertical force.

**(b)** Since $x = 0$ when $t = 0$, and $v = dx/dt = \sqrt{160 - 64x/3}$, we have $v(0) = \sqrt{160} \approx 12.65$ ft/s.

**(c)** Since $x$ should always be positive, we solve $x(t) = 0$, getting $t = 0$ and $t = \frac{3}{2}\sqrt{5/2} \approx 2.3717$. Since the graph of $x(t)$ is a parabola, the maximum value occurs at $t_m = \frac{3}{4}\sqrt{5/2}$. (This can also be obtained by solving $x'(t) = 0$.) At this time the height of the chain is $x(t_m) \approx 7.5$ ft. This is higher than predicted because of the momentum generated by the force. When the chain is 5 feet high it still has a positive velocity of about 7.3 ft/s, which keeps it going higher for a while.

**(d)** As discussed in the solution to part (**c**) of this problem, the chain has momentum generated by the force applied to it that will cause it to go higher than expected. It will then fall back to below the expected maximum height, again due to momentum. This, in turn, will cause it to next go higher than expected, and so on.

**16. (a)** Setting $dx/dt = v$, the differential equation becomes $(L - x)dv/dt - v^2 = Lg$. But, by the Chain Rule, $dv/dt = (dv/dx)(dx/dt) = v\,dv/dx$, so $(L - x)v\,dv/dx - v^2 = Lg$. Separating variables and integrating we obtain

$$\frac{v}{v^2 + Lg}\,dv = \frac{1}{L - x}\,dx \quad \text{and} \quad \frac{1}{2}\ln(v^2 + Lg) = -\ln(L - x) + \ln c,$$

so $\sqrt{v^2 + Lg} = c/(L - x)$. When $x = 0$, $v = 0$, and $c = L\sqrt{Lg}$. Solving for $v$ and simplifying we get

$$\frac{dx}{dt} = v(x) = \frac{\sqrt{Lg(2Lx - x^2)}}{L - x}.$$

Again, separating variables and integrating we obtain

$$\frac{L - x}{\sqrt{Lg(2Lx - x^2)}}\,dx = dt \quad \text{and} \quad \frac{\sqrt{2Lx - x^2}}{\sqrt{Lg}} = t + c_1.$$

Since $x(0) = 0$, we have $c_1 = 0$ and $\sqrt{2Lx - x^2}/\sqrt{Lg} = t$. Solving for $x$ we get

$$x(t) = L - \sqrt{L^2 - Lgt^2} \quad \text{and} \quad v(t) = \frac{dx}{dt} = \frac{\sqrt{Lg}t}{\sqrt{L - gt^2}}.$$

**(b)** The chain will be completely on the ground when $x(t) = L$ or $t = \sqrt{L/g}$.

**(c)** The predicted velocity of the upper end of the chain when it hits the ground is infinity.

**17. (a)** Let $(x, y)$ be the coordinates of $S_2$ on the curve $C$. The slope at $(x, y)$ is then

$$\frac{dy}{dx} = \frac{v_1 t - y}{0 - x} = \frac{y - v_1 t}{x} \quad \text{or} \quad xy' - y = -v_1 t.$$

**(b)** Differentiating with respect to $x$ and using $r = v_1/v_2$ gives

$$\begin{aligned} xy'' + y' - y' &= -v_1 \frac{dt}{dx} \\ xy'' &= -v_1 \frac{dt}{ds}\frac{ds}{dx} \\ xy'' &= -v_1 \frac{1}{v_2}\left(-\sqrt{1 + (y')^2}\right) \\ xy'' &= r\sqrt{1 + (y')^2}\,. \end{aligned}$$

Letting $u = y'$ and separating variables, we obtain

$$\begin{aligned} x\frac{du}{dx} &= r\sqrt{1 + u^2} \\ \frac{du}{\sqrt{1 + u^2}} &= \frac{r}{x}\,dx \\ \sinh^{-1} u &= r\ln x + \ln c = \ln(cx^r) \\ u &= \sinh(\ln cx^r) \\ \frac{dy}{dx} &= \frac{1}{2}\left(cx^r - \frac{1}{cx^r}\right). \end{aligned}$$

At $t = 0$, $dy/dx = 0$ and $x = a$, so $0 = ca^r - 1/ca^r$. Thus $c = 1/a^r$ and

$$\frac{dy}{dx} = \frac{1}{2}\left[\left(\frac{x}{a}\right)^r - \left(\frac{a}{x}\right)^r\right] = \frac{1}{2}\left[\left(\frac{x}{a}\right)^r - \left(\frac{x}{a}\right)^{-r}\right].$$

If $r > 1$ or $r < 1$, integrating gives

$$y = \frac{a}{2}\left[\frac{1}{1+r}\left(\frac{x}{a}\right)^{1+r} - \frac{1}{1-r}\left(\frac{x}{a}\right)^{1-r}\right] + c_1.$$

When $t = 0$, $y = 0$ and $x = a$, so $0 = (a/2)[1/(1+r) - 1/(1-r)] + c_1$. Thus $c_1 = ar/(1-r^2)$ and

$$y = \frac{a}{2}\left[\frac{1}{1+r}\left(\frac{x}{a}\right)^{1+r} - \frac{1}{1-r}\left(\frac{x}{a}\right)^{1-r}\right] + \frac{ar}{1-r^2}\,.$$

If $r = 1$, then integration gives

$$y = \frac{1}{2}\left[\frac{x^2}{2a} - \frac{1}{a}\ln x\right] + c_2.$$

When $t = 0$, $y = 0$ and $x = a$, so $0 = (1/2)[a/2 - (1/a)\ln a] + c_2$. Thus $c_2 = -(1/2)[a/2 - (1/a)\ln a]$ and

$$y = \frac{1}{2}\left[\frac{x^2}{2a} - \frac{1}{a}\ln x\right] - \frac{1}{2}\left[\frac{a}{2} - \frac{1}{a}\ln a\right] = \frac{1}{2}\left[\frac{1}{2a}(x^2 - a^2) + \frac{1}{a}\ln\frac{a}{x}\right].$$

**(c)** To see if the paths ever intersect we first note that if $r > 1$, then $v_1 > v_2$ and $y \to \infty$ as $x \to 0^+$. In other words, $S_2$ always lags behind $S_1$. Next, if $r < 1$, then $v_1 < v_2$ and $y = ar/(1 - r^2)$ when $x = 0$. In other words, when the submarine's speed is greater than the ship's, their paths will intersect at the point $(0, ar/(1 - r^2))$. Finally, if $r = 1$, then $y \to \infty$ as $x \to 0^+$, meaning $S_2$ will never catch up with $S_1$.

**18.** **(a)** Let $(r, \theta)$ denote the polar coordinates of the destroyer $S_1$. When $S_1$ travels the 6 miles from $(9, 0)$ to $(3, 0)$ it stands to reason, since $S_2$ travels half as fast as $S_1$, that the polar coordinates of $S_2$ are $(3, \theta_2)$, where $\theta_2$ is unknown. In other words, the distances of the ships from $(0, 0)$ are the same and $r(t) = 15t$ then gives the radial distance of both ships. This is necessary if $S_1$ is to intercept $S_2$.

**(b)** The differential of arc length in polar coordinates is $(ds)^2 = (r\,d\theta)^2 + (dr)^2$, so that

$$\left(\frac{ds}{dt}\right)^2 = r^2\left(\frac{d\theta}{dt}\right)^2 + \left(\frac{dr}{dt}\right)^2.$$

Using $ds/dt = 30$ and $dr/dt = 15$ then gives

$$\begin{aligned}
900 &= 225t^2\left(\frac{d\theta}{dt}\right)^2 + 225\\
675 &= 225t^2\left(\frac{d\theta}{dt}\right)^2\\
\frac{d\theta}{dt} &= \frac{\sqrt{3}}{t}\\
\theta(t) &= \sqrt{3}\ln t + c = \sqrt{3}\ln\frac{r}{15} + c.
\end{aligned}$$

When $r = 3$, $\theta = 0$, so $c = -\sqrt{3}\ln\frac{1}{5}$ and

$$\theta(t) = \sqrt{3}\left(\ln\frac{r}{15} - \ln\frac{1}{5}\right) = \sqrt{3}\,\ln\frac{r}{3}.$$

Thus $r = 3e^{\theta/\sqrt{3}}$, whose graph is a logarithmic spiral.

**(c)** The time for $S_1$ to go from $(9, 0)$ to $(3, 0) = \frac{1}{5}$ hour. Now $S_1$ must intercept the path of $S_2$ for some angle $\beta$, where $0 < \beta < 2\pi$. At the time of interception $t_2$ we have $15t_2 = 3e^{\beta/\sqrt{3}}$ or $t = \frac{1}{5}e^{\beta/\sqrt{3}}$. The total time is then

$$t = \frac{1}{5} + \frac{1}{5}e^{\beta/\sqrt{3}} < \frac{1}{5}(1 + e^{2\pi/\sqrt{3}}).$$

**19.** **(a)** The auxiliary equation is $m^2 + g/l = 0$, so the general solution of the differential equation is

$$\theta(t) = c_1\cos\sqrt{\frac{g}{l}}\,t + c_2\sin\sqrt{\frac{g}{l}}\,t.$$

The initial condtion $\theta(0)=0$ implies $c_1=0$ and $\theta'(0)=\omega_0$ implies $c_2=\omega_0\sqrt{l/g}$. Thus,

$$\theta(t)=\omega_0\sqrt{\frac{l}{g}}\,\sin\sqrt{\frac{g}{l}}\,t.$$

**(b)** At $\theta_{\max}$, $\sin\sqrt{g/l}\,t=1$, so

$$\theta_{\max}=\omega_0\sqrt{\frac{l}{g}}=\frac{m_b}{m_w+m_b}\,\frac{v_b}{l}\sqrt{\frac{l}{g}}=\frac{m_b}{m_w+m_b}\,\frac{v_b}{\sqrt{lg}}$$

and

$$v_b=\frac{m_w+m_b}{m_b}\sqrt{lg}\;\theta_{\max}.$$

**(c)** We have $\cos\theta_{\max}=(l-h)/l=1-h/l$. Then

$$\cos\theta_{\max}\approx 1-\frac{1}{2}\theta_{\max}^2=1-\frac{h}{l}$$

and

$$\theta_{\max}^2=\frac{2h}{l}\qquad\text{or}\qquad\theta_{\max}=\sqrt{\frac{2h}{l}}\,.$$

Thus

$$v_b=\frac{m_w+m_b}{m_b}\,\sqrt{lg}\sqrt{\frac{2h}{l}}=\frac{m_w+m_b}{m_b}\,\sqrt{2gh}\,.$$

**(d)** When $m_b=5$ g, $m_w=1$ kg, and $h=6$ cm, we have

$$v_b=\frac{1005}{5}\,\sqrt{2\,(980)\,(6)}\approx 21,797\text{ cm/s}.$$

**20. (a)** Substituting $u=dx/dt$ into

$$m\,\frac{d^2x}{dt^2}=-k\left(\frac{dx}{dt}\right)^2,\qquad x(0)=0,\;x'(0)=v_0$$

gives

$$\frac{du}{dt}=-\frac{k}{m}\,u^2\quad\text{or}\quad u^{-1}=\frac{k}{m}\,t+c_1.$$

The initial conditions imply $c_1=1/v_0$, so

$$u=\frac{dx}{dt}=\frac{1}{kt/m+1/v_0}=\frac{v_0}{kv_0t/m+1}.$$

Integrating and applying the intial conditions gives

$$x(t)=\frac{m}{k}\,\ln\left|\frac{kv_0}{m}\,t+1\right|+c_2=\frac{m}{k}\,\ln\left|\frac{kv_0}{m}\,t+1\right|.$$

Similarly, substituting $u=dy/dt$ into

$$m\,\frac{d^2y}{dt^2}=mg-k\left(\frac{dy}{dt}\right)^2,\qquad y(0)=0,\;y'(0)=0$$

gives

$$\frac{du}{dt} = g - \frac{k}{m}u^2 \quad \text{or} \quad \frac{m}{k}\frac{du}{mg/k - u^2} = dt.$$

Integrating, we obtain

$$\frac{m}{k}\frac{1}{\sqrt{mg/k}}\tanh^{-1}\frac{u}{\sqrt{mg/k}} = t + c_3.$$

The intial conditions imply $c_3 = 0$ so

$$\frac{dy}{dt} = u = \sqrt{\frac{mg}{k}}\tanh\sqrt{\frac{kg}{m}}\,t.$$

Integrating and applying the initial conditions gives

$$y(t) = \sqrt{\frac{mg}{k}}\sqrt{\frac{m}{kg}}\ln\left(\cosh\sqrt{\frac{kg}{m}}\,t\right) + c_4 = \frac{m}{k}\ln\left(\cosh\sqrt{\frac{kg}{m}}\,t\right).$$

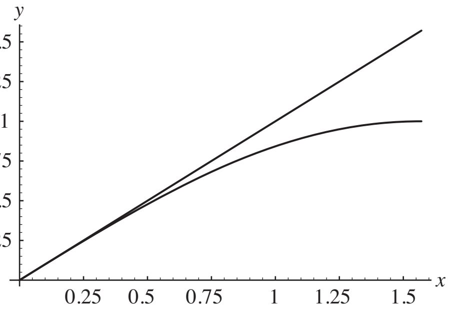
**(b)** Using the fact that 256 lbs is equivalent to 8 slugs of mass, we solve (using numerical procedure)

$$1000 = \frac{8}{0.0053}\ln\left(\cosh\sqrt{\frac{0.0053(32)}{8}}\,t\right),$$

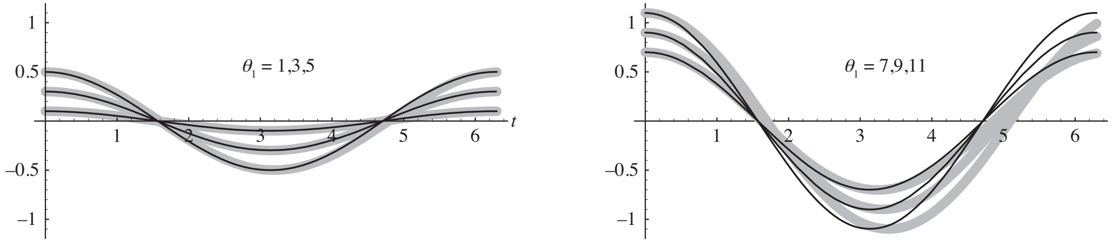
which gives $t = 8.8$ seconds. Then $x(8.8) = 1919$ ft is the horizontal distance travelled by the supply pack.

**21.** Since $(dx/dt)^2$ is always positive, it is necessary to use $|dx/dt|\,(dx/dt)$ in order to account for the fact that the motion is oscillatory and the velocity (or its square) should be negative when the spring is contracting.

**22.** **(a)** From the graph we see that the approximations appears to be quite good for $0 \le x \le 0.4$. Using an equation solver to solve $\sin x - x = 0.05$ and $\sin x - x = 0.005$, we find that the approximation is accurate to one decimal place for $\theta_1 = 0.67$ and to two decimal places for $\theta_1 = 0.31$.



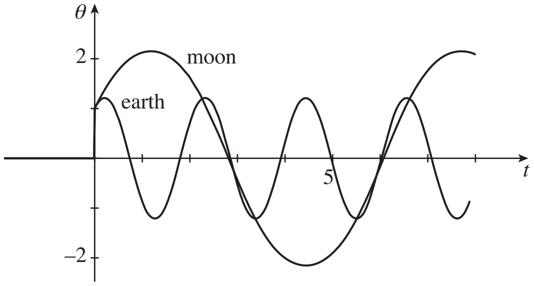
**(b)**



**23. (a)** Write the differential equation as

$$\frac{d^2\theta}{dt^2}+\omega^2\sin\theta=0,$$

where $\omega^2=g/l$. To determine the differences between the Earth and the Moon we take $l=3$, $\theta(0)=1$, and $\theta'(0)=2$. Using $g=32$ on the Earth and $g=32\times 0.165$ on the Moon we obtain the graphs shown in the figure.



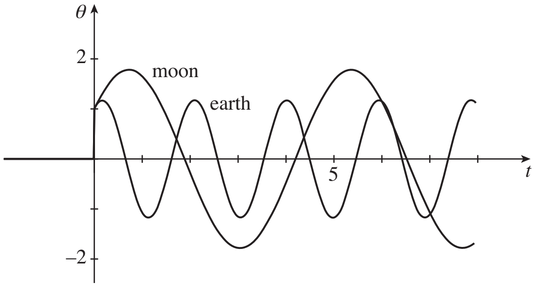
**(b)** Comparing the apparent periods of the graphs, we see that the pendulum oscillates faster on the Earth than on the Moon and the amplitude is greater on the Moon than on the Earth.

**24.** The linear model is

$$\frac{d^2\theta}{dt^2}+\omega^2\theta=0,$$

where $\omega^2=g/l$. When $g=32$, $l=3$, $\theta(0)=1$, and $\theta'(0)=2$, the solution is

$$\theta(t)=\cos 3.2660t+0.6124\sin 3.2660t.$$



When $g=32\times 0.1652$ the solution is

$$\theta(t)=\cos 1.3267t+1.5076\sin 1.3267t.$$

As in the nonlinear case, the pendulum oscillates faster on the Earth than on the Moon and still has greater amplitude on the Moon.

**25. (a)** The general solution of

$$\frac{d^2\theta}{dt^2}+\theta=0$$

is $\theta(t)=c_1\cos t+c_2\sin t$. From $\theta(0)=\pi/12$ and $\theta'(0)=-1/3$ we find

$$\theta(t)=\left(\frac{\pi}{12}\right)\cos t-\left(\frac{1}{3}\right)\sin t.$$

Setting $\theta(t)=0$ we have $\tan t=\pi/4$ which implies $t_1=\tan^{-1}(\pi/4)\approx 0.66577$.

**(b)** We set $\theta(t)=\theta(0)+\theta'(0)t+\frac{1}{2}\theta''(0)t^2+\frac{1}{6}\theta'''(0)t^3+\cdots$ and use $\theta''(t)=-\sin\theta(t)$ together with $\theta(0)=\pi/12$ and $\theta'(0)=-1/3$. Then

$$\theta''(0)=-\sin\left(\frac{\pi}{12}\right)=-\sqrt{2}\,\frac{(\sqrt{3}-1)}{4}$$

and

$$\theta'''(0) = -\cos\theta(0)\cdot\theta'(0) = -\cos\left(\frac{\pi}{12}\right)\left(-\frac{1}{3}\right) = \sqrt{2}\,\frac{\sqrt{3}+1}{12}.$$

Thus

$$\theta(t) = \frac{\pi}{12} - \frac{1}{3}t - \frac{\sqrt{2}\,(\sqrt{3}-1)}{8}\,t^2 + \frac{\sqrt{2}\,(\sqrt{3}+1)}{72}\,t^3 + \cdots.$$

**(c)** Setting $\pi/12 - t/3 = 0$ we obtain $t_1 = \pi/4 \approx 0.785398$.

**(d)** Setting

$$\frac{\pi}{12} - \frac{1}{3}t - \frac{\sqrt{2}\,(\sqrt{3}-1)}{8}\,t^2 = 0$$

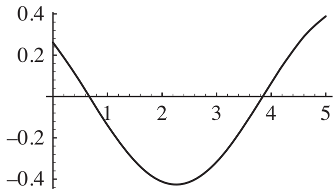
and using the positive root we obtain $t_1 \approx 0.63088$.

**(e)** Setting

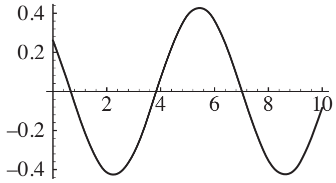
$$\frac{\pi}{12} - \frac{1}{3}t - \frac{\sqrt{2}\,(\sqrt{3}-1)}{8}\,t^2 + \frac{\sqrt{2}\,(\sqrt{3}+1)}{72}\,t^3 = 0$$

we find with the help of a CAS that $t_1 \approx 0.661973$ is the first positive root.

**(f)** From the output we see that $y(t)$ is an interpolating function on the interval $0 \le t \le 5$, whose graph is shown. The positive root of $y(t) = 0$ near $t = 1$ is $t_1 = 0.666404$.



**(f)** To find the next two positive roots we change the interval used in **NDSolve** and **Plot** from $\{\mathbf{t},\mathbf{0},\mathbf{5}\}$ to $\{\mathbf{t},\mathbf{0},\mathbf{10}\}$. We see from the graph that the second and third positive roots are near 4 and 7, respectively. Replacing $\{\mathbf{t},\mathbf{1}\}$ in **FindRoot** with $\{\mathbf{t},\mathbf{4}\}$ and then $\{\mathbf{t},\mathbf{7}\}$ we obtain $t_2 = 3.84411$ and $t_3 = 7.0218$.



**26.** From the table below we see that the pendulum first passes the vertical position between 1.6 and 1.7 seconds. To refine our estimate of $t_1$ we estimate the solution of the differential equation on $[1.6, 1.7]$ using a step size of $h = 0.01$. From the resulting table we see that $t_1$ is between 1.63 and 1.64 seconds. Repeating the process with $h = 0.001$ we conclude that $t_1 \approx 1.634$. Then the period of the pendulum is approximately $4t_1 = 6.536$. The error when using $t_1 = 2\pi$ is $6.536 - 6.283 = 0.253$ and the percentage relative error is $(0.253/6.536)100 = 3.87$.

h = 0.1

| $t_n$ | $\theta_n$ |
|---|---|
| 0.00 | 0.78540 |
| 0.10 | 0.78187 |
| 0.20 | 0.77129 |
| 0.30 | 0.75375 |
| 0.40 | 0.72937 |
| 0.50 | 0.69833 |
| 0.60 | 0.66086 |
| 0.70 | 0.61726 |
| 0.80 | 0.56788 |
| 0.90 | 0.51313 |
| 1.00 | 0.45347 |
| 1.10 | 0.38943 |
| 1.20 | 0.32161 |
| 1.30 | 0.25062 |
| 1.40 | 0.17716 |
| 1.50 | 0.10194 |
| 1.60 | 0.02570 |
| 1.70 | −0.05080 |
| 1.80 | −0.12678 |
| 1.90 | −0.20151 |
| 2.00 | −0.27423 |

h = 0.01

| $t_n$ | $\theta_n$ |
|---|---|
| 1.60 | 0.02570 |
| 1.61 | 0.01805 |
| 1.62 | 0.01040 |
| 1.63 | 0.00274 |
| 1.64 | −0.00491 |
| 1.65 | −0.01256 |
| 1.66 | −0.02021 |
| 1.67 | −0.02786 |
| 1.68 | −0.03551 |
| 1.69 | −0.04315 |
| 1.70 | −0.05080 |

h = 0.001

| | |
|---|---|
| 1.630 | 0.00274 |
| 1.631 | 0.00198 |
| 1.632 | 0.00121 |
| 1.633 | 0.00045 |
| 1.634 | −0.00032 |
| 1.635 | −0.00108 |
| 1.636 | −0.00185 |
| 1.637 | −0.00261 |

## Chapter 5 in Review

**1.** 8 ft, since $k = 4$

**2.** $2\pi/5$, since $\frac{1}{4}x'' + 6.25x = 0$

**3.** 5/4 m, since $x = -\cos 4t + \frac{3}{4}\sin 4t$

**4.** True

**5.** False; since an external force may exist

**6.** False; since the equation of motion in this case is $x(t) = e^{-\lambda t}(c_1 + c_2t)$ and $x(t) = 0$ can have at most one real solution

**7.** overdamped

**8.** From $x(0) = (\sqrt{2}/2)\sin\phi = -1/2$ we see that $\sin\phi = -1/\sqrt{2}$, so $\phi$ is an angle in the third or fourth quadrant. Since $x'(t) = \sqrt{2}\cos(2t + \phi)$, $x'(0) = \sqrt{2}\cos\phi = 1$ and $\cos\phi > 0$. Thus $\phi$ is in the fourth quadrant and $\phi = -\pi/4$.

**9.** $y = 0$ because $\lambda = 8$ is not an eigenvalue

**10.** $y = \cos 6x$ because $\lambda = (6)^2 = 36$ is an eigenvalue

**11.** The period of a spring/mass system is given by $T = 2\pi/\omega$ where $\omega^2 = k/m = kg/W$, where $k$ is the spring constant, $W$ is the weight of the mass attached to the spring, and $g$

is the acceleration due to gravity. Thus, the period of oscillation is $T = (2\pi/\sqrt{kg}\,)\sqrt{W}$. If the weight of the original mass is $W$, then $(2\pi/\sqrt{kg}\,)\sqrt{W} = 3$ and $(2\pi/\sqrt{kg}\,)\sqrt{W-8} = 2$. Dividing, we get $\sqrt{W}/\sqrt{W-8} = 3/2$ or $W = \frac{9}{4}(W-8)$. Solving for $W$ we find that the weight of the original mass was 14.4 pounds.

**12.** **(a)** Solving $\frac{3}{8}x'' + 6x = 0$ subject to $x(0) = 1$ and $x'(0) = -4$ we obtain

$$x = \cos 4t - \sin 4t = \sqrt{2}\sin(4t + 3\pi/4).$$

**(b)** The amplitude is $\sqrt{2}$, period is $\pi/2$, and frequency is $2/\pi$.

**(c)** If $x = 1$ then $t = n\pi/2$ and $t = -\pi/8 + n\pi/2$ for $n = 1, 2, 3, \ldots$.

**(d)** If $x = 0$ then $t = \pi/16 + n\pi/4$ for $n = 0, 1, 2, \ldots$. The motion is upward for $n$ even and downward for $n$ odd.

**(e)** $x'(3\pi/16) = 0$

**(f)** If $x' = 0$ then $4t + 3\pi/4 = \pi/2 + n\pi$ or $t = 3\pi/16 + n\pi$.

**13.** We assume that the spring is initially compressed by 4 inches and that the positive direction on the $x$-axis is in the direction of elongation of the spring. Then, from $\frac{1}{4}x'' + \frac{3}{2}x' + 2x = 0$, $x(0) = -1/3$, and $x'(0) = 0$ we obtain $x = -\frac{2}{3}e^{-2t} + \frac{1}{3}e^{-4t}$.

**14.** From $x'' + \beta x' + 64x = 0$ we see that oscillatory motion results if $\beta^2 - 256 < 0$ or $0 \le \beta < 16$.

**15.** From $mx'' + 4x' + 2x = 0$ we see that nonoscillatory motion results if $16 - 8m \ge 0$ or $0 < m \le 2$.

**16.** From $\frac{1}{4}x'' + x' + x = 0$, $x(0) = 4$, and $x'(0) = 2$ we obtain $x = 4e^{-2t} + 10te^{-2t}$. If $x'(t) = 0$, then $t = 1/10$, so that the maximum displacement is $x = 5e^{-0.2} \approx 4.094$.

**17.** Writing $\frac{1}{8}x'' + \frac{8}{3}x = \cos\gamma t + \sin\gamma t$ in the form $x'' + \frac{64}{3}x = 8\cos\gamma t + 8\sin\gamma t$ we identify $\omega^2 = \frac{64}{3}$. The system is in a state of pure resonance when $\gamma = \omega = \sqrt{64/3} = 8/\sqrt{3}$.

**18.** Clearly $x_p = A/\omega^2$ suffices.

**19.** From $\frac{1}{8}x'' + x' + 3x = e^{-t}$, $x(0) = 2$, and $x'(0) = 0$ we obtain

$$x_c = e^{-4t}\left(c_1\cos 2\sqrt{2}\,t + c_2\sin 2\sqrt{2}\,t\right), \quad x_p = \frac{8}{17}e^{-t}, \quad \text{and}$$

$$x = e^{-4t}\left(\frac{26}{17}\cos 2\sqrt{2}\,t + \frac{28\sqrt{2}}{17}\sin 2\sqrt{2}\,t\right) + \frac{8}{17}e^{-t}.$$

**20.** **(a)** Let $k$ be the effective spring constant and $x_1$ and $x_2$ the elongation of springs $k_1$ and $k_2$. The restoring forces satisfy $k_1x_1 = k_2x_2$ so $x_2 = (k_1/k_2)x_1$. From $k(x_1 + x_2) = k_1x_1$ we have

$$\begin{aligned} k\left(x_1 + \frac{k_1}{k_2}\,x_2\right) &= k_1x_1 \\ k\left(\frac{k_2 + k_1}{k_2}\right) &= k_1 \\ k &= \frac{k_1k_2}{k_1 + k_2} \\ \frac{1}{k} &= \frac{1}{k_1} + \frac{1}{k_2}\,. \end{aligned}$$

From $k_1 = 2W$ and $k_2 = 4W$ we find $1/k = 1/2W + 1/4W = 3/4W$. Then $k = 4W/3 = 4mg/3$. The differential equation $mx'' + kx = 0$ then becomes $x'' + (4g/3)x = 0$. The solution is

$$x(t) = c_1 \cos 2\sqrt{\frac{g}{3}}\,t + c_2 \sin 2\sqrt{\frac{g}{3}}\,t.$$

The initial conditions $x(0) = 1$ and $x'(0) = 2/3$ imply $c_1 = 1$ and $c_2 = 1/\sqrt{3g}\,$.

**(b)** To compute the maximum speed of the mass we compute

$$x'(t) = 2\sqrt{\frac{g}{3}}\sin 2\sqrt{\frac{g}{3}}\,t + \frac{2}{3}\cos 2\sqrt{\frac{g}{3}}\,t \quad \text{and} \quad |x'(t)| = \sqrt{4\,\frac{g}{3} + \frac{4}{9}} = \frac{2}{3}\sqrt{3g + 1}\,.$$

**21.** From $q'' + 10^4q = 100\sin 50t$, $q(0) = 0$, and $q'(0) = 0$ we obtain $q_c = c_1\cos 100t + c_2\sin 100t$, $q_p = \frac{1}{75}\sin 50t$, and

**(a)** $q(t) = -\frac{1}{150}\sin 100t + \frac{1}{75}\sin 50t$,

**(b)** $i(t) = -\frac{2}{3}\cos 100t + \frac{2}{3}\cos 50t$, and

**(c)** $q(t) = 0$ when $\sin 50t(1 - \cos 50t) = 0$ or $t = n\pi/50$ for $n = 0, 1, 2, \ldots$.

**22.** By Kirchhoff's second law,

$$L\,\frac{d^2q}{dt^2} + R\,\frac{dq}{dt} + \frac{1}{C}\,q = E(t).$$

Using $q'(t) = i(t)$ we can write the differential equation in the form

$$L\,\frac{di}{dt} + Ri + \frac{1}{C}\,q = E(t).$$

Then differentiating we obtain

$$L\,\frac{d^2i}{dt^2} + R\,\frac{di}{dt} + \frac{1}{C}\,i = E'(t).$$

**23.** For $\lambda = \alpha^2 > 0$ the general solution is $y = c_1 \cos \alpha x + c_2 \sin \alpha x$. Now

$$y(0) = c_1 \qquad \text{and} \qquad y(2\pi) = c_1 \cos 2\pi\alpha + c_2 \sin 2\pi\alpha,$$

so the condition $y(0) = y(2\pi)$ implies

$$c_1 = c_1 \cos 2\pi\alpha + c_2 \sin 2\pi\alpha$$

which is true when $\alpha = \sqrt{\lambda} = n$ or $\lambda = n^2$ for $n = 1, 2, 3, \ldots$. Since

$$y' = -\alpha c_1 \sin \alpha x + \alpha c_2 \cos \alpha x = -nc_1 \sin nx + nc_2 \cos nx,$$

we see that $y'(0) = nc_2 = y'(2\pi)$ for $n = 1, 2, 3, \ldots$ Thus, the eigenvalues are $n^2$ for $n = 1, 2, 3, \ldots$, with corresponding eigenfunctions $\cos nx$ and $\sin nx$. When $\lambda = 0$, the general solution is $y = c_1 x + c_2$ and the corresponding eigenfunction is $y = 1$.

For $\lambda = -\alpha^2 < 0$ the general solution is $y = c_1 \cosh \alpha x + c_2 \sinh \alpha x$. In this case $y(0) = c_1$ and $y(2\pi) = c_1 \cosh 2\pi\alpha + c_2 \sinh 2\pi\alpha$, so $y(0) = y(2\pi)$ can only be valid for $\alpha = 0$. Thus, there are no eigenvalues corresponding to $\lambda < 0$.

**24.** **(a)** The differential equation is $d^2r/dt^2 - \omega^2 r = -g \sin \omega t$. The auxiliary equation is $m^2 - \omega^2 = 0$, so $r_c = c_1 e^{\omega t} + c_2 e^{-\omega t}$. A particular solution has the form $r_p = A \sin \omega t + B \cos \omega t$. Substituting into the differential equation we find $-2A\omega^2 \sin \omega t - 2B\omega^2 \cos \omega t = -g \sin \omega t$. Thus, $B = 0$, $A = g/2\omega^2$, and $r_p = (g/2\omega^2) \sin \omega t$. The general solution of the differential equation is $r(t) = c_1 e^{\omega t} + c_2 e^{-\omega t} + (g/2\omega^2) \sin \omega t$. The initial conditions imply $c_1 + c_2 = r_0$ and $g/2\omega - \omega c_1 + \omega c_2 = v_0$. Solving for $c_1$ and $c_2$ we get

$$c_1 = (2\omega^2 r_0 + 2\omega v_0 - g)/4\omega^2 \quad \text{and} \quad c_2 = (2\omega^2 r_0 - 2\omega v_0 + g)/4\omega^2,$$

so that

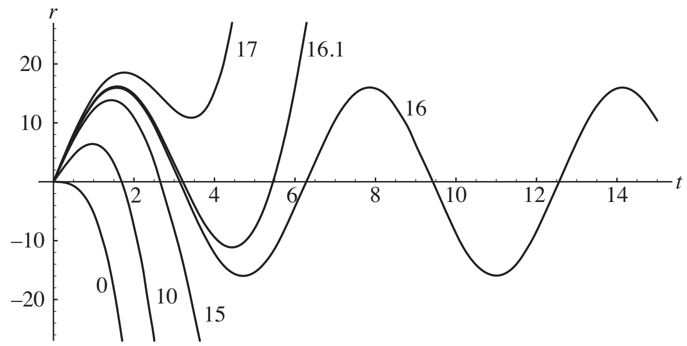
$$r(t) = \frac{2\omega^2 r_0 + 2\omega v_0 - g}{4\omega^2} e^{\omega t} + \frac{2\omega^2 r_0 - 2\omega v_0 + g}{4\omega^2} e^{-\omega t} + \frac{g}{2\omega^2} \sin \omega t.$$

**(b)** The bead will exhibit simple harmonic motion when the exponential terms are missing. Solving $c_1 = 0$, $c_2 = 0$ for $r_0$ and $v_0$ we find $r_0 = 0$ and $v_0 = g/2\omega$.

To find the minimum length of rod that will accommodate simple harmonic motion we determine the amplitude of $r(t)$ and double it. Thus $L = g/\omega^2$.

**(c)** As $t$ increases, $e^{\omega t}$ approaches infinity and $e^{-\omega t}$ approaches 0. Since $\sin \omega t$ is bounded, the distance, $r(t)$, of the bead from the pivot point increases without bound and the distance of the bead from $P$ will eventually exceed $L/2$.

**(d)**



**(e)** For each $v_0$ we want to find the smallest value of $t$ for which $r(t) = \pm 20$. Whether we look for $r(t) = -20$ or $r(t) = 20$ is determined by looking at the graphs in part (d). The total times that the bead stays on the rod is shown in the table below.

| $v_0$ | 0 | 10 | 15 | 16.1 | 17 |
|---|---|---|---|---|---|
| $r$ | –20 | –20 | –20 | 20 | 20 |
| $t$ | 1.55007 | 2.35494 | 3.43088 | 6.11627 | 4.22339 |

When $v_0 = 16$ the bead never leaves the rod.

**25.** Unlike the derivation given in Section 3.8 in the text, the weight $mg$ of the mass $m$ does not appear in the net force since the spring is not stretched by the weight of the mass when it is in the equilibrium position (i.e., there is no $mg - ks$ term in the net force). The only force acting on the mass when it is in motion is the restoring force of the spring. By Newton's second law,

$$m\frac{d^2x}{dt^2} = -kx \qquad \text{or} \qquad \frac{d^2x}{dt^2} + \frac{k}{m}x = 0.$$

**26.** By Newton's second law of motion:

$$m\frac{d^2x}{dt^2} = -k_1x - k_2x \quad \text{or} \quad m\frac{d^2x}{dt^2} + (k_1 + k_2)\,x = 0.$$

Here $-k_1x$ is the force to the left due to the elongation $x$ of the spring with constant $k_1$ and $-k_2x$ is the force to the left due to the compression $x$ of the spring with constant $k_2$.

**27.** The force of kinetic friction opposing the motion of the mass in $\mu N$, where $\mu$ is the coefficient of sliding friction and $N$ is the normal component of the weight. Since friction is a force opposite to the direction of motion and since $N$ is pointed directly downward (it is simply the weight of the mass), Newton's second law gives, for motion to the right ($x' > 0$),

$$m\frac{d^2x}{dt^2} = -kx - \mu mg,$$

and for motion to the left ($x' < 0$),

$$m\frac{d^2x}{dt^2} = -kx + \mu mg.$$

Traditionally, these two equations are written as one expression

$$m\frac{d^2x}{dt^2} + f_k\,\text{sgn}(x') + kx = 0,$$

where $f_k = \mu mg$ and

$$\text{sgn}(x') = \begin{cases} 1, & x' > 0 \\ -1, & x' < 0 \end{cases}$$

**28. (a)** The differential equation is $x'' + \text{sgn}(x') + x = 0$ or

$$x'' + x = \begin{cases} 1, & \text{motion to the left} \\ -1, & \text{motion to the right.} \end{cases}$$

Correspondingly

$$x(t) = \begin{cases} c_3\cos t + c_4\sin t + 1, & \text{motion to the left} \\ c_1\cos t + c_2\sin t - 1, & \text{motion to the right} \end{cases}$$

For motion to the left $x'' + x = 1$, $x(0) = 5.5$, $x'(0) = 0$ gives $x(t) = 4.5\cos t + 1$. From $x'(t) = -4.5\sin t$ we see that the mass is at rest ($x'(t) = 0$) at $t = \pi$ so the interval of definition is $[0, \pi]$. Note that $x'(t) < 0$ for $0 < t < \pi$. The mass is now on the left at $x(\pi) = -3.5$.

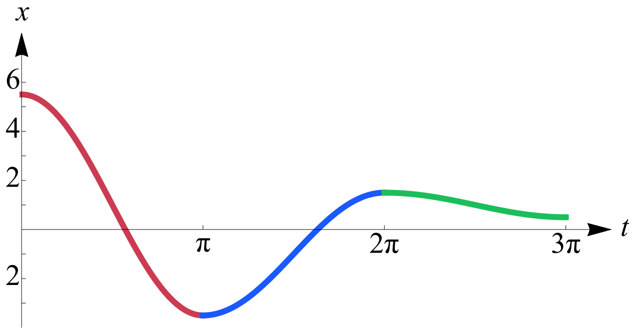
**(b)** For motion to the right we solve the initial value problem $x'' + x = -1$, $x(\pi) = -3.5$, $x'(\pi) = 0$ gives $x(t) = 2.5\cos t - 1$. From $x'(t) = -2.5\sin t$ we see that the velocity is 0 next at $t = 2\pi$ so the interval of definition is $[\pi, 2\pi]$. Note that $x'(t) > 0$ for $\pi < t < 2\pi$. The mass is once again on the right at $x(2\pi) = 1.5$.

**(c)** For motion to the left we solve the initial-value problem $x'' + x = 1$, $x(2\pi) = 1.5$, $x'(2\pi) = 0$ gives $x(t) = 0.5\cos t + 1$. From $x'(t) = -0.5\sin t$ we see that the velocity is 0 next at $t = 3\pi$ so the interval of definition is $[2\pi, 3\pi]$. Note that $x'(t) < 0$ for $2\pi < t < 3\pi$. The mass is still on the right at $x(3\pi) = 0.5$.

**(d)** Now if there is further motion to the right we solve the initial-value problem $x'' + x = -1$, $x(3\pi) = 0.5$, $x'(3\pi) = 0$, which gives $x(t) = -1.5\cos t - 1$. From $x'(t) = 1.5\sin t$ we see that the velocity is 0 next at $t = 4\pi$ so the interval of definition is $[3\pi, 4\pi]$. But for $3\pi < t < 4\pi$, $x'(t) < 0$ which contradicts the assumption of motion to the right. This indicates that the solution $x(t) = c_1\cos t + c_2\sin t - 1$ is no longer applicable. The other solution for motion to the left is also not applicable since it implies $x'(t) > 0$. The mass has undoubtedly stopped at $x(3\pi) = 0.5$.

Motion on the interval $[0, 3\pi]$:

**(e)** $$x(t) = \begin{cases} 4.5\cos t + 1, & 0 \le t < \pi \\ 2.5\cos t - 1, & \pi \le t < 2\pi \\ 0.5\cos t + 1, & 2\pi \le t < 3\pi \\ 0.5, & t \ge 3\pi \end{cases}$$



**29.** The given initial-value problem is

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0, \qquad \theta(0) = \pi/6, \quad \theta'(0) = 0$$

Differentiating with respect to time $t$ the differential equation yields

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta, \quad \frac{d^3\theta}{dt^3} = -\frac{g}{l}\cos\theta\,\frac{d\theta}{dt}, \quad \text{and} \quad \frac{d^4\theta}{dt^4} = -\frac{g}{l}\cos\theta\,\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta\left(\frac{d\theta}{dt}\right)^2, \ \ldots$$

From the differential equation, the initial conditions, and the foregoing derivatives we see at $t = 0$ that $\sin(\theta(0)) = \sin\pi/6 = 1/2$ and $\cos(\theta(0)) = \cos\pi/6 = \sqrt{3}/2$ and so:

$$\theta(0) = \frac{\pi}{6}, \quad \theta'(0) = 0, \quad \theta''(0) = -\frac{g}{2l}, \quad \theta'''(0) = 0, \quad \theta^{(4)}(0) = \frac{\sqrt{3}\,g^2}{4l^2}, \ \ldots$$

The Maclaurin expansion

$$\theta(t) = \theta(0) + \theta'(0)t + \frac{\theta''(0)}{2!}t^2 + \frac{\theta'''(0)}{3!}t^3 + \frac{\theta^{(4)}(0)}{4!}t^4 + \cdots$$

is the

$$\theta(t) = \frac{\pi}{6} - \frac{g}{4l}t^2 + \frac{\sqrt{3}\,g^2}{96l^2}t^4 + \cdots.$$

**30.** **(a)** If $m$ and $l$ are constant, then (1) in the text is simply

$$ml^2\frac{d}{dt}\left(\frac{d\theta}{dt}\right) = -mgl\sin\theta$$

$$ml^2\frac{d^2\theta}{dt^2} = -mgl\sin\theta$$

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0$$

**(b)** If $m$ is constant and $l$ is a function of time $t$ then by the product rule of differentiation,

$$\begin{aligned}\frac{d}{dt}\left(ml^2\,\frac{d\theta}{dt}\right) &= -mgl\sin\theta \\ ml^2\,\frac{d^2\theta}{dt^2} + m\left(2l\,\frac{dl}{dt}\right)\frac{d\theta}{dl} + mgl\sin\theta &= 0 \\ ml^2\,\frac{d^2\theta}{dt^2} + 2ml\,\frac{dl}{dt}\,\frac{d\theta}{dt} + mgl\sin\theta &= 0 \\ l\,\frac{d^2\theta}{dt^2} + 2\,\frac{dl}{dt}\,\frac{d\theta}{dt} + g\sin\theta &= 0\end{aligned}$$

When $l(t) = l_0 + x(t)$ the differential equation becomes

$$(l_0 + x)\,\frac{d^2\theta}{dt^2} + 2\,\frac{dx}{dt}\,\frac{d\theta}{dt} + g\sin\theta = 0$$

**31. (a)** The linear initial-value problem is

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta_1 = 0, \qquad \theta_1(0) = \theta_0,\ \theta_1'(0) = 0.$$

Applying the initial conditions to $\theta_1(t) = c_1\cos\sqrt{g/l}\,t + c_2\sin\sqrt{g/l}\,t$ gives $c_1 = \theta_0$ and $c_2 = 0$. Therefore the solution is

$$\theta_1(t) = \theta_0\cos\sqrt{g/l}\,t.$$

The string hits the nail when $\theta_1(t_1) = 0$ or $\cos\sqrt{g/l}\,t_1 = 0$. Thus

$$\begin{aligned}\sqrt{\frac{g}{l}}\,t_1 &= \frac{\pi}{2} \\ t_1 &= \frac{\pi}{2}\sqrt{\frac{l}{g}}\ \text{seconds}\end{aligned}$$

The interval over which the solution $\theta_1(t)$ is defined is $\left[0, \frac{\pi}{2}\sqrt{\frac{l}{g}}\right]$.

**(b)** Now $\theta_2(t_1) = \theta_1(t_1) = 0$. Also, the initial angular speed when the string hits the nail satisfies $\theta_2'(t_1) = \theta_1'(t_1)$. Using $t_1$ found in part (a)

$$\theta_1'(t_1) = -\theta_0\sqrt{\frac{g}{l}}\sin\left(\sqrt{\frac{g}{l}}\,\frac{\pi}{2}\sqrt{\frac{l}{g}}\right) = -\theta_0\sqrt{\frac{g}{l}}\sin\frac{\pi}{2} = -\theta_0\sqrt{\frac{g}{l}}$$

Since the length of the string is $l/4$ the initial-value problem for $\theta_2(t)$ is

$$\frac{d^2}{dt^2} + \frac{g}{l/4}\theta_2 = 0, \quad \theta_2(T) = 0, \quad \theta_2'(T) = -\theta_0\sqrt{\frac{g}{l}}$$

Applying the initial condition $\theta_2(t_1) = 0$ to $\theta_2(t) = c_3 \cos 2\sqrt{g/l}\, t + c_2 \sin 2\sqrt{g/l}\, t$ gives $c_3 = 0$. Therefore

$$\theta_2(t) = c_4 \sin 2\sqrt{\frac{g}{l}}\, t$$

$$\theta_2'(t) = c_4 2\sqrt{\frac{g}{l}} \cos 2\sqrt{\frac{g}{l}}\, t$$

and so

$$\theta_2'(t_1) = c_4 2\sqrt{\frac{g}{l}} \cos \overbrace{\left(2\sqrt{\frac{g}{l}}\frac{\pi}{2}\sqrt{\frac{l}{g}}\right)}^{\pi} = -2c_4\sqrt{\frac{g}{l}} = -\theta_0\sqrt{\frac{g}{l}}$$

So $c_4 = \theta_0/2$ and

$$\theta_2(t) = \frac{1}{2}\theta_0 \sin 2\sqrt{\frac{g}{l}}\, t.$$

At the time $t_2$ we have $\theta_2(t) = 0$ for the second time:

$$\theta(t_2) = \frac{1}{2}\,\theta_0 \sin 2\sqrt{\frac{g}{l}}\, t_2 = 0$$

$$2\sqrt{\frac{g}{l}\, t_2} = 2\pi$$

$$t_2 = \pi\sqrt{\frac{l}{g}}$$

The interval over which $\theta_2(t)$ is defined is $\left[\frac{\pi}{2}\sqrt{\frac{l}{g}}, \pi\sqrt{\frac{l}{g}}\right]$.

**32. (a)** The solution of the differential equation

$$\frac{d^2x}{dt^2} + 4x = \sin 4t$$

is $x(t) = c_1 \cos 2t + c_2 \sin 2t - \frac{1}{12}\sin 4t$. The intial conditions $x(0) = 0$, $x'(0) = v_0$ give $c_1 = 0$ and $c_2 = \frac{1}{6} + \frac{1}{2}\,v_0$. Therefore $x(t) = \left(\frac{1}{6} + \frac{1}{2}\,v_0\right)\sin 2t - \frac{1}{12}\sin 4t$.

Using $\sin 2\theta = 2\sin theta \cos\theta$ we can write $\sin 4t = 2\sin 2t \cos 2t$ and so

$$x(t) = \left(\frac{1}{6} + \frac{1}{2}\,v_0\right)\sin 2t - \frac{1}{6}\,\sin 2t \cos 2t = \sin 2t\left(\frac{1}{6} + \frac{1}{2}\,v_0 - \frac{1}{6}\,\cos 2t\right)$$

The first positive solution $t_1$ of $x(t) = 0$ is then obtained from $\sin 2t = 0$ or $t_1 = \pi/2$. Thes the solution of the initial-value problemis defined on $[0, \pi/2]$.

**(b)** The solution of the differential equation

$$\frac{d^2x}{dt^2} + x = \sin 4t$$

is $x(t) = c_3 \cos t + c_4 \sin t - \frac{1}{15} \sin 4t$. The initial conditions $x(\pi/2) = 0$, $x'(\pi/2) = -\frac{2}{3} - v_0$ give $c_4 = 0$ and $c_3 = \frac{2}{5} + v_0$. So

$$x(t) = \left(\frac{2}{5} + v_0\right) \cos t - \frac{1}{15} \sin 4t.$$

By rewriting:

$$\begin{aligned} x(t) &= \left(\frac{2}{5} + v_0\right) \cos t - \frac{1}{15} \sin 4t = \left(\frac{2}{5} + v_0\right) \cos t - \frac{2}{15} \sin 2t \cos 2t \\ &= \left(\frac{2}{5} + v_0\right) \cos t - \frac{4}{15} \sin t \cos t \cos 2t = \cos t \left(\frac{2}{5} + v_0 - \frac{4}{15} \sin t \cos 2t\right) \end{aligned}$$

we see $x(t) = 0$ for $t = 3\pi/2$. Thus the solution is defined on $[\pi/2, 3\pi/2]$.

**(c)** Because the mass is again below the equilibrium position $x = 0$ the differential equation in the next initial-value problem is again

$$\frac{d^2x}{dt^2} + 4x = \sin 4t$$

and its solution is $x(t) = c_5 \cos 2t + c_6 \sin 2t - \frac{1}{12} \sin 4t$. The intial condition are $x(3\pi/2) = 0$, $x'(3\pi/2) = \frac{2}{15} + v_0$ and give $c_5 = 0$ and $c_6 = -\frac{7}{30} - \frac{1}{2} v_0$. So the solution is

$$x(t) = -\left(\frac{7}{30} + \frac{1}{2} v_0\right) \sin 2t - \frac{1}{12} \sin 4t.$$

From the alternative form

$$\begin{aligned} x(t) &= \left(-\frac{7}{30} - \frac{1}{2} v_0\right) \sin 2t - \frac{1}{12} \sin 4t = \left(-\frac{7}{30} - \frac{1}{2} v_0\right) \sin 2t - \frac{1}{6} \sin 2t \cos 2t \\ &= \sin 2t \left(-\frac{7}{30} - \frac{1}{2} v_0 - \frac{1}{6} \cos 2t\right) \end{aligned}$$

we see that $x(t) = 0$ for $t_3 = 2\pi$. The the solution is defined on $[3\pi/2, 2\pi]$.

**(d)** Proceeding as in part (b) we find $x(t) = c_7 \cos t + c_8 \sin t - \frac{1}{15} \sin 4t$ The initial conditions $x(2\pi) = 0$, $x'(2\pi) = -\frac{4}{5} - v_0$ give $c_7 = 0$ and $c_8 = -\frac{8}{15} - v_0$. So

$$x(t) = -\left(\frac{8}{15} + v_0\right) \sin t - \frac{1}{15} \sin 4t.$$

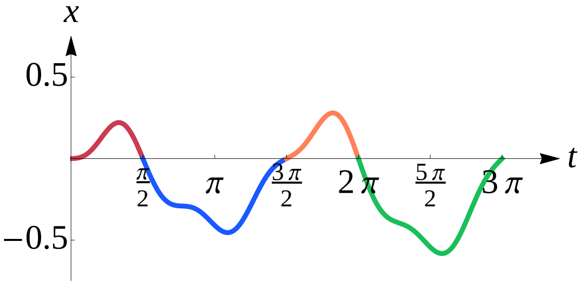
From the form $x(t) = \sin\left(-\frac{8}{15} - v_0 - \frac{4}{15} \sin t \cos 2t\right)$ we see $x(t) = 0$ when $t_4 = 3\pi$. The solution is defined on $[2\pi, 3\pi]$.

**(e)** The velocity of the mass at the beginning of the first cycle is $x'(0) = v_0$, the velocity at the beginning of the second cycle is $x'(3\pi/2) = \frac{2}{15} + v_0$, the velocity at the beginning of the third cycle is $x'(3\pi) = \frac{4}{15} + v_0$. and so on. It stands to reason that the amplitudes of oscillation will increase because the velocity of the mass is increasing by $\frac{2}{15}$ at the beginning of each cycle.

**(f)** For $v_0 = 0.01$ the graph of the piecewise-defined function

$$x(t) = \begin{cases} \left(\dfrac{1}{6} + 0.005\right)\sin 2t - \dfrac{1}{12}\sin 4t, & 0 \le t \le \pi/2 \\ \left(\dfrac{2}{5} + 0.01\right)\cos t - \dfrac{1}{15}\sin 4t, & \pi/2 < t \le 3\pi/2 \\ -\left(\dfrac{7}{30} + 0.005\right)\sin 2t - \dfrac{1}{12}\sin 4t, & 3\pi/2 < t \le 2\pi \\ -\left(\dfrac{8}{15} + 0.01\right)\sin t - \dfrac{1}{15}\sin 4t, & 2\pi < t \le 3\pi \end{cases}$$

is shown below.



# Chapter 6

# Series Solutions of Linear Equations

## 6.1 Review of Power Series

**1.** $\lim\limits_{n\to\infty}\dfrac{a_{n+1}}{a_n}=\lim\limits_{n\to\infty}\left|\dfrac{x^{n+1}/(n+1)}{x^n/n}\right|=\lim\limits_{n\to\infty}\dfrac{n}{n+1}|x|=|x|$

The series is absolutely convergent on $(-1,1)$. At $x=-1$, the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ is the harmonic series which diverges. At $x=1$, the series $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ converges by the alternating series test. Thus, the given series converges on $(-1,1]$, and the radius of converegence is 1.

**2.** $\lim\limits_{n\to\infty}\dfrac{a_{n+1}}{a_n}=\lim\limits_{n\to\infty}\left|\dfrac{x^{n+1}/(n+1)^2}{x^n/n^2}\right|=\lim\limits_{n\to\infty}\dfrac{n}{n+1}|x|=|x|$

The series is absolutely convergent on $(-1,1)$. At $x=-1$, the series $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$ converges by the alternating series test. At $x=1$, the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ is a convergent $p$-series. Thus, the given series converges on $[-1,1]$, and the radius of converegence is 1.

**3.** $\lim\limits_{n\to\infty}\dfrac{a_{n+1}}{a_n}=\lim\limits_{n\to\infty}\left|\dfrac{2^{n+1}x^{n+1}/(n+1)}{2^nx^n/n}\right|=\lim\limits_{n\to\infty}\dfrac{2n}{n+1}|x|=2|x|$

The series is absolutely convergent for $2|x|<1$ or $|x|<\frac{1}{2}$. The radius of convergence is $R=\frac{1}{2}$. At $x=-\frac{1}{2}$, the series $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ converges by the alternating series test. At $x=\frac{1}{2}$, the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ is the harmonic series which diverges. Thus, the given series converges on $[-\frac{1}{2},\frac{1}{2})$, and the radius of convergence is $\frac{1}{2}$.

**4.** $\lim\limits_{n\to\infty}\dfrac{a_{n+1}}{a_n}=\lim\limits_{n\to\infty}\left|\dfrac{5^{n+1}x^{n+1}/(n+1)!}{5^nx^n/n!}\right|=\lim\limits_{n\to\infty}\dfrac{5}{n+1}|x|=0$

The series is absolutely convergent on $(-\infty,\infty)$, and the radius of convergence is $R=\infty$.

**5.** $$\lim_{k\to\infty}\frac{a_{k+1}}{a_k}=\lim_{k\to\infty}\left|\frac{(x-5)^{k+1}/10^{k+1}}{(x-5)^k/10^k}\right|=\lim_{k\to\infty}\frac{1}{10}|x-5|=\frac{1}{10}|x-5|$$

The series is absolutely convergent for $\frac{1}{10}|x-5|<1$, $|x-5|<10$, or on $(-5,15)$. At $x=-5$, the series $\sum_{k=1}^{\infty}(-1)^k\frac{(-10)^n}{10^n}=\sum_{k=1}^{\infty}1$ diverges by the $n$th term test. At $x=15$, the series $\sum_{k=1}^{\infty}(-1)^k\frac{10^k}{10^k}=\sum_{k=1}^{\infty}(-1)^k$ diverges by the $n$th term test. Thus, the series converges on $(-5,15)$, and the radius of convergence is 10.

**6.** $$\lim_{k\to\infty}\frac{a_{k+1}}{a_k}=\lim_{k\to\infty}\left|\frac{(k+1)!(x-1)^{k+1}}{k!(x-1)^k}\right|=\lim_{k\to\infty}(k+1)|x-1|=\begin{cases}\infty, & x\neq 1\\ 0, & x=1\end{cases}$$

The radius of convergence is $R=0$ and the series converges only for $x=1$.

**7.** $$\lim_{k\to\infty}\frac{a_{k+1}}{a_k}=\lim_{k\to\infty}\left|\frac{(3x-1)^{n+1}/\left[(n+1)^2+(n+1)\right]}{(3x-1)^n/(n^2+n)}\right|=\lim_{k\to\infty}\frac{n^2+n}{n^2+3n+2}|3x-1|=|3x-1|$$

The series is absolutely convergent for $|3x-1|<1$ or on $(0,2/3)$. At $x=0$, the series $\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2+k}$ converges by the alternating series test. At $x=2/3$, the series $\sum_{k=1}^{\infty}\frac{1}{k^2+k}$ converges by comparison with the $p$-series $\sum_{k=1}^{\infty}\frac{1}{k^2}$. Thus, the series converges on $[0,2/3]$, and the radius of convergence is $1/3$.

**8.** $$\lim_{k\to\infty}\frac{a_{k+1}}{a_k}=\lim_{k\to\infty}\left|\frac{(4x-5)^{n+1}/3^{n+1}}{(4x-5)^n/3^n}\right|=\lim_{k\to\infty}\frac{1}{3}|4x-5|=\frac{1}{3}|4x-5|$$

The series is absolutely convergent for $\frac{1}{3}|4x-5|<1$, $|4x-5|<3$ or on $(1/2,2)$. At $x=\frac{1}{2}$, the series $\sum_{k=0}^{\infty}\frac{(-3)^k}{3^k}=\sum_{k=0}^{\infty}(-1)^k$ diverges by the $n$th term test. At $x=2$, the series $\sum_{k=0}^{\infty}\frac{3^k}{3^k}=\sum_{k=0}1$ diverges by $n$th term test. Thus, the series converges on $(1/2,2)$, and the radius of convergence is $5/4$.

**9.** Write the series as $\sum_{k=1}^{\infty}\left(\frac{32}{75}\right)^k x^k$.Then

$$\lim_{k\to\infty}\frac{a_{k+1}}{a_k}=\lim_{k\to\infty}\left|\frac{(32/75)^{k+1}x^{k+1}}{(32/75)^k x^k}\right|=\lim_{k\to\infty}\frac{32}{75}|x|=\frac{32}{75}|x|$$

The series is absolutely convergent for $\frac{32}{75}|x|<1$, or on $(-75/32,75/32)$. At $x=-75/32$, the series $\sum_{k=1}^{\infty}(-1)^k$ diverges by the $n$th term test. At $x=75/32$, the series $\sum_{k=1}1$ diverges by $n$th term test. Thus, the series converges on $(-75/32,75/32)$, and the radius of convergence is $75/32$.

**10.** $\lim\limits_{k\to\infty} \dfrac{a_{k+1}}{a_k} = \lim\limits_{k\to\infty} \left| \dfrac{x^{2(k+1)+1}/9^{k+1}}{x^{2n+1}/9^k} \right| = \lim\limits_{k\to\infty} \dfrac{1}{9}x^2 = \dfrac{1}{9}x^2$

The series is absolutely convergent for $\frac{1}{9}x^2 < 1$, or on $(-3, 3)$. At $x = -3$, the series $\sum\limits_{k=1}^{\infty} (-1)^k(-3)$ diverges by the $n$th term test. At $x = 3$, the series $\sum\limits_{k=1} (-1)^k 3$ diverges by $n$th term test. Thus, the series converges on $(-3, 3)$, and the radius of convergence is 3.

**11.** We replace $x$ by $-x/2$ in the Maclaurin series of $e^x$.

$$e^{-x/2} = \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{-x}{2} \right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!2^n}\, x^n$$

**12.** We replace $x$ by $3x$ in the Maclaurin series of $e^x$ and multiply the result by $x$.

$$xe^{3x} = x \cdot \sum_{n=0}^{\infty} \frac{1}{n!}(3x)^n = \sum_{n=0}^{\infty} \frac{3^n}{n!}\, x^{n+1}$$

**13.** We factor out a $\dfrac{1}{2}$ and replace $x$ by $-\dfrac{x}{2}$ in the Maclaurin series of $\dfrac{1}{1-x}$.

$$\frac{1}{2+x} = \frac{1}{2} \cdot \frac{1}{1-(-x/2)} = \frac{1}{2}\left[1 + \left(-\frac{x}{2}\right) + \left(-\frac{x}{2}\right)^2 + \cdots\right] = \frac{1}{2}\left[1 - \frac{x}{2} + \frac{x^2}{2^2} - \cdots\right]$$

$$= \frac{1}{2} - \frac{x}{2^2} + \frac{x^2}{2^3} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}}\, x^n$$

**14.** We replace $x$ by $-x^2$ in the Maclaurin series of $\frac{1}{1-x}$ and multiply the result by $x$.

$$\frac{x}{1+x^2} = x \cdot \frac{1}{1-(-x^2)} = x\left[1 + \left(-x^2\right) + \left(-x^2\right)^2 + \left(-x^2\right)^3 + \cdots\right]$$

$$= x - x^3 + x^5 - x^7 + \cdots = \sum_{n=0}^{\infty} (-1)^n x^{2n+1}$$

**15.** We replace $x$ by $-x$ in the Maclaurin series of $\ln(1+x)$.

$$\ln(1-x) = -x - \frac{(-x)^2}{2} + \frac{(-x)^3}{3} + \cdots = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots = \sum_{n=1}^{\infty} \frac{-1}{n}\, x^n$$

**16.** We replace $x$ by $x^2$ in the Maclaurin series of $\sin x$.

$$\sin x^2 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(x^2\right)^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\, x^{4n+2}$$

**17.** By periodicity $\sin x = \sin[(x - 2\pi) + 2\pi] = \sin(x - 2\pi)$, so

$$\sin x = \sin(x - 2\pi) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\,(x - 2\pi)^{2n+1}$$

**18.** We first note that

$$\ln x = \ln\left[2\left(1+\frac{x-2}{2}\right)\right] = \ln 2 + \ln\left(1+\frac{x-2}{2}\right).$$

Next use the Maclaurin series of $\ln(1+x)$ with $x$ replaces by $\dfrac{x-2}{2}$.

$$\begin{aligned}\ln x &= \ln 2 + \frac{x-2}{2} - \frac{1}{2}\left(\frac{x-2}{2}\right)^2 + \frac{1}{3}\left(\frac{x-2}{2}\right)^3 - \frac{1}{4}\left(\frac{x-2}{2}\right)^4 + \cdots \\ &= \ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n2^n}(x-2)^n\end{aligned}$$

**19.** $$\begin{aligned}\sin x\cos x &= \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots\right)\left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots\right) \\ &= x - \frac{2x^3}{3} + \frac{2x^5}{15} - \frac{4x^7}{315} + \cdots\end{aligned}$$

**20.** $$e^{-x}\cos x = \left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \cdots\right)\left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots\right) = 1 - x + \frac{x^3}{3} - \frac{x^4}{6} + \cdots$$

**21.** $$\sec x = \frac{1}{\cos x} = \frac{1}{1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots} = 1 + \frac{x^2}{2!} + \frac{5x^4}{4!} + \frac{61x^6}{6!} + \cdots$$

Since $\cos(\pi/2) = \cos(-\pi/2) = 0$, the series converges on $(-\pi/2, \pi/2)$.

**22.** $$\tan x = \frac{\sin x}{\cos x} = \frac{x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots}{1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots} = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \cdots$$

Since $\cos(\pi/2) = \cos(-\pi/2) = 0$, the series converges on $(-\pi/2, \pi/2)$.

**23.** Let $k = n+2$ so that $n = k-2$ and

$$\sum_{n=1}^{\infty} nc_n x^{n+2} = \sum_{k=3}^{\infty}(k-2)c_{k-2}x^k.$$

**24.** Let $k = n-3$ so that $n = k+3$ and

$$\sum_{n=3}^{\infty}(2n-1)c_n x^{n-3} = \sum_{k=0}^{\infty}(2k+5)c_{k+3}x^k.$$

**25.** In the first summation let $k = n-1$ so that $n = k+1$ and

$$\sum_{n=1}^{\infty} nc_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n = \sum_{k=0}^{\infty}(k+1)\,c_{k+1}x^k - \sum_{k=0}^{\infty} c_k x^k = \sum_{n=0}^{\infty}\left[(k+1)c_{k+1} - c_k\right]x^k.$$

**26.** In the first summation let $k = n - 1$ and in the second summation let $k = n + 2$. Then $n = k + 1$ in the first summation, $n = k - 2$ in the second summation, and

$$\begin{aligned}\sum_{n=1}^{\infty} nc_n x^{n-1} + 3\sum_{n=0}^{\infty} c_n x^{n+2} &= \sum_{n=0}^{\infty} (k+1)c_{k+1}x^k + 3\sum_{k=2}^{\infty} c_{k-2}x^k \\ &= c_2 + 2c_2 x + \sum_{k=2}^{\infty} (k+1)c_{k+1}x^k + 3\sum_{k=2}^{\infty} c_{k-2}x^k \\ &= c_1 + 2c_2 x + \sum_{k=2}^{\infty} \left[(k+1)c_{k+1} + 3c_{k-2}\right] x^k.\end{aligned}$$

**27.** In the first summation let $k = n - 1$ and in the second summation let $k = n + 1$. Then $n = k + 1$ in the first summation, $n = k - 1$ in the second summation, and

$$\begin{aligned}\sum_{n=1}^{\infty} 2nc_n x^{n-1} + \sum_{n=0}^{\infty} 6c_n x^{n+1} &= 2 \cdot 1 \cdot c_1 x^0 + \sum_{n=2}^{\infty} 2nc_n x^{n-1} + \sum_{n=0}^{\infty} 6c_n x^{n+1} \\ &= 2c_1 + \sum_{k=1}^{\infty} 2(k+1)c_{k+1}x^k + \sum_{k=1}^{\infty} 6c_{k-1}x^k \\ &= 2c_1 + \sum_{k=1}^{\infty} [2(k+1)c_{k+1} + 6c_{k-1}]x^k\end{aligned}$$

**28.** In the first summation let $k = n - 2$ and in the second summation let $k = n + 2$. Then $n = k + 2$ in the first summation, $n = k - 2$ in the second summation, and

$$\begin{aligned}\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^{n+2} &= \sum_{k=0}^{\infty} (k+2)(k+1)c_{k+2}x^k + \sum_{k=2}^{\infty} c_{k-2}x^k \\ &= 2c_2 + 6c_3 x + \sum_{k=2}^{\infty} (k+2)(k+1)c_{k+2}x^k + \sum_{k=2}^{\infty} c_{k-2}x^k \\ &= 2c_2 + 6c_3 x + \sum_{k=2}^{\infty} [(k+2)(k+1)c_{k+2} + c_{k-2}]x^k\end{aligned}$$

**29.** In the first summation let $k = n-2$ and in the second and third summations let $k = n$. Then $n = k+2$ in the first summation, $n = k$ in the second and third summations, and

$$\begin{aligned}\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - 2\sum_{n=1}^{\infty} nc_n x^n + \sum_{n=0}^{\infty} c_n x^n\\ &= \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k - \sum_{k=1}^{\infty} 2kc_k x^k + \sum_{k=0}^{\infty} c_k x^k\\ &= 2c_2 + 6c_3 x + \sum_{k=2}^{\infty}(k+2)(k+1)c_{k+2}x^k + \sum_{k=2}^{\infty} c_{k-2}x^k\\ &= c_0 + 2c_2 + \sum_{k=1}^{\infty}[(k+2)(k+1)c_{k+2} - (2k-1)c_k]x^k = 0\end{aligned}$$

**30.** In the first and third summations let $k = n$ and in the second summation let $k = n-2$. Then $n = k$ in the first and third summations, $n = k+2$ in the second summation, and

$$\begin{aligned}&\sum_{n=2}^{\infty} n(n-1)c_n x^n + 2\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + 3\sum_{n=1}^{\infty} nc_n x^n\\ &= 2\cdot 2\cdot 1c_2 x^0 + 2\cdot 3\cdot 2c_3 x^1 + 3\cdot 1\cdot c_1 x^1\\ &\qquad + \sum_{n=2}^{\infty} n(n-1)c_n x^n + 2\sum_{n=4}^{\infty} n(n-1)c_n x^{n-2} + 3\sum_{n=2}^{\infty} nc_n x^n\\ &= 4c_2 + (3c_1 + 12c_3)x + \sum_{k=2}^{\infty} k(k-1)c_k x^k + 2\sum_{k=2}^{\infty}(k+2)(k+1)c_{k+2}x^k + 3\sum_{k=2}^{\infty} kc_k x^k\\ &= 4c_2 + (3c_1 + 12c_3)x + \sum_{k=2}^{\infty}\left[(k(k-1) + 3k)\,c_k + 2(k+2)(k+1)c_{k+2}\right]x^k\\ &= 4c_2 + (3c_1 + 12c_3)x + \sum_{k=2}^{\infty}\left[k(k+2)c_k + 2(k+1)(k+2)c_{k+2}\right]x^k\end{aligned}$$

**31.** Since $y' = \displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n 2n}{n!}x^{2n-1}$, we have

$$\begin{aligned}y' + 2xy &= \sum_{n=1}^{\infty}\frac{(-1)^n 2n}{n!}x^{2n-1} + 2x\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}x^{2n} = 2\left[\underbrace{\sum_{n=1}^{\infty}\frac{(-1)^n}{(n-1)!}x^{2n-1}}_{k=n} + \underbrace{\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}x^{2n+1}}_{k=n+1}\right]\\ &= 2\left[\sum_{k=1}^{\infty}\frac{(-1)^k}{(k-1)!}x^{2k-1} + \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(k-1)!}x^{2k-1}\right] = 2\sum_{k=1}^{\infty}\left[\frac{(-1)^k}{(k-1)!} + \frac{(-1)^{k-1}}{(k-1)!}\right]x^{2k-1}\\ &= 2\sum_{k=1}^{\infty}\left[\frac{(-1)^k}{(k-1)!} - \frac{(-1)^k}{(k-1)!}\right]x^{2k-1} = 0\end{aligned}$$

**32.** Since $y' = \sum_{n=1}^{\infty}(-1)^n 2nx^{2n-1}$, we have

$$\left(1+x^2\right)y' + 2xy$$

$$= \left(1+x^2\right)\sum_{n=1}^{\infty}(-1)^n 2nx^{2n-1} + 2x\sum_{n=0}^{\infty}(-1)^n x^{2n}$$

$$= \sum_{n=1}^{\infty}(-1)^n 2nx^{2n-1} + \sum_{n=1}^{\infty}(-1)^n 2nx^{2n+1} + 2x\sum_{n=0}^{\infty}(-1)^n x^{2n}$$

$$= \left[\underbrace{\sum_{n=1}^{\infty}(-1)^n x^{2n-1}}_{k=n} + \underbrace{\sum_{n=1}^{\infty}(-1)^n x^{2n+1}}_{k=n+1} + \underbrace{\sum_{n=1}^{\infty}2(-1)^n x^{2n+1}}_{k=n+1}\right]$$

$$= \sum_{k=1}^{\infty}(-1)^k x^{2k-1} + \sum_{k=2}^{\infty}(-1)^{k-1}(2k-2)x^{2k-1} + \sum_{k=1}^{\infty}2(-1)^{k-1}x^{2k-1}$$

$$= -2x + \sum_{k=2}^{\infty}(-1)^k 2kx^{2k-1} + \sum_{k=2}^{\infty}(-1)^{k-1}(2k-2)x^{2k-1} + 2x + \sum_{k=2}^{\infty}2(-1)^{k-1}x^{2k-1}$$

$$= \sum_{k=2}^{\infty}\left[(-1)^k 2k + (-1)^{k-1}2k\right]x^{2k-1} = \sum_{k=2}^{\infty}\left[(-1)^k 2k - (-1)^k 2k\right]x^{2k-1} = 0$$

**33.** In this problem we must take special care with starting values for the indices of summation. Normally when a power series is given in summation notation, successive derivatives of the power series of the unknown function start with an index that is one higher than the preceding one. In this case, the power series starts with $n = 1$ to avoid division by zero. From the first derivative on this is no longer necessary and the index of summation starts again with $n = 1$ for $y'$. To justify this to yourself you could simply write out the first few term of the power series for $y$.

Since $y' = \sum_{n=1}^{\infty}(-1)^{n+1}x^{n-1}$ and $y'' = \sum_{n=2}^{\infty}(-1)^{n+1}(n-1)x^{n-2}$

$$(x+1)y'' + y' = (x+1)\sum_{n=2}^{\infty}(-1)^{n+1}(n-1)x^{n-2} + \sum_{n=1}^{\infty}(-1)^{n+1}x^{n-1}$$

$$= \sum_{n=2}^{\infty}(-1)^{n+1}(n-1)x^{n-1} + \sum_{n=2}^{\infty}(-1)^{n+1}(n-1)x^{n-2} + \sum_{n=1}^{\infty}(-1)^{n+1}x^{n-1}$$

$$= -x^0 + x^0 + \underbrace{\sum_{n=2}^{\infty}(-1)^{n+1}(n-1)x^{n-1}}_{k=n-1} + \underbrace{\sum_{n=3}^{\infty}(-1)^{n+1}(n-1)x^{n-2}}_{k=n-2} + \underbrace{\sum_{n=2}^{\infty}(-1)^{n+1}x^{n-1}}_{k=n-1}$$

$$= \sum_{k=1}^{\infty} (-1)^{k+2} k x^k + \sum_{k=1}^{\infty} (-1)^{k+3} (k+1) x^k + \sum_{k=1}^{\infty} (-1)^{k+2} x^k$$

$$= \sum_{k=1}^{\infty} \left[ (-1)^{k+2} k - (-1)^{k+2} k - (-1)^{k+2} + (-1)^{k+2} \right] x^k = 0$$

**34.** SInce $y' = \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 2n}{2^{2n} (n!)^2} x^{2n-1}$ and $y'' = \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 2n(2n-1)}{2^{2n} (n!)^2} x^{2n-2}$

$$xy'' + y' + xy = \underbrace{\sum_{n=1}^{\infty} \frac{(-1)^n 2n(2n-1)}{2^{2n} (n!)^2} x^{2n-1}}_{k=n} + \underbrace{\sum_{n=1}^{\infty} \frac{(-1)^n 2n}{2^{2n} (n!)^2} x^{2n-1}}_{k=n} + \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n} (n!)^2} x^{2n+1}}_{k=n+1}$$

$$= \sum_{k=1}^{\infty} \left[ \frac{(-1)^k 2k(2k-1)}{2^{2k} (k!)^2} + \frac{(-1)^k 2k}{2^{2k} (k!)^2} + \frac{(-1)^{k-1}}{2^{2k-2} [(k-1)!]^2} \right] x^{2k-1}$$

$$= \sum_{k=1}^{\infty} \left[ \frac{(-1)^k (2k)^2}{2^{2k} (k!)^2} - \frac{(-1)^k}{2^{2k-2} [(k-1)!]^2} \right] x^{2k-1}$$

$$= \sum_{k=1}^{\infty} (-1)^k \left[ \frac{(2k)^2 - 2^2 k^2}{2^{2k} (k!)^2} \right] x^{2k-1} = 0$$

*In Problems 3538 we start with the assumption that* $y = \displaystyle\sum_{n=0}^{\infty} c_n x^n$*, substitute into the differential equation, and finally find some values of* $c_n$*. The solution is then written in terms of elementary functions. (One of the points of power series solutions of differential equations however is that it wont always be possible to express the power series in terms of elementary functions.)*

**35.** Substituting into the differential equation we have

$$y' - 5y = \sum_{n=1}^{\infty} n c_n x^{n-1} - \sum_{n=0}^{\infty} 5 c_n x^n = \sum_{k=0}^{\infty} (k+1) c_{k+1} x^k - \sum_{k=0}^{\infty} 5 c_k x^k$$

$$= \sum_{k=0}^{\infty} \left[ (k+1) c_{k+1} - 5 c_k \right] x^k = 0.$$

Thus $c_{k+1} = \dfrac{5}{k+1}\,c_k$, for $k = 01, 1, 2, \ldots$, and

$$\begin{aligned} c_1 &= \frac{5}{1}\,c_0 = 5c_0 \\ c_2 &= \frac{5}{2}\,c_1 = \frac{5^2}{2}\,c_0 \\ c_3 &= \frac{5}{3}\,c_2 = \frac{5^3}{3\cdot 2}\,c_0 \\ c_4 &= \frac{5}{4}\,c_3 = \frac{5^4}{4\cdot 3\cdot 2}\,c_0 \\ &\vdots \end{aligned}$$

Hence,

$$y = c_0 + 5c_0x + \frac{5^2}{2}\,c_0x^2 + \frac{5^3}{3\cdot 2}\,c_0x^3 + \frac{5^4}{4\cdot 3\cdot 2}\,c_0x^4 + \cdots$$

and

$$y = c_0\sum_{k=0}^{\infty}\frac{1}{k!}\,(5x)^k = c_0e^{5x}.$$

**36.** Substituting into the differential equation we have

$$\begin{aligned} 4y' + y &= 4\sum_{n=1}^{\infty} nc_nx^{n-1} + \sum_{n=0}^{\infty} c_nx^n = 4\sum_{k=0}^{\infty}(k+1)c_{k+1}x^k + \sum_{k=0}^{\infty} c_kx^k \\ &= \sum_{k=0}^{\infty}\left[4(k+1)c_{k+1} + c_k\right]x^k = 0. \end{aligned}$$

Thus $c_{k+1} = -\dfrac{1}{4(k+1)}\,c_k$, for $k = 0, 1, 2, \ldots$, and

$$\begin{aligned} c_1 &= -\frac{1}{4\cdot 1}\,c_0 \\ c_2 &= -\frac{1}{4\cdot 2}\,c_1 = \frac{1}{4^2\cdot 1\cdot 2}\,c_0 \\ c_3 &= -\frac{1}{4\cdot 3}\,c_2 = -\frac{1}{4^3\cdot 1\cdot 2\cdot 3}\,c_0 \\ c_4 &= -\frac{1}{4\cdot 4}\,c_3 = \frac{1}{4^4\cdot 1\cdot 2\cdot 3\cdot 4}c_0 \\ &\vdots \end{aligned}$$

Hence,

$$y = c_0 - \frac{1}{4\cdot 1}\,c_0x + \frac{1}{4^2\cdot 1\cdot 2}\,c_0x^2 - \frac{1}{4^3\cdot 1\cdot 2\cdot 3}\,c_0x^3 + \frac{1}{4^4\cdot 1\cdot 2\cdot 3\cdot 4}\,c_0x^4 - \cdots$$

and

$$y = c_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{4^k k!}\, x^k = c_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(\frac{x}{4}\right)^k = c_0 e^{-x/4}\,.$$

**37.** Substituting into the differential equation we have

$$\begin{aligned} y' - xy &= \sum_{n=1}^{\infty} nc_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^{n+1} = \sum_{k=0}^{\infty} (k+1)c_{k+1}x^k - \sum_{k=1}^{\infty} c_{k-1}x^k \\ &= c_1 + \sum_{k=1}^{\infty} (k+1)c_{k+1}x^k - \sum_{k=1}^{\infty} c_{k+1}x^k \\ &= c_1 + \sum_{k=1}^{\infty} \left[(k+1)c_{k+1} - c_{k-1}\right] x^k = 0. \end{aligned}$$

Thus $c_1 = 0$ and $c_{k+1} = -\dfrac{1}{4(k+1)}\, c_k$, for $k = 0, 1, 2, \ldots$, and

$$\begin{aligned} c_2 &= \frac{1}{2}\, c_0 \\ c_3 &= \frac{1}{3}\, c_1 = 0 \\ c_4 &= \frac{1}{4}\, c_2 = \frac{1}{4}\left(\frac{1}{2}\, c_0\right) = \frac{1}{2^2 2!}\, c_0 \\ c_5 &= \frac{1}{5}\, c_3 = 0 \\ c_6 &= \frac{1}{6}\, c_4 = \frac{1}{6}\left(\frac{1}{2^2 2!}\, c_0\right) = \frac{1}{2^3 3!}\, c_0 \\ c_7 &= \frac{1}{7}\, c_5 = 0 \\ c_8 &= \frac{1}{8}\, c_6 = \frac{1}{8}\left(\frac{1}{2^3 3!}\, c_0\right) = \frac{1}{2^4 4!}\, c_0 \\ &\ \vdots \end{aligned}$$

Hence,

$$\begin{aligned} y &= c_0 + \frac{1}{2}\, c_0 x^2 + \frac{1}{2^2 2!}\, c_0 x^4 + \frac{1}{2^3 3!}\, c_0 x^6 + \frac{1}{2^4 4!}\, c_0 x^8 + \cdots \\ &= c_0 \left[1 + \left(\frac{x^2}{2}\right) + \frac{1}{2!}\left(\frac{x^2}{2}\right)^2 + \frac{1}{3!}\left(\frac{x^2}{2}\right)^3 + \frac{1}{4!}\left(\frac{x^2}{2}\right)^4 + \cdots\right], \end{aligned}$$

and

$$y = c_0 \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{x^2}{2}\right)^k = c_0 e^{x^2/2}.$$

**38.** Substituting into the differential equatoin we have

$$\begin{aligned}(1+x)y'+y &= \sum_{n=1}^{\infty} nc_nx^{n-1} + \sum_{n=1}^{\infty} c_nnx^n + \sum_{n=0}^{\infty} c_nx^n \\ &= \sum_{k=0}^{\infty} (k+1)c_{k+1}x^k + \sum_{k=1}^{\infty} c_kkx^k + \sum_{k=0}^{\infty} c_kx^k \\ &= c_1 + c_0 + \sum_{k=1}^{\infty} (k+1)c_{k+1}x^k + \sum_{k=1}^{\infty} c_kkx^k + \sum_{k=1}^{\infty} c_kx^k \\ &= c_1 + c_0 + \sum_{k=1}^{\infty} \left[(k+1)c_{k+1} + (k+1)c_k\right]x^k = 0\end{aligned}$$

Thus $c_1 + c_0 = 0$ and $c_{k+1} = -c_k$, for $k = 1, 2, 3 \ldots$, so

$$\begin{aligned}c_1 &= -c_0 \\ c_2 &= -c_1 = c_0 \\ c_3 &= -c_2 = -c_0 \\ c_4 &= -c_3 = c_0 \\ &\vdots\end{aligned}$$

Hence,

$$y = c_0 - c_0x + c_0x^2 - c_0x^3 + c_0x^4 - \cdots = c_0\left[1 - x + x^2 - x^3 + x^4 - \cdots\right],$$

and

$$y = c_0\sum_{k=0}^{\infty}(-1)^lx^l = \frac{c_0}{1+x}\,.$$

**39.** From the double-angle formula

$$\sin 2x = 2\sin x\cos x \qquad \text{and} \qquad \sin x\cos x = \frac{1}{2}\sin 2x.$$

Therefore we replace $x$ by $2x$ in the Maclaurin series for $\sin x$. This gives

$$\begin{aligned}\sin x\cos x = \frac{1}{2}\sin 2x &= \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!(2x)^{2n+1}} = \sum_{n=0}^{\infty}\frac{(-4)^n}{(2n+1)!}x^{2n+1} \\ &= x - \frac{2}{3}x^3 + \frac{2}{15}x^5 - \cdots .\end{aligned}$$

**40.** Even though the interval of convergence of $\cos x$ is $(-\infty, \infty)$, the power series for $\sec x$ cannot converge on that interval because $\sec x = 1/\cos x$ is discontinuous at odd integer multiples of $\pi/2$. Since the series is centered at 0 it makes sense, and can be proved, that the interval of convergence is the open interval $(-\pi/2, \pi/2)$.

## 6.2 Solutions About Ordinary Points

**1.** The singular points of $(x^2-25)y''+2xy'+y=0$ are $-5$ and $5$. The distance from 0 to either of these points is 5. The distance from 1 to the closest of these points is 4.

**2.** The singular points of $(x^2-2x+10)y''+xy'-4y=0$ are $1+3i$ and $1-3i$. The distance from 0 to either of these points is $\sqrt{10}$. The distance from 1 to either of these points is 3.

*In Problems 3-6 we use*

$$y=\sum_{n=0}^{\infty}c_nx^n\,, \qquad y=\sum_{n=1}^{\infty}nc_nx^{n-1}\,, \qquad \textit{and} \quad y''=\sum_{n=2}^{\infty}n(n-1)c_nx^{n-2}\,.$$

**3.** We have

$$y''+y=\underbrace{\sum_{n=2}^{\infty}n(n-1)c_nx^{n-2}}_{k=n-2}+\underbrace{\sum_{n=0}^{\infty}c_nx^n}_{k=n}=\sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k+\sum_{k=0}^{\infty}c_kx^k$$

$$=\sum_{k=0}^{\infty}\left[(k+2)(k+1)c_{k+2}+c_k\right]x^k=0.$$

Thus $c_{k+2}=-\dfrac{c_k}{(k+2)(k+1)}$, for $k=0,1,2,\ldots$, and for $k=0,2,4,6,\ldots$, we get

$$c_2=-\frac{c_0}{2!}$$
$$c_4=\frac{c_0}{4!}$$
$$c_6=-\frac{c_0}{6!}$$
$$\vdots$$

For $k=1,3,5,7,\ldots$ we get

$$c_3=-\frac{c_1}{3!}$$
$$c_5=\frac{c_1}{5!}$$
$$c_7=-\frac{c_1}{7!}$$
$$\vdots$$

Hence,

$$y_1(x)=c_0\left[1-\frac{1}{2!}x^2+\frac{1}{4!}x^4-\frac{1}{6!}x^6+\cdots\right]$$

and

$$y_2(x) = c_1 \left[ x - \frac{1}{3!}\,x^3 + \frac{1}{5!}\,x^5 - \frac{1}{7!}\,x^7 + \cdots \right].$$

The solution $y_1(x)$ is recognized as $y_1(x) = c_0 \cos x$, and the solution $y_2(x)$ is recognized as $y_2(x) = c_1 \sin x$.

**4.** We have

$$y'' - y = \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{k=n-2} - \underbrace{\sum_{n=0}^{\infty} c_n x^n}_{k=n} = \sum_{k=0}^{\infty} (k+2)(k+1)c_{k+2}x^k - \sum_{k=0}^{\infty} c_k x^k$$

$$= \sum_{k=0}^{\infty} \left[(k+2)(k+1)c_{k+2} - c_k\right] x^k = 0.$$

Thus $c_{k+2} = \dfrac{c_k}{(k+2)(k+1)}$, for $k = 0, 1, 2, \ldots$, and for $k = 0, 2, 4, 6, \ldots$, we get

$$c_2 = \frac{c_0}{2!}$$
$$c_4 = \frac{c_0}{4!}$$
$$c_6 = \frac{c_0}{6!}$$
$$\vdots$$

For $k = 1, 3, 5, 7, \ldots$ we get

$$c_3 = \frac{c_1}{3!}$$
$$c_5 = \frac{c_1}{5!}$$
$$c_7 = \frac{c_1}{7!}$$
$$\vdots$$

Hence,

$$y_1(x) = c_0 \left[ 1 + \frac{1}{2!}\,x^2 + \frac{1}{4!}\,x^4 + \frac{1}{6!}\,x^6 + \cdots \right]$$

and

$$y_2(x) = c_1 \left[ x + \frac{1}{3!}\,x^3 + \frac{1}{5!}\,x^5 + \frac{1}{7!}\,x^7 + \cdots \right].$$

The solution $y_1(x)$ is recognized as $y_1(x) = c_0 \cosh x$, and the solution $y_2(x)$ is recognized as $y_2(x) = c_1 \sinh x$.

**5.** We have

$$y'' - y' = \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{k=n-2} - \underbrace{\sum_{n=0}^{\infty} nc_n x^{n-1}}_{k=n-1} = \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k - \sum_{k=0}^{\infty}(k+1)c_{k+1}x^k$$

$$= \sum_{k=0}^{\infty}\left[(k+2)(k+1)c_{k+2} - (k+1)c_{k+1}\right]x^k = 0.$$

Thus $c_{k+2} = \dfrac{(k+1)c_{k+1}}{(k+2)(k+1)} = \dfrac{c_{k+1}}{k+2}$, for $k = 0, 1, 2, \ldots$, so

$$\begin{aligned} c_2 &= \frac{c_1}{2!} \\ c_3 &= \frac{c_2}{3} = \frac{c_1}{3!} \\ c_7 &= \frac{c_3}{4} = \frac{c_1}{4!} \\ &\vdots \end{aligned}$$

Hence, the solution of the differential equation is

$$\begin{aligned} y(x) = y_1(x) + y_2(x) &= c_0 + c_1\left[x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots\right] \\ &= c_0 + c_1\left[-1 + 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots\right] \\ &= c_0 - c_1 + c_1\sum_{k=0}^{\infty}\frac{1}{k!}x^k. \end{aligned}$$

The solutions $y_1(x)$ and $y_2(x)$ are recognized as

$$y_1(x) = c_0 \qquad \text{and} \qquad y_2(x) = -c_1 + c_1e^x$$

**6.** We have

$$y'' + 2y' = \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{k=n-2} + 2\underbrace{\sum_{n=0}^{\infty} nc_n x^{n-1}}_{k=n-1} = \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k + \sum_{k=0}^{\infty}2(k+1)c_{k+1}x^k$$

$$= \sum_{k=0}^{\infty}\left[(k+2)(k+1)c_{k+2} + 2(k+1)c_{k+1}\right]x^k = 0.$$

Thus $c_{k+2} = -\dfrac{2(k+1)c_{k+1}}{(k+2)(k+1)} = -\dfrac{2c_{k+1}}{k+2}$, for $k = 0, 1, 2, \ldots$, so

$$\begin{aligned} c_2 &= -\frac{2c_1}{2!} \\ c_3 &= \frac{2^2 c_2}{3} = \frac{2^2 c_1}{3!} \\ c_7 &= -\frac{2^3 c_3}{4} = -\frac{2^3 c_1}{4!} \\ &\vdots \end{aligned}$$

Hence, the solution of the differential equation is

$$\begin{aligned} y(x) = y_1(x) + y_2(x) &= c_0 + c_1\left[x - \frac{2}{2!}x^2 + \frac{2^2}{3!}x^3 - \frac{2^3}{4!}x^4 + \cdots\right] \\ &= c_0 + \frac{1}{2}c_1\left[2x - \frac{2^2}{2!}x^2 + \frac{2^3}{3!}x^3 - \frac{2^4}{4!}x^4 + \cdots\right] \\ &= c_0 + \frac{1}{2}c_1\left[1 - 1 + 2x - \frac{2^2}{2!}x^2 + \frac{2^3}{3!}x^3 - \frac{2^4}{4!}x^4 + \cdots\right] \\ &= c_0 + \frac{1}{2}c_1 - \frac{1}{2}c_1\sum_{k=0}^{\infty}\frac{1}{k!}(2x)^k. \end{aligned}$$

The solutions $y_1(x)$ and $y_2(x)$ are recognized as

$$y_1(x) = c_0 \qquad \text{and} \qquad y_2(x) = \frac{1}{2}c_1 - \frac{1}{2}c_1 e^{-2x}$$

**7.** Substituting $y = \sum_{n=0}^{\infty} c_n x^n$ into the differential equation we have

$$\begin{aligned} y'' + xy &= \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{k=n-2} + \underbrace{\sum_{n=0}^{\infty} c_n x^{n+1}}_{k=n+1} = \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k + \sum_{k=1}^{\infty} c_{k-1}x^k \\ &= 2c_2 + \sum_{k=1}^{\infty}[(k+2)(k+1)c_{k+2} + c_{k-1}]x^k = 0. \end{aligned}$$

Thus

$$\begin{aligned} c_2 &= 0 \\ (k+2)(k+1)c_{k+2} + c_{k-1} &= 0 \end{aligned}$$

and

$$c_{k+2} = -\frac{c_{k-1}}{(k+2)(k+1)}, \quad k = 1, 2, 3, \ldots .$$

Choosing $c_0 = 1$ and $c_1 = 0$ we find

$$c_3 = -\frac{1}{6} \qquad c_4 = c_5 = 0 \qquad c_6 = \frac{1}{180}$$

and so on. For $c_0 = 0$ and $c_1 = 1$ we obtain

$$c_3 = 0 \qquad c_4 = -\frac{1}{12} \qquad c_5 = c_6 = 0 \qquad c_7 = \frac{1}{504}$$

and so on. Thus, two solutions are

$$y_1 = 1 - \frac{1}{6}x^3 + \frac{1}{180}x^6 - \cdots \qquad \text{and} \qquad y_2 = x - \frac{1}{12}x^4 + \frac{1}{504}x^7 - \cdots .$$

**8.** Substituting $y = \sum_{n=0}^{\infty} c_n x^n$ into the differential equation we have

$$y'' + x^2 y = \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{k=n-2} + \underbrace{\sum_{n=0}^{\infty} c_n x^{n+2}}_{k=n+2} = \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k + \sum_{k=2}^{\infty} c_{k-2}x^k$$

$$= 2c_2 + 6c_3 x + \sum_{k=2}^{\infty}[(k+2)(k+1)c_{k+2} + c_{k-2}]x^k = 0.$$

Thus

$$c_2 = c_3 = 0$$
$$(k+2)(k+1)c_{k+2} + c_{k-2} = 0$$

and

$$c_{k+2} = -\frac{1}{(k+2)(k+1)}c_{k-2}, \qquad k = 2, 3, 4, \ldots .$$

Choosing $c_0 = 1$ and $c_1 = 0$ we find

$$c_4 = -\frac{1}{12} \qquad c_5 = c_6 = c_7 = 0 \qquad c_8 = \frac{1}{672}$$

and so on. For $c_0 = 0$ and $c_1 = 1$ we obtain

$$c_4 = 0 \qquad c_5 = -\frac{1}{20} \qquad c_6 = c_7 = c_8 = 0 \qquad c_9 = \frac{1}{1440}$$

and so on. Thus, two solutions are

$$y_1 = 1 - \frac{1}{12}x^4 + \frac{1}{672}x^8 - \cdots \qquad \text{and} \qquad y_2 = x - \frac{1}{20}x^5 + \frac{1}{1440}x^9 - \cdots .$$

**9.** Substituting $y = \sum_{n=0}^{\infty} c_n x^n$ into the differential equation we have

$$\begin{aligned} y'' - 2xy' + y &= \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{k=n-2} - 2\underbrace{\sum_{n=1}^{\infty} nc_n x^n}_{k=n} + \underbrace{\sum_{n=0}^{\infty} c_n x^n}_{k=n} \\ &= \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k - 2\sum_{k=1}^{\infty} kc_k x^k + \sum_{k=0}^{\infty} c_k x^k \\ &= 2c_2 + c_0 + \sum_{k=1}^{\infty}[(k+2)(k+1)c_{k+2} - (2k-1)c_k]x^k = 0. \end{aligned}$$

Thus

$$\begin{aligned} 2c_2 + c_0 &= 0 \\ (k+2)(k+1)c_{k+2} - (2k-1)c_k &= 0 \end{aligned}$$

and

$$\begin{aligned} c_2 &= -\frac{1}{2}c_0 \\ c_{k+2} &= \frac{2k-1}{(k+2)(k+1)}\,c_k, \quad k = 1, 2, 3, \dots\,. \end{aligned}$$

Choosing $c_0 = 1$ and $c_1 = 0$ we find

$$c_2 = -\frac{1}{2} \qquad c_3 = c_5 = c_7 = \cdots = 0 \qquad c_4 = -\frac{1}{8} \qquad c_6 = -\frac{7}{240}$$

and so on. For $c_0 = 0$ and $c_1 = 1$ we obtain

$$c_2 = c_4 = c_6 = \cdots = 0 \qquad c_3 = \frac{1}{6} \qquad c_5 = \frac{1}{24} \qquad c_7 = 1112$$

and so on. Thus, two solutions are

$$y_1 = 1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{7}{240}x^6 - \cdots \qquad \text{and} \qquad y_2 = x + \frac{1}{6}x^3 + \frac{1}{24}x^5 + \frac{1}{112}x^7 + \cdots\,.$$

**10.** Substituting $y = \sum_{n=0}^{\infty} c_n x^n$ into the differential equation we have

$$\begin{aligned} y'' - xy' + 2y &= \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{k=n-2} - \underbrace{\sum_{n=1}^{\infty} nc_n x^n}_{k=n} + 2\underbrace{\sum_{n=0}^{\infty} c_n x^n}_{k=n} \\ &= \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k - \sum_{k=1}^{\infty} kc_k x^k + 2\sum_{k=0}^{\infty} c_k x^k \\ &= 2c_2 + 2c_0 + \sum_{k=1}^{\infty}[(k+2)(k+1)c_{k+2} - (k-2)c_k]x^k = 0. \end{aligned}$$

Thus

$$2c_2 + 2c_0 = 0$$
$$(k+2)(k+1)c_{k+2} - (k-2)c_k = 0$$

and

$$c_2 = -c_0$$
$$c_{k+2} = \frac{k-2}{(k+2)(k+1)}\,c_k, \quad k = 1, 2, 3, \ldots .$$

Choosing $c_0 = 1$ and $c_1 = 0$ we find

$$c_2 = -1 \qquad c_3 = c_5 = c_7 = \cdots = 0 \qquad c_4 = 0 \qquad c_6 = c_8 = c_{10} = \cdots = 0.$$

For $c_0 = 0$ and $c_1 = 1$ we obtain

$$c_2 = c_4 = c_6 = \cdots = 0 \qquad c_3 = -\frac{1}{6} \qquad c_5 = -\frac{1}{120}$$

and so on. Thus, two solutions are

$$y_1 = 1 - x^2 \qquad \text{and} \qquad y_2 = x - \frac{1}{6}x^3 - \frac{1}{120}x^5 - \cdots .$$

**11.** Substituting $y = \sum_{n=0}^{\infty} c_n x^n$ into the differential equation we have

$$y'' + x^2y' + xy = \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{k=n-2} + \underbrace{\sum_{n=1}^{\infty} nc_n x^{n+1}}_{k=n+1} + \underbrace{\sum_{n=0}^{\infty} c_n x^{n+1}}_{k=n+1}$$

$$= \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k + \sum_{k=2}^{\infty}(k-1)c_{k-1}x^k + \sum_{k=1}^{\infty} c_{k-1}x^k$$

$$= 2c_2 + (6c_3 + c_0)x + \sum_{k=2}^{\infty}[(k+2)(k+1)c_{k+2} + kc_{k-1}]x^k = 0.$$

Thus

$$c_2 = 0$$
$$6c_3 + c_0 = 0$$
$$(k+2)(k+1)c_{k+2} + kc_{k-1} = 0$$

and

$$c_2 = 0$$
$$c_3 = -\frac{1}{6}c_0$$
$$c_{k+2} = -\frac{k}{(k+2)(k+1)}\,c_{k-1}, \quad k = 2, 3, 4, \ldots .$$

Choosing $c_0 = 1$ and $c_1 = 0$ we find

$$c_3 = -\frac{1}{6} \qquad c_4 = c_5 = 0 \qquad c_6 = \frac{1}{45}$$

and so on. For $c_0 = 0$ and $c_1 = 1$ we obtain

$$c_3 = 0 \qquad c_4 = -\frac{1}{6} \qquad c_5 = c_6 = 0 \qquad c_7 = \frac{5}{252}$$

and so on. Thus, two solutions are

$$y_1 = 1 - \frac{1}{6}x^3 + \frac{1}{45}x^6 - \cdots \qquad \text{and} \qquad y_2 = x - \frac{1}{6}x^4 + \frac{5}{252}x^7 - \cdots .$$

**12.** Substituting $y = \sum_{n=0}^{\infty} c_n x^n$ into the differential equation we have

$$\begin{aligned} y'' + 2xy' + 2y &= \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{k=n-2} + 2\underbrace{\sum_{n=1}^{\infty} nc_n x^n}_{k=n} + 2\underbrace{\sum_{n=0}^{\infty} c_n x^n}_{k=n} \\ &= \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k + 2\sum_{k=1}^{\infty} kc_k x^k + 2\sum_{k=0}^{\infty} c_k x^k \\ &= 2c_2 + 2c_0 + \sum_{k=1}^{\infty}[(k+2)(k+1)c_{k+2} + 2(k+1)c_k]x^k = 0. \end{aligned}$$

Thus

$$\begin{aligned} 2c_2 + 2c_0 &= 0 \\ (k+2)(k+1)c_{k+2} + 2(k+1)c_k &= 0 \end{aligned}$$

and

$$\begin{aligned} c_2 &= -c_0 \\ c_{k+2} &= -\frac{2}{k+2}\,c_k, \quad k = 1, 2, 3, \dots . \end{aligned}$$

Choosing $c_0 = 1$ and $c_1 = 0$ we find

$$c_2 = -1 \qquad c_3 = c_5 = c_7 = \cdots = 0 \qquad c_4 = \frac{1}{2} \qquad c_6 = -\frac{1}{6}$$

and so on. For $c_0 = 0$ and $c_1 = 1$ we obtain

$$c_2 = c_4 = c_6 = \cdots = 0 \qquad c_3 = -\frac{2}{3} \qquad c_5 = 415 \qquad c_7 = -8105$$

and so on. Thus, two solutions are

$$y_1 = 1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6 + \cdots \quad \text{and} \quad y_2 = x - \frac{2}{3}x^3 + \frac{4}{15}x^5 - \frac{8}{105}x^7 + \cdots .$$

**13.** Substituting $y = \sum_{n=0}^{\infty} c_n x^n$ into the differential equation we have

$$\begin{aligned}(x-1)y'' + y' &= \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-1}}_{k=n-1} - \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{k=n-2} + \underbrace{\sum_{n=1}^{\infty} nc_n x^{n-1}}_{k=n-1} \\ &= \sum_{k=1}^{\infty}(k+1)kc_{k+1}x^k - \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k + \sum_{k=0}^{\infty}(k+1)c_{k+1}x^k \\ &= -2c_2 + c_1 + \sum_{k=1}^{\infty}[(k+1)kc_{k+1} - (k+2)(k+1)c_{k+2} + (k+1)c_{k+1}]x^k = 0.\end{aligned}$$

Thus

$$\begin{gathered}-2c_2 + c_1 = 0 \\ (k+1)^2 c_{k+1} - (k+2)(k+1)c_{k+2} = 0\end{gathered}$$

and

$$\begin{aligned}c_2 &= \frac{1}{2}c_1 \\ c_{k+2} &= \frac{k+1}{k+2}\,c_{k+1}, \quad k = 1, 2, 3, \ldots .\end{aligned}$$

Choosing $c_0 = 1$ and $c_1 = 0$ we find $c_2 = c_3 = c_4 = \cdots = 0$. For $c_0 = 0$ and $c_1 = 1$ we obtain

$$c_2 = \frac{1}{2}, \qquad c_3 = \frac{1}{3}, \qquad c_4 = \frac{1}{4},$$

and so on. Thus, two solutions are

$$y_1 = 1 \qquad \text{and} \qquad y_2 = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \cdots .$$

**14.** Substituting $y = \sum_{n=0}^{\infty} c_n x^n$ into the differential equation we have

$$\begin{aligned}(x+2)y'' + xy' - y &= \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-1}}_{k=n-1} + \underbrace{\sum_{n=2}^{\infty} 2n(n-1)c_n x^{n-2}}_{k=n-2} + \underbrace{\sum_{n=1}^{\infty} nc_n x^n}_{k=n} - \underbrace{\sum_{n=0}^{\infty} c_n x^n}_{k=n} \\ &= \sum_{k=1}^{\infty}(k+1)kc_{k+1}x^k + \sum_{k=0}^{\infty}2(k+2)(k+1)c_{k+2}x^k + \sum_{k=1}^{\infty}kc_kx^k - \sum_{k=0}^{\infty}c_kx^k \\ &= 4c_2 - c_0 + \sum_{k=1}^{\infty}[(k+1)kc_{k+1} + 2(k+2)(k+1)c_{k+2} + (k-1)c_k]\,x^k = 0.\end{aligned}$$

Thus

$$4c_2 - c_0 = 0$$
$$(k+1)kc_{k+1} + 2(k+2)(k+1)c_{k+2} + (k-1)c_k = 0, \quad k = 1, 2, 3, \ldots$$

and

$$c_2 = \frac{1}{4}c_0$$
$$c_{k+2} = -\frac{(k+1)kc_{k+1} + (k-1)c_k}{2(k+2)(k+1)}, \quad k = 1, 2, 3, \ldots .$$

Choosing $c_0 = 1$ and $c_1 = 0$ we find

$$c_1 = 0, \qquad c_2 = \frac{1}{4}, \qquad c_3 = -\frac{1}{24}, \qquad c_4 = 0, \qquad c_5 = \frac{1}{480}$$

and so on. For $c_0 = 0$ and $c_1 = 1$ we obtain

$$c_2 = 0 \qquad c_3 = 0 \qquad c_4 = c_5 = c_6 = \cdots = 0.$$

Thus, two solutions are

$$y_1 = c_0\left[1 + \frac{1}{4}x^2 - \frac{1}{24}x^3 + \frac{1}{480}x^5 + \cdots\right] \qquad \text{and} \qquad y_2 = c_1 x.$$

**15.** Substituting $y = \sum_{n=0}^{\infty} c_n x^n$ into the differential equation we have

$$y'' - (x+1)y' - y = \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{k=n-2} - \underbrace{\sum_{n=1}^{\infty} nc_n x^n}_{k=n} - \underbrace{\sum_{n=1}^{\infty} nc_n x^{n-1}}_{k=n-1} - \underbrace{\sum_{n=0}^{\infty} c_n x^n}_{k=n}$$

$$= \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k - \sum_{k=1}^{\infty} kc_k x^k - \sum_{k=0}^{\infty}(k+1)c_{k+1}x^k - \sum_{k=0}^{\infty} c_k x^k$$

$$= 2c_2 - c_1 - c_0 + \sum_{k=1}^{\infty}[(k+2)(k+1)c_{k+2} - (k+1)c_{k+1} - (k+1)c_k]x^k = 0.$$

Thus

$$2c_2 - c_1 - c_0 = 0$$
$$(k+2)(k+1)c_{k+2} - (k+1)(c_{k+1} + c_k) = 0$$

and

$$c_2 = \frac{c_1 + c_0}{2}$$
$$c_{k+2} = \frac{c_{k+1} + c_k}{k+2}, \quad k = 1, 2, 3, \ldots .$$

Choosing $c_0 = 1$ and $c_1 = 0$ we find

$$c_2 = \frac{1}{2}, \qquad c_3 = \frac{1}{6}, \qquad c_4 = \frac{1}{6},$$

and so on. For $c_0 = 0$ and $c_1 = 1$ we obtain

$$c_2 = \frac{1}{2}, \qquad c_3 = \frac{1}{2}, \qquad c_4 = \frac{1}{4},$$

and so on. Thus, two solutions are

$$y_1 = 1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{6}x^4 + \cdots \qquad \text{and} \qquad y_2 = x + \frac{1}{2}x^2 + \frac{1}{2}x^3 + \frac{1}{4}x^4 + \cdots .$$

**16.** Substituting $y = \sum_{n=0}^{\infty} c_n x^n$ into the differential equation we have

$$\begin{aligned}
\left(x^2+1\right)y'' - 6y &= \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^n}_{k=n} + \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{k=n-2} \underbrace{-6\sum_{n=0}^{\infty} c_n x^n}_{k=n} \\
&= \sum_{k=2}^{\infty} k(k-1)c_k x^k + \sum_{k=0}^{\infty} (k+2)(k+1)c_{k+2}x^k - 6\sum_{k=0}^{\infty} c_k x^k \\
&= 2c_2 - 6c_0 + (6c_3 - 6c_1)x + \sum_{k=2}^{\infty} \left[\left(k^2 - k - 6\right)c_k + (k+2)(k+1)c_{k+2}\right]x^k = 0.
\end{aligned}$$

Thus

$$\begin{aligned}
2c_2 - 6c_0 &= 0 \\
6c_3 - 6c_1 &= 0 \\
(k-3)(k+2)c_k + (k+2)(k+1)c_{k+2} &= 0
\end{aligned}$$

and

$$c_2 = 3c_0 \qquad c_3 = c_1 \qquad c_{k+2} = -\frac{k-3}{k+1}c_k, \quad k = 2, 3, 4, \dots .$$

Choosing $c_0 = 1$ and $c_1 = 0$ we find

$$c_2 = 3 \qquad c_3 = c_5 = c_7 = \cdots = 0 \qquad c_4 = 1 \qquad c_6 = -\frac{1}{5}$$

and so on. For $c_0 = 0$ and $c_1 = 1$ we obtain

$$c_2 = c_4 = c_6 = \cdots = 0 \qquad c_3 = 1 \qquad c_5 = c_7 = c_9 = \cdots = 0.$$

Thus, two solutions are

$$y_1 = 1 + 3x^2 + x^4 - \frac{1}{5}x^6 + \cdots \qquad \text{and} \qquad y_2 = x + x^3.$$

**17.** Substituting $y=\sum_{n=0}^{\infty} c_nx^n$ into the differential equation we have

$$\left(x^2+2\right)y''+3xy'-y=\underbrace{\sum_{n=2}^{\infty}n(n-1)c_nx^n}_{k=n}+2\underbrace{\sum_{n=2}^{\infty}n(n-1)c_nx^{n-2}}_{k=n-2}+3\underbrace{\sum_{n=1}^{\infty}nc_nx^n}_{k=n}-\underbrace{\sum_{n=0}^{\infty}c_nx^n}_{k=n}$$

$$=\sum_{k=2}^{\infty}k(k-1)c_kx^k+2\sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k+3\sum_{k=1}^{\infty}kc_kx^k-\sum_{k=0}^{\infty}c_kx^k$$

$$=(4c_2-c_0)+(12c_3+2c_1)x+\sum_{k=2}^{\infty}\left[2(k+2)(k+1)c_{k+2}+\left(k^2+2k-1\right)c_k\right]x^k=0.$$

Thus

$$\begin{aligned}4c_2-c_0&=0\\12c_3+2c_1&=0\\2(k+2)(k+1)c_{k+2}+\left(k^2+2k-1\right)c_k&=0\end{aligned}$$

and

$$c_2=\frac{1}{4}c_0 \qquad c_3=-\frac{1}{6}c_1 \qquad c_{k+2}=-\frac{k^2+2k-1}{2(k+2)(k+1)}c_k, \quad k=2,3,4,\ldots .$$

Choosing $c_0=1$ and $c_1=0$ we find

$$c_2=\frac{1}{4} \qquad c_3=c_5=c_7=\cdots=0 \qquad c_4=-\frac{7}{96}$$

and so on. For $c_0=0$ and $c_1=1$ we obtain

$$c_2=c_4=c_6=\cdots=0 \qquad c_3=-\frac{1}{6} \qquad c_5=\frac{7}{120}$$

and so on. Thus, two solutions are

$$y_1=1+\frac{1}{4}x^2-\frac{7}{96}x^4+\cdots \qquad \text{and} \qquad y_2=x-\frac{1}{6}x^3+\frac{7}{120}x^5-\cdots .$$

**18.** Substituting $y=\sum_{n=0}^{\infty} c_nx^n$ into the differential equation we have

$$\left(x^2-1\right)y''+xy'-y=\underbrace{\sum_{n=2}^{\infty}n(n-1)c_nx^n}_{k=n}-\underbrace{\sum_{n=2}^{\infty}n(n-1)c_nx^{n-2}}_{k=n-2}+\underbrace{\sum_{n=1}^{\infty}nc_nx^n}_{k=n}-\underbrace{\sum_{n=0}^{\infty}c_nx^n}_{k=n}$$

$$=\sum_{k=2}^{\infty}k(k-1)c_kx^k-\sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k+\sum_{k=1}^{\infty}kc_kx^k-\sum_{k=0}^{\infty}c_kx^k$$

$$=(-2c_2-c_0)-6c_3x+\sum_{k=2}^{\infty}\left[-(k+2)(k+1)c_{k+2}+\left(k^2-1\right)c_k\right]x^k=0.$$

Thus

$$\begin{aligned} -2c_2 - c_0 &= 0 \\ -6c_3 &= 0 \\ -(k+2)(k+1)c_{k+2} + (k-1)(k+1)c_k &= 0 \end{aligned}$$

and

$$c_2 = -\frac{1}{2}c_0 \qquad c_3 = 0 \qquad c_{k+2} = \frac{k-1}{k+2}\,c_k, \quad k = 2, 3, 4, \ldots .$$

Choosing $c_0 = 1$ and $c_1 = 0$ we find

$$c_2 = -\frac{1}{2} \qquad c_3 = c_5 = c_7 = \cdots = 0 \qquad c_4 = -\frac{1}{8}$$

and so on. For $c_0 = 0$ and $c_1 = 1$ we obtain

$$c_2 = c_4 = c_6 = \cdots = 0 \qquad c_3 = c_5 = c_7 = \cdots = 0.$$

Thus, two solutions are

$$y_1 = 1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \cdots \qquad \text{and} \qquad y_2 = x.$$

**19.** Substituting $y = \sum_{n=0}^{\infty} c_n x^n$ into the differential equation we have

$$\begin{aligned} (x-1)y'' - xy' + y &= \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-1}}_{k=n-1} - \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{k=n-2} - \underbrace{\sum_{n=1}^{\infty} nc_n x^n}_{k=n} + \underbrace{\sum_{n=0}^{\infty} c_n x^n}_{k=n} \\ &= \sum_{k=1}^{\infty}(k+1)kc_{k+1}x^k - \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k - \sum_{k=1}^{\infty} kc_k x^k + \sum_{k=0}^{\infty} c_k x^k \\ &= -2c_2 + c_0 + \sum_{k=1}^{\infty}[-(k+2)(k+1)c_{k+2} + (k+1)kc_{k+1} - (k-1)c_k]x^k = 0. \end{aligned}$$

Thus

$$\begin{aligned} -2c_2 + c_0 &= 0 \\ -(k+2)(k+1)c_{k+2} + (k-1)kc_{k+1} - (k-1)c_k &= 0 \end{aligned}$$

and

$$\begin{aligned} c_2 &= \frac{1}{2}c_0 \\ c_{k+2} &= \frac{kc_{k+1}}{k+2} - \frac{(k-1)c_k}{(k+2)(k+1)}, \quad k = 1, 2, 3, \ldots . \end{aligned}$$

Choosing $c_0 = 1$ and $c_1 = 0$ we find

$$c_2 = \frac{1}{2}, \qquad c_3 = \frac{1}{6}, \qquad c_4 = 0,$$

and so on. For $c_0 = 0$ and $c_1 = 1$ we obtain $c_2 = c_3 = c_4 = \cdots = 0$. Thus,

$$y = C_1\left(1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots\right) + C_2 x$$

and

$$y' = C_1\left(x + \frac{1}{2}x^2 + \cdots\right) + C_2.$$

The initial conditions imply $C_1 = -2$ and $C_2 = 6$, so

$$y = -2\left(1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots\right) + 6x = 8x - 2e^x\,.$$

**20.** Substituting $y = \sum_{n=0}^{\infty} c_n x^n$ into the differential equation we have

$$(x+1)y'' - (2-x)y' + y$$

$$= \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-1}}_{k=n-1} + \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{k=n-2} - 2\underbrace{\sum_{n=1}^{\infty} nc_n x^{n-1}}_{k=n-1} + \underbrace{\sum_{n=1}^{\infty} nc_n x^n}_{k=n} + \underbrace{\sum_{n=0}^{\infty} c_n x^n}_{k=n}$$

$$= \sum_{k=1}^{\infty}(k+1)kc_{k+1}x^k + \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k - 2\sum_{k=0}^{\infty}(k+1)c_{k+1}x^k + \sum_{k=1}^{\infty}kc_k x^k + \sum_{k=0}^{\infty}c_k x^k$$

$$= 2c_2 - 2c_1 + c_0 + \sum_{k=1}^{\infty}[(k+2)(k+1)c_{k+2} + (k-2)(k+1)c_{k+1} + (k+1)c_k]x^k = 0.$$

Thus

$$2c_2 - 2c_1 + c_0 = 0$$
$$(k+2)(k+1)c_{k+2} - (k-2)(k+1)c_{k+1} + (k+1)c_k = 0$$

and

$$c_2 = c_1 - \frac{1}{2}c_0 \qquad c_{k+2} = \frac{k-2}{k+2}\,c_{k+1} - \frac{1}{k+2}\,c_k, \quad k = 1, 2, 3, \ldots\,.$$

Choosing $c_0 = 1$ and $c_1 = 0$ we find

$$c_2 = -\frac{1}{2}, \qquad c_3 = \frac{1}{6}, \qquad c_4 = \frac{1}{8},$$

and so on. For $c_0 = 0$ and $c_1 = 1$ we obtain

$$c_2 = 1, \qquad c_3 = -\frac{2}{3}, \qquad c_4 = -\frac{1}{4},$$

and so on. Thus,

$$y = C_1\left(1 - \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{8}x^4 + \cdots\right) + C_2\left(x + x^2 - \frac{2}{3}x^3 - \frac{1}{4}x^4 + \cdots\right)$$

and

$$y' = C_1\left(-x + \frac{1}{2}x^2 + \frac{1}{2}x^3 + \cdots\right) + C_2\left(1 + 2x - 2x^2 - x^3 + \cdots\right).$$

The initial conditions imply $C_1 = 2$ and $C_2 = -1$, so

$$\begin{aligned} y &= 2\left(1 - \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{8}x^4 + \cdots\right) - \left(x + x^2 - \frac{2}{3}x^3 - \frac{1}{4}x^4 + \cdots\right) \\ &= 2 - x - 2x^2 + x^3 + \frac{1}{2}x^4 + \cdots . \end{aligned}$$

**21.** Substituting $y = \sum_{n=0}^{\infty} c_n x^n$ into the differential equation we have

$$\begin{aligned} y'' - 2xy' + 8y &= \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{k=n-2} - 2\underbrace{\sum_{n=1}^{\infty} nc_n x^n}_{k=n} + 8\underbrace{\sum_{n=0}^{\infty} c_n x^n}_{k=n} \\ &= \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k - 2\sum_{k=1}^{\infty} kc_k x^k + 8\sum_{k=0}^{\infty} c_k x^k \\ &= 2c_2 + 8c_0 + \sum_{k=1}^{\infty}[(k+2)(k+1)c_{k+2} + (8-2k)c_k]x^k = 0. \end{aligned}$$

Thus

$$\begin{aligned} 2c_2 + 8c_0 &= 0 \\ (k+2)(k+1)c_{k+2} + (8-2k)c_k &= 0 \end{aligned}$$

and

$$c_2 = -4c_0 \qquad\qquad c_{k+2} = \frac{2(k-4)}{(k+2)(k+1)}c_k, \quad k = 1, 2, 3, \ldots .$$

Choosing $c_0 = 1$ and $c_1 = 0$ we find

$$c_2 = -4 \qquad c_3 = c_5 = c_7 = \cdots = 0 \qquad c_4 = \frac{4}{3} \qquad c_6 = c_8 = c_{10} = \cdots = 0.$$

For $c_0 = 0$ and $c_1 = 1$ we obtain

$$c_2 = c_4 = c_6 = \cdots = 0 \qquad c_3 = -1 \qquad c_5 = \frac{1}{10}$$

and so on. Thus,

$$y = C_1\left(1 - 4x^2 + \frac{4}{3}x^4\right) + C_2\left(x - x^3 + \frac{1}{10}x^5 + \cdots\right)$$

and

$$y' = C_1\left(-8x + \frac{16}{3}x^3\right) + C_2\left(1 - 3x^2 + \frac{1}{2}x^4 + \cdots\right).$$

The initial conditions imply $C_1 = 3$ and $C_2 = 0$, so

$$y = 3\left(1 - 4x^2 + \frac{4}{3}x^4\right) = 3 - 12x^2 + 4x^4.$$

**22.** Substituting $y = \sum_{n=0}^{\infty} c_n x^n$ into the differential equation we have

$$\begin{aligned}(x^2+1)y'' + 2xy' &= \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^n}_{k=n} + \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{k=n-2} + \underbrace{\sum_{n=1}^{\infty} 2nc_n x^n}_{k=n} \\ &= \sum_{k=2}^{\infty} k(k-1)c_k x^k + \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k + \sum_{k=1}^{\infty} 2kc_k x^k \\ &= 2c_2 + (6c_3 + 2c_1)x + \sum_{k=2}^{\infty}\left[k(k+1)c_k + (k+2)(k+1)c_{k+2}\right]x^k = 0.\end{aligned}$$

Thus

$$\begin{aligned}2c_2 &= 0 \\ 6c_3 + 2c_1 &= 0 \\ k(k+1)c_k + (k+2)(k+1)c_{k+2} &= 0\end{aligned}$$

and

$$c_2 = 0 \qquad c_3 = -\frac{1}{3}c_1 \qquad c_{k+2} = -\frac{k}{k+2}c_k, \quad k = 2, 3, 4, \ldots .$$

Choosing $c_0 = 1$ and $c_1 = 0$ we find $c_3 = c_4 = c_5 = \cdots = 0$. For $c_0 = 0$ and $c_1 = 1$ we obtain

$$c_3 = -\frac{1}{3} \qquad c_4 = c_6 = c_8 = \cdots = 0 \qquad c_5 = -\frac{1}{5} \qquad c_7 = \frac{1}{7}$$

and so on. Thus

$$y = C_0 + C_1\left(x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots\right)$$

and

$$y' = c_1\left(1 - x^2 + x^4 - x^6 + \cdots\right).$$

The initial conditions imply $c_0 = 0$ and $c_1 = 1$, so

$$y = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots .$$

**23.** Substituting $y = \sum_{n=0}^{\infty} c_n x^n$ into the differential equation we have

$$\begin{aligned} y'' + (\sin x)y &= \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \cdots\right)\left(c_0 + c_1x + c_2x^2 + \cdots\right) \\ &= \left[2c_2 + 6c_3x + 12c_4x^2 + 20c_5x^3 + \cdots\right] + \left[c_0x + c_1x^2 + \left(c_2 - \frac{1}{6}c_0\right)x^3 + \cdots\right] \\ &= 2c_2 + (6c_3 + c_0)x + (12c_4 + c_1)x^2 + \left(20c_5 + c_2 - \frac{1}{6}c_0\right)x^3 + \cdots = 0. \end{aligned}$$

Thus

$$2c_2 = 0 \qquad 6c_3 + c_0 = 0 \qquad 12c_4 + c_1 = 0 \qquad 20c_5 + c_2 - \frac{1}{6}c_0 = 0$$

and

$$c_2 = 0 \qquad c_3 = -\frac{1}{6}c_0 \qquad c_4 = -\frac{1}{12}c_1 \qquad c_5 = -\frac{1}{20}c_2 + \frac{1}{120}c_0.$$

Choosing $c_0 = 1$ and $c_1 = 0$ we find

$$c_2 = 0, \qquad c_3 = -\frac{1}{6}, \qquad c_4 = 0, \qquad c_5 = \frac{1}{120}$$

and so on. For $c_0 = 0$ and $c_1 = 1$ we obtain

$$c_2 = 0, \qquad c_3 = 0, \qquad c_4 = -\frac{1}{12}, \qquad c_5 = 0$$

and so on. Thus, two solutions are

$$y_1 = 1 - \frac{1}{6}x^3 + \frac{1}{120}x^5 + \cdots \qquad \text{and} \qquad y_2 = x - \frac{1}{12}x^4 + \cdots.$$

**24.** Substituting $y = \sum_{n=0}^{\infty} c_n x^n$ into the differential equation we have

$$\begin{aligned} y'' + e^x y' - y &= \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} \\ &\quad + \left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots\right)\left(c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + \cdots\right) - \sum_{n=0}^{\infty} c_n x^n \\ &= \left[2c_2 + 6c_3x + 12c_4x^2 + 20c_5x^3 + \cdots\right] \\ &\quad + \left[c_1 + (2c_2 + c_1)x + \left(3c_3 + 2c_2 + \frac{1}{2}c_1\right)x^2 + \cdots\right] - \left[c_0 + c_1x + c_2x^2 + \cdots\right] \\ &= (2c_2 + c_1 - c_0) + (6c_3 + 2c_2)x + \left(12c_4 + 3c_3 + c_2 + \frac{1}{2}c_1\right)x^2 + \cdots = 0. \end{aligned}$$

Thus

$$\begin{aligned} 2c_2 + c_1 - c_0 &= 0 \\ 6c_3 + 2c_2 &= 0 \\ 12c_4 + 3c_3 + c_2 + \frac{1}{2}c_1 &= 0 \end{aligned}$$

and

$$c_2 = \frac{1}{2}c_0 - \frac{1}{2}c_1 \qquad c_3 = -\frac{1}{3}c_2 \qquad c_4 = -\frac{1}{4}c_3 + \frac{1}{12}c_2 - \frac{1}{24}c_1.$$

Choosing $c_0 = 1$ and $c_1 = 0$ we find

$$c_2 = \frac{1}{2}, \qquad c_3 = -\frac{1}{6}, \qquad c_4 = 0$$

and so on. For $c_0 = 0$ and $c_1 = 1$ we obtain

$$c_2 = -\frac{1}{2}, \qquad c_3 = \frac{1}{6}, \qquad c_4 = -\frac{1}{24}$$

and so on. Thus, two solutions are

$$y_1 = 1 + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \cdots \qquad \text{and} \qquad y_2 = x - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{24}x^4 + \cdots .$$

**25.** The singular points of $(\cos x)y'' + y' + 5y = 0$ are odd integer multiples of $\pi/2$. The distance from 0 to either $\pm\pi/2$ is $\pi/2$. The singular point closest to 1 is $\pi/2$. The distance from 1 to the closest singular point is then $\pi/2 - 1$.

**26.** Substituting $y = \sum_{n=0}^{\infty} c_n x^n$ into the first differential equation leads to

$$y'' - xy = \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{k=n-2} - \underbrace{\sum_{n=0}^{\infty} c_n x^{n+1}}_{k=n+1} = \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k - \sum_{k=1}^{\infty} c_{k-1}x^k$$

$$= 2c_2 + \sum_{k=1}^{\infty}[(k+2)(k+1)c_{k+2} - c_{k-1}]x^k = 1.$$

Thus

$$\begin{aligned} 2c_2 &= 1 \\ (k+2)(k+1)c_{k+2} - c_{k-1} &= 0 \end{aligned}$$

and

$$c_2 = \frac{1}{2} \qquad c_{k+2} = \frac{c_{k-1}}{(k+2)(k+1)}, \quad k = 1, 2, 3, \ldots .$$

Let $c_0$ and $c_1$ be arbitrary and iterate to find

$$c_2 = \frac{1}{2} \qquad c_3 = \frac{1}{6}c_0 \qquad c_4 = \frac{1}{12}c_1 \qquad c_5 = \frac{1}{20}c_2 = \frac{1}{40}$$

and so on. The solution is

$$\begin{aligned} y &= c_0 + c_1 x + \frac{1}{2}x^2 + \frac{1}{6}c_0 x^3 + \frac{1}{12}c_1 x^4 + \frac{1}{40}c_5 + \cdots \\ &= c_0\left(1 + \frac{1}{6}x^3 + \cdots\right) + c_1\left(x + \frac{1}{12}x^4 + \cdots\right) + \frac{1}{2}x^2 + \frac{1}{40}x^5 + \cdots . \end{aligned}$$

Substituting $y = \sum_{n=0}^{\infty} c_n x^n$ into the second differential equation leads to

$$\begin{aligned} y'' - 4xy' - 4y &= \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{k=n-2} - \underbrace{\sum_{n=1}^{\infty} 4nc_n x^n}_{k=n} - \underbrace{\sum_{n=0}^{\infty} 4c_n x^n}_{k=n} \\ &= \sum_{k=0}^{\infty} (k+2)(k+1)c_{k+2}x^k - \sum_{k=1}^{\infty} 4kc_k x^k - \sum_{k=0}^{\infty} 4c_k x^k \\ &= 2c_2 - 4c_0 + \sum_{k=1}^{\infty} \left[(k+2)(k+1)c_{k+2} - 4(k+1)c_k\right] x^k \\ &= e^x = 1 + \sum_{k=1}^{\infty} \frac{1}{k!}x^k. \end{aligned}$$

Thus

$$\begin{gathered} 2c_2 - 4c_0 = 1 \\ (k+2)(k+1)c_{k+2} - 4(k+1)c_k = \frac{1}{k!} \end{gathered}$$

and

$$\begin{aligned} c_2 &= \frac{1}{2} + 2c_0 \\ c_{k+2} &= \frac{1}{(k+2)!} + \frac{4}{k+2}c_k, \qquad k = 1,\, 2,\, 3,\, \ldots . \end{aligned}$$

Let $c_0$ and $c_1$ be arbitrary and iterate to find

$$\begin{aligned} c_2 &= \frac{1}{2} + 2c_0 \\ c_3 &= \frac{1}{3!} + \frac{4}{3}c_1 = \frac{1}{3!} + \frac{4}{3}c_1 \\ c_4 &= \frac{1}{4!} + \frac{4}{4}c_2 = \frac{1}{4!} + \frac{1}{2} + 2c_0 = \frac{13}{4!} + 2c_0 \\ c_5 &= \frac{1}{5!} + \frac{4}{5}c_3 = \frac{1}{5!} + \frac{4}{5\cdot 3!} + \frac{16}{15}c_1 = \frac{17}{5!} + \frac{16}{15}c_1 \\ c_6 &= \frac{1}{6!} + \frac{4}{6}c_4 = \frac{1}{6!} + \frac{4\cdot 13}{6\cdot 4!} + \frac{8}{6}c_0 = \frac{261}{6!} + \frac{4}{3}c_0 \\ c_7 &= \frac{1}{7!} + \frac{4}{7}c_5 = \frac{1}{7!} + \frac{4\cdot 17}{7\cdot 5!} + \frac{64}{105}c_1 = \frac{409}{7!} + \frac{64}{105}c_1 \end{aligned}$$

and so on. The solution is

$$\begin{aligned} y &= c_0 + c_1x + \left(\frac{1}{2} + 2c_0\right)x^2 + \left(\frac{1}{3!} + \frac{4}{3}c_1\right)x^3 + \left(\frac{13}{4!} + 2c_0\right)x^4 + \left(\frac{17}{5!} + \frac{16}{15}c_1\right)x^5 \\ &\quad + \left(\frac{261}{6!} + \frac{4}{3}c_0\right)x^6 + \left(\frac{409}{7!} + \frac{64}{105}c_1\right)x^7 + \cdots \\ &= c_0\left[1 + 2x^2 + 2x^4 + \frac{4}{3}x^6 + \cdots\right] + c_1\left[x + \frac{4}{3}x^3 + \frac{16}{15}x^5 + \frac{64}{105}x^7 + \cdots\right] \\ &\quad + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{13}{4!}x^4 + \frac{17}{5!}x^5 + \frac{261}{6!}x^6 + \frac{409}{7!}x^7 + \cdots . \end{aligned}$$

**27.** We identify $P(x) = 0$ and $Q(x) = \sin x/x$. The Taylor series representation for $\sin x/x$ is $1 - x^2/3! + x^4/5! - \cdots$, for $|x| < \infty$. Thus, $Q(x)$ is analytic at $x = 0$ and $x = 0$ is an ordinary point of the differential equation.

**28.** Since $\sqrt{x}$ is continuous at $x = 0$, but derivatives of all orders are discontinuous at this point, $x = 0$ is a singular point of the differential equation; not an ordinary point.

**29.** **(a)** Substituting $y = \sum_{n=0}^{\infty} c_nx^n$ into the differential equation we have

$$\begin{aligned} y'' + xy' + y &= \underbrace{\sum_{n=2}^{\infty} n(n-1)c_nx^{n-2}}_{k=n-2} + \underbrace{\sum_{n=1}^{\infty} nc_nx^n}_{k=n} + \underbrace{\sum_{n=0}^{\infty} c_nx^n}_{k=n} \\ &= \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k + \sum_{k=1}^{\infty} kc_kx^k + \sum_{k=0}^{\infty} c_kx^k \\ &= (2c_2 + c_0) + \sum_{k=1}^{\infty}\left[(k+2)(k+1)c_{k+2} + (k+1)c_k\right]x^k = 0. \end{aligned}$$

Thus

$$\begin{gathered} 2c_2 + c_0 = 0 \\ (k+2)(k+1)c_{k+2} + (k+1)c_k = 0 \end{gathered}$$

and

$$c_2 = -\frac{1}{2}c_0 \qquad\qquad c_{k+2} = -\frac{1}{k+2}c_k, \quad k = 1, 2, 3, \dots .$$

Choosing $c_0 = 1$ and $c_1 = 0$ we find

$$\begin{aligned} c_2 &= -\frac{1}{2} \\ c_3 &= c_5 = c_7 = \cdots = 0 \\ c_4 &= -\frac{1}{4}\left(-\frac{1}{2}\right) = \frac{1}{2^2 \cdot 2} \\ c_6 &= -\frac{1}{6}\left(\frac{1}{2^2 \cdot 2}\right) = -\frac{1}{2^3 \cdot 3!} \end{aligned}$$

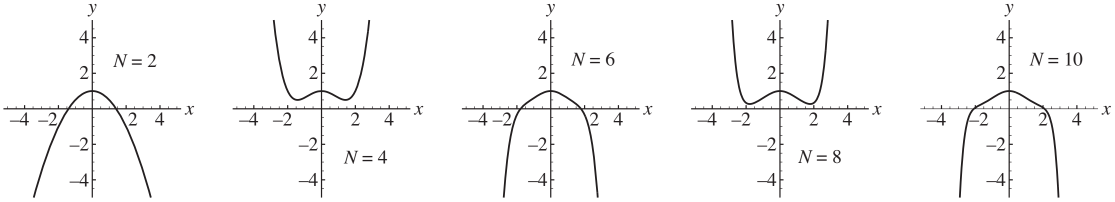
and so on. For $c_0 = 0$ and $c_1 = 1$ we obtain

$$\begin{aligned} c_2 &= c_4 = c_6 = \cdots = 0 \\ c_3 &= -\frac{1}{3} = -\frac{2}{3!} \\ c_5 &= -\frac{1}{5}\left(-\frac{1}{3}\right) = \frac{1}{5 \cdot 3} = \frac{4 \cdot 2}{5!} \\ c_7 &= -\frac{1}{7}\left(\frac{4 \cdot 2}{5!}\right) = -\frac{6 \cdot 4 \cdot 2}{7!} \end{aligned}$$

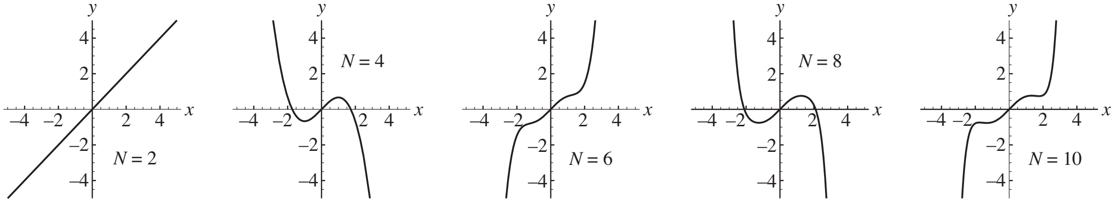
and so on. Thus, two solutions are

$$y_1 = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k \cdot k!} x^{2k} \qquad \text{and} \qquad y_2 = \sum_{k=0}^{\infty} \frac{(-1)^k 2^k k!}{(2k+1)!} x^{2k+1}.$$

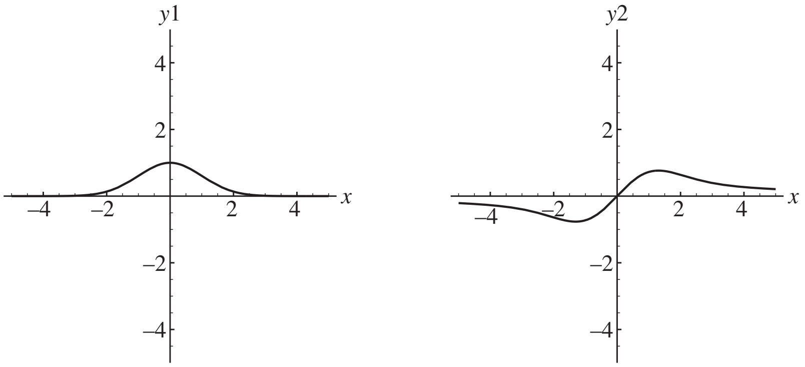
**(b)** For $y_1$, $S_3 = S_2$ and $S_5 = S_4$, so we plot $S_2$, $S_4$, $S_6$, $S_8$, and $S_{10}$.



For $y_2$, $S_3 = S_4$ and $S_5 = S_6$, so we plot $S_2$, $S_4$, $S_6$, $S_8$, and $S_{10}$.



**(c)**



The graphs of $y_1$ and $y_2$ obtained from a numerical solver are shown. We see that the partial sum representations indicate the even and odd natures of the solution, but don't really give a very accurate representation of the true solution. Increasing $N$ to about 20 gives a much more accurate representation on $[-4, 4]$.

**(d)** From $e^x = \sum_{k=0}^{\infty} x^k/k!$ we see that $e^{-x^2/2} = \sum_{k=0}^{\infty} (-x^2/2)^k/k! = \sum_{k=0}^{\infty} (-1)^k x^{2k}/2^k k!$. From (5) of Section 3.2 we have

$$y_2 = y_1 \int \frac{e^{-\int x\,dx}}{y_1^2}\,dx = e^{-x^2/2} \int \frac{e^{-x^2/2}}{(e^{-x^2/2})^2}\,dx = e^{-x^2/2} \int \frac{e^{-x^2/2}}{e^{-x^2}}\,dx = e^{-x^2/2} \int e^{x^2/2}\,dx$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k k!}\,x^{2k} \int \sum_{k=0}^{\infty} \frac{1}{2^k k!}\,x^{2k}\,dx = \left(\sum_{k=0}^{\infty} \frac{(-1)^k}{2^k k!}\,x^{2k}\right)\left(\sum_{k=0}^{\infty} \int \frac{1}{2^k k!}\,x^{2k}\,dx\right)$$

$$= \left(\sum_{k=0}^{\infty} \frac{(-1)^k}{2^k k!}\,x^{2k}\right)\left(\sum_{k=0}^{\infty} \frac{1}{(2k+1)2^k k!}\,x^{2k+1}\right)$$

$$= \left(1 - \frac{1}{2}x^2 + \frac{1}{2^2 \cdot 2}x^4 - \frac{1}{2^3 \cdot 3!}x^6 + \cdots\right)\left(x + \frac{1}{3 \cdot 2}x^3 + \frac{1}{5 \cdot 2^2 \cdot 2}x^5 + \frac{1}{7 \cdot 2^3 \cdot 3!}x^7 + \cdots\right)$$

$$= x - \frac{2}{3!}x^3 + \frac{4 \cdot 2}{5!}x^5 - \frac{6 \cdot 4 \cdot 2}{7!}x^7 + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k 2^k k!}{(2k+1)!}\,x^{2k+1}.$$

**30. (a)** We have

$$y'' + (\cos x)y = 2c_2 + 6c_3 x + 12c_4 x^2 + 20c_5 x^3 + 30c_6 x^4 + 42c_7 x^5 + \cdots$$

$$+ \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)(c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + \cdots)$$

$$= (2c_2 + c_0) + (6c_3 + c_1)x + \left(12c_4 + c_2 - \frac{1}{2}c_0\right)x^2 + \left(20c_5 + c_3 - \frac{1}{2}c_1\right)x^3$$

$$+ \left(30c_6 + c_4 + \frac{1}{24}c_0 - \frac{1}{2}c_2\right)x^4 + \left(42c_7 + c_5 + \frac{1}{24}c_1 - \frac{1}{2}c_3\right)x^5 + \cdots.$$

Then

$$30c_6 + c_4 + \frac{1}{24}c_0 - \frac{1}{2}c_2 = 0 \qquad \text{and} \qquad 42c_7 + c_5 + \frac{1}{24}c_1 - \frac{1}{2}c_3 = 0,$$

which gives $c_6 = -c_0/80$ and $c_7 = -19c_1/5040$. Thus

$$y_1(x) = 1 - \frac{1}{2}x^2 + \frac{1}{12}x^4 - \frac{1}{80}x^6 + \cdots$$

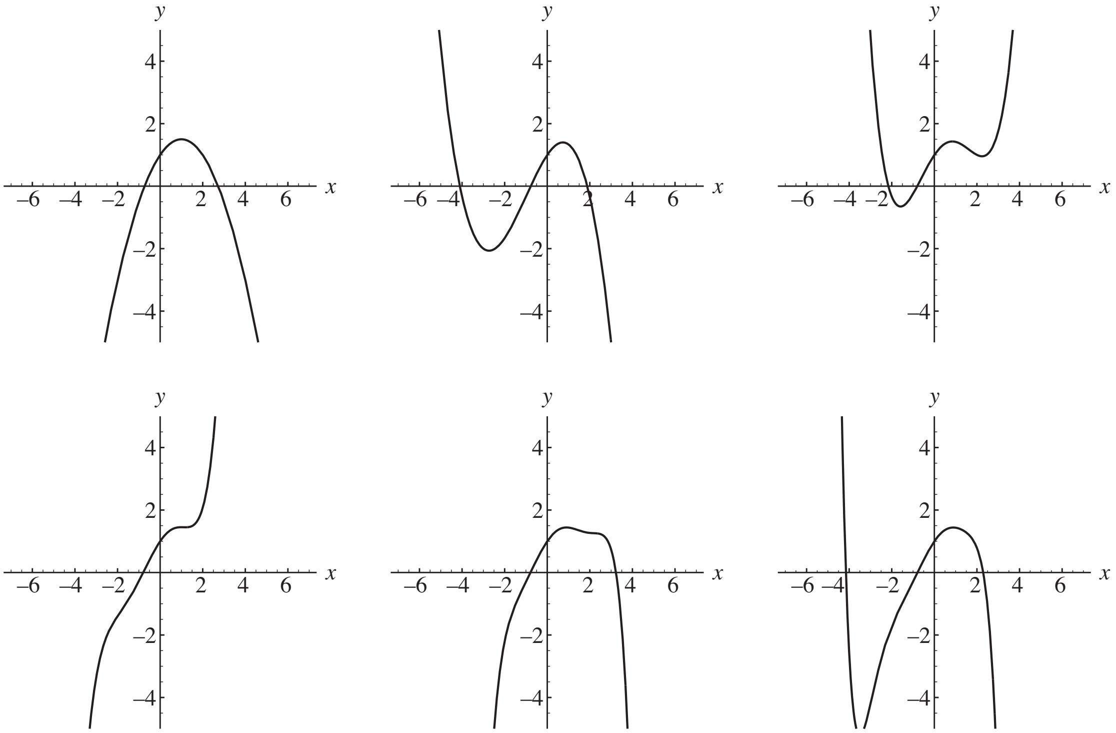
and

$$y_2(x) = x - \frac{1}{6}x^3 + \frac{1}{30}x^5 - \frac{19}{5040}x^7 + \cdots .$$

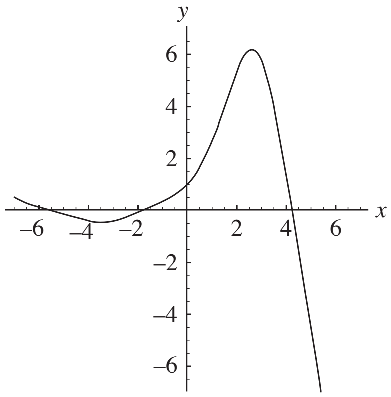
**(b)** From part (a) the general solution of the differential equation is $y = c_1y_1 + c_2y_2$. Then $y(0) = c_1 + c_2 \cdot 0 = c_1$ and $y'(0) = c_1 \cdot 0 + c_2 = c_2$, so the solution of the initial-value problem is

$$y = y_1 + y_2 = 1 + x - \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{1}{30}x^5 - \frac{1}{80}x^6 - \frac{19}{5040}x^7 + \cdots .$$

**(c)**



**(d)**



## 6.3 Solutions About Singular Points

**1.** Irregular singular point: $x = 0$

**2.** Regular singular points: $x = 0, -3$

**3.** Irregular singular point: $x = 3$; regular singular point: $x = -3$

**4.** Irregular singular point: $x = 1$; regular singular point: $x = 0$

**5.** Regular singular points: $x = 0, \pm 2i$

**6.** Irregular singular point: $x = 5$; regular singular point: $x = 0$

**7.** Regular singular points: $x = -3, 2$

**8.** Regular singular points: $x = 0, \pm i$

**9.** Irregular singular point: $x = 0$; regular singular points: $x = 2, \pm 5$

**10.** Irregular singular point: $x = -1$; regular singular points: $x = 0, 3$

**11.** Writing the differential equation in the form

$$y'' + \frac{5}{x-1}\,y' + \frac{x}{x+1}\,y = 0$$

we see that $x_0 = 1$ and $x_0 = -1$ are regular singular points. For $x_0 = 1$ the differential equation can be put in the form

$$(x-1)^2y'' + 5(x-1)y' + \frac{x(x-1)^2}{x+1}\,y = 0.$$

In this case $p(x) = 5$ and $q(x) = x(x-1)^2/(x+1)$. For $x_0 = -1$ the differential equation can be put in the form

$$(x+1)^2y'' + 5(x+1)\frac{x+1}{x-1}\,y' + x(x+1)y = 0.$$

In this case $p(x) = (x+1)/(x-1)$ and $q(x) = x(x+1)$.

**12.** Writing the differential equation in the form

$$y'' + \frac{x+3}{x}\,y' + 7xy = 0$$

we see that $x_0 = 0$ is a regular singular point. Multiplying by $x^2$, the differential equation can be put in the form

$$x^2y'' + x(x+3)y' + 7x^3y = 0.$$

We identify $p(x) = x + 3$ and $q(x) = 7x^3$.

**13.** We identify $P(x) = 5/3x + 1$ and $Q(x) = -1/3x^2$, so that $p(x) = xP(x) = \frac{5}{3} + x$ and $q(x) = x^2Q(x) = -\frac{1}{3}$. Then $a_0 = \frac{5}{3}$, $b_0 = -\frac{1}{3}$, and the indicial equation is

$$r(r-1) + \frac{5}{3}r - \frac{1}{3} = r^2 + \frac{2}{3}r - \frac{1}{3} = \frac{1}{3}(3r^2 + 2r - 1) = \frac{1}{3}(3r-1)(r+1) = 0.$$

The indicial roots are $\frac{1}{3}$ and $-1$. Since these do not differ by an integer we expect to find two series solutions using the method of Frobenius.

**14.** We identify $P(x) = 1/x$ and $Q(x) = 10/x$, so that $p(x) = xP(x) = 1$ and $q(x) = x^2Q(x) = 10x$. Then $a_0 = 1$, $b_0 = 0$, and the indicial equation is

$$r(r-1) + r = r^2 = 0.$$

The indicial roots are 0 and 0. Since these are equal, we expect the method of Frobenius to yield a single series solution.

**15.** Substituting $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ into the differential equation and collecting terms, we obtain

$$2xy'' - y' + 2y = \left(2r^2 - 3r\right) c_0 x^{r-1} + \sum_{k=1}^{\infty} \left[2(k+r-1)(k+r)c_k - (k+r)c_k + 2c_{k-1}\right] x^{k+r-1}$$
$$= 0,$$

which implies

$$2r^2 - 3r = r(2r-3) = 0$$

and

$$(k+r)(2k+2r-3)c_k + 2c_{k-1} = 0.$$

The indicial roots are $r = 0$ and $r = 3/2$. For $r = 0$ the recurrence relation is

$$c_k = -\frac{2c_{k-1}}{k(2k-3)}, \quad k = 1, 2, 3, \ldots,$$

and

$$c_1 = 2c_0, \qquad c_2 = -2c_0, \qquad c_3 = \frac{4}{9}c_0,$$

and so on. For $r = 3/2$ the recurrence relation is

$$c_k = -\frac{2c_{k-1}}{(2k+3)k}, \quad k = 1, 2, 3, \ldots,$$

and

$$c_1 = -\frac{2}{5}c_0, \qquad c_2 = \frac{2}{35}c_0, \qquad c_3 = -\frac{4}{945}c_0,$$

and so on. The general solution on $(0, \infty)$ is

$$y = C_1\left(1 + 2x - 2x^2 + \frac{4}{9}x^3 + \cdots\right) + C_2 x^{3/2}\left(1 - \frac{2}{5}x + \frac{2}{35}x^2 - \frac{4}{945}x^3 + \cdots\right).$$

**16.** Substituting $y=\sum_{n=0}^{\infty}c_nx^{n+r}$ into the differential equation and collecting terms, we obtain

$$\begin{aligned}2xy''+5y'+xy&=\left(2r^2+3r\right)c_0x^{r-1}+\left(2r^2+7r+5\right)c_1x^r\\&\qquad+\sum_{k=2}^{\infty}[2(k+r)(k+r-1)c_k+5(k+r)c_k+c_{k-2}]x^{k+r-1}\\&=0,\end{aligned}$$

which implies

$$2r^2+3r=r(2r+3)=0,$$
$$\left(2r^2+7r+5\right)c_1=0,$$

and

$$(k+r)(2k+2r+3)c_k+c_{k-2}=0.$$

The indicial roots are $r=-3/2$ and $r=0$, so $c_1=0$ . For $r=-3/2$ the recurrence relation is

$$c_k=-\frac{c_{k-2}}{(2k-3)k}\,,\quad k=2,3,4,\dots,$$

and

$$c_2=-\frac{1}{2}c_0,\qquad c_3=0,\qquad c_4=\frac{1}{40}c_0,$$

and so on. For $r=0$ the recurrence relation is

$$c_k=-\frac{c_{k-2}}{k(2k+3)}\,,\quad k=2,3,4,\dots,$$

and

$$c_2=-\frac{1}{14}c_0,\qquad c_3=0,\qquad c_4=\frac{1}{616}c_0,$$

and so on. The general solution on $(0,\infty)$ is

$$y=C_1x^{-3/2}\left(1-\frac{1}{2}x^2+\frac{1}{40}x^4+\cdots\right)+C_2\left(1-\frac{1}{14}x^2+\frac{1}{616}x^4+\cdots\right).$$

**17.** Substituting $y=\sum_{n=0}^{\infty}c_nx^{n+r}$ into the differential equation and collecting terms, we obtain

$$\begin{aligned}4xy''+\frac{1}{2}y'+y&=\left(4r^2-\frac{7}{2}r\right)c_0x^{r-1}+\sum_{k=1}^{\infty}\left[4(k+r)(k+r-1)c_k+\frac{1}{2}(k+r)c_k+c_{k-1}\right]x^{k+r-1}\\&=0,\end{aligned}$$

which implies

$$4r^2-\frac{7}{2}r=r\left(4r-\frac{7}{2}\right)=0$$

and

$$\frac{1}{2}(k+r)(8k+8r-7)c_k + c_{k-1} = 0.$$

The indicial roots are $r = 0$ and $r = 7/8$. For $r = 0$ the recurrence relation is

$$c_k = -\frac{2c_{k-1}}{k(8k-7)}, \qquad k = 1, 2, 3, \ldots,$$

and

$$c_1 = -2c_0, \qquad c_2 = \frac{2}{9}c_0, \qquad c_3 = -\frac{4}{459}c_0,$$

and so on. For $r = 7/8$ the recurrence relation is

$$c_k = -\frac{2c_{k-1}}{(8k+7)k}, \qquad k = 1, 2, 3, \ldots,$$

and

$$c_1 = -\frac{2}{15}c_0, \qquad c_2 = 2345c_0, \qquad c_3 = -\frac{4}{32,085}c_0,$$

and so on. The general solution on $(0, \infty)$ is

$$y = C_1\left(1 - 2x + \frac{2}{9}x^2 - \frac{4}{459}x^3 + \cdots\right) + C_2x^{7/8}\left(1 - \frac{2}{15}x + \frac{2}{345}x^2 - \frac{4}{32{,}085}x^3 + \cdots\right).$$

**18.** Substituting $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ into the differential equation and collecting terms, we obtain

$$\begin{aligned} 2x^2y'' - xy' + \left(x^2+1\right)y &= \left(2r^2 - 3r + 1\right)c_0x^r + \left(2r^2 + r\right)c_1x^{r+1} \\ &\quad + \sum_{k=2}^{\infty}[2(k+r)(k+r-1)c_k - (k+r)c_k + c_k + c_{k-2}]x^{k+r} \\ &= 0, \end{aligned}$$

which implies

$$\begin{aligned} 2r^2 - 3r + 1 = (2r-1)(r-1) &= 0, \\ \left(2r^2 + r\right)c_1 &= 0, \end{aligned}$$

and

$$[(k+r)(2k+2r-3)+1]c_k + c_{k-2} = 0.$$

The indicial roots are $r = 1/2$ and $r = 1$, so $c_1 = 0$. For $r = 1/2$ the recurrence relation is

$$c_k = -\frac{c_{k-2}}{k(2k-1)}, \qquad k = 2, 3, 4, \ldots,$$

and

$$c_2 = -\frac{1}{6}c_0, \qquad c_3 = 0, \qquad c_4 = \frac{1}{168}c_0,$$

and so on. For $r = 1$ the recurrence relation is

$$c_k = -\frac{c_{k-2}}{k(2k+1)}, \quad k = 2, 3, 4, \dots,$$

and

$$c_2 = -\frac{1}{10}c_0, \qquad c_3 = 0, \qquad c_4 = \frac{1}{360}c_0,$$

and so on. The general solution on $(0, \infty)$ is

$$y = C_1 x^{1/2}\left(1 - \frac{1}{6}x^2 + \frac{1}{168}x^4 + \cdots\right) + C_2 x\left(1 - \frac{1}{10}x^2 + \frac{1}{360}x^4 + \cdots\right).$$

**19.** Substituting $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ into the differential equation and collecting terms, we obtain

$$\begin{aligned}3xy'' + (2-x)y' - y &= \left(3r^2 - r\right)c_0 x^{r-1} \\ &\quad + \sum_{k=1}^{\infty}[3(k+r-1)(k+r)c_k + 2(k+r)c_k - (k+r)c_{k-1}]x^{k+r-1} \\ &= 0,\end{aligned}$$

which implies

$$3r^2 - r = r(3r-1) = 0$$

and

$$(k+r)(3k+3r-1)c_k - (k+r)c_{k-1} = 0.$$

The indicial roots are $r = 0$ and $r = 1/3$. For $r = 0$ the recurrence relation is

$$c_k = \frac{c_{k-1}}{3k-1}, \quad k = 1, 2, 3, \dots,$$

and

$$c_1 = \frac{1}{2}c_0, \qquad c_2 = \frac{1}{10}c_0, \qquad c_3 = \frac{1}{80}c_0,$$

and so on. For $r = 1/3$ the recurrence relation is

$$c_k = \frac{c_{k-1}}{3k}, \quad k = 1, 2, 3, \dots,$$

and

$$c_1 = \frac{1}{3}c_0, \qquad c_2 = \frac{1}{18}c_0, \qquad c_3 = \frac{1}{162}c_0,$$

and so on. The general solution on $(0, \infty)$ is

$$y = C_1\left(1 + \frac{1}{2}x + \frac{1}{10}x^2 + \frac{1}{80}x^3 + \cdots\right) + C_2 x^{1/3}\left(1 + \frac{1}{3}x + \frac{1}{18}x^2 + \frac{1}{162}x^3 + \cdots\right).$$

**20.** Substituting $y=\sum_{n=0}^{\infty}c_nx^{n+r}$ into the differential equation and collecting terms, we obtain

$$x^2y''-\left(x-\frac{2}{9}\right)y=\left(r^2-r+\frac{2}{9}\right)c_0x^r+\sum_{k=1}^{\infty}\left[(k+r)(k+r-1)c_k+\frac{2}{9}c_k-c_{k-1}\right]x^{k+r}$$
$$=0,$$

which implies

$$r^2-r+\frac{2}{9}=\left(r-\frac{2}{3}\right)\left(r-\frac{1}{3}\right)=0$$

and

$$\left[(k+r)(k+r-1)+\frac{2}{9}\right]c_k-c_{k-1}=0.$$

The indicial roots are $r=2/3$ and $r=1/3$. For $r=2/3$ the recurrence relation is

$$c_k=\frac{3c_{k-1}}{3k^2+k}, \quad k=1,2,3,\ldots,$$

and

$$c_1=\frac{3}{4}c_0, \qquad c_2=\frac{9}{56}c_0, \qquad c_3=\frac{9}{560}c_0,$$

and so on. For $r=1/3$ the recurrence relation is

$$c_k=\frac{3c_{k-1}}{3k^2-k}, \quad k=1,2,3,\ldots,$$

and

$$c_1=\frac{3}{2}c_0, \qquad c_2=\frac{9}{20}c_0, \qquad c_3=\frac{9}{160}c_0,$$

and so on. The general solution on $(0,\infty)$ is

$$y=C_1x^{2/3}\left(1+\frac{3}{4}x+\frac{9}{56}x^2+\frac{9}{560}x^3+\cdots\right)+C_2x^{1/3}\left(1+\frac{3}{2}x+\frac{9}{20}x^2+\frac{9}{160}x^3+\cdots\right).$$

**21.** Substituting $y=\sum_{n=0}^{\infty}c_nx^{n+r}$ into the differential equation and collecting terms, we obtain

$$2xy''-(3+2x)y'+y=\left(2r^2-5r\right)c_0x^{r-1}+\sum_{k=1}^{\infty}[2(k+r)(k+r-1)c_k$$
$$-3(k+r)c_k-2(k+r-1)c_{k-1}+c_{k-1}]x^{k+r-1}$$
$$=0,$$

which implies

$$2r^2-5r=r(2r-5)=0$$

and

$$(k+r)(2k+2r-5)c_k-(2k+2r-3)c_{k-1}=0.$$

The indicial roots are $r=0$ and $r=5/2$. For $r=0$ the recurrence relation is

$$c_k=\frac{(2k-3)c_{k-1}}{k(2k-5)}, \quad k=1,2,3,\dots,$$

and

$$c_1=\frac{1}{3}c_0, \qquad c_2=-\frac{1}{6}c_0, \qquad c_3=-\frac{1}{6}c_0,$$

and so on. For $r=5/2$ the recurrence relation is

$$c_k=\frac{2(k+1)c_{k-1}}{k(2k+5)}, \quad k=1,2,3,\dots,$$

and

$$c_1=\frac{4}{7}c_0, \qquad c_2=\frac{4}{21}c_0, \qquad c_3=\frac{32}{693}c_0,$$

and so on. The general solution on $(0,\infty)$ is

$$y=C_1\left(1+\frac{1}{3}x-\frac{1}{6}x^2-\frac{1}{6}x^3+\cdots\right)+C_2x^{5/2}\left(1+\frac{4}{7}x+\frac{4}{21}x^2+\frac{32}{693}x^3+\cdots\right).$$

**22.** Substituting $y=\sum_{n=0}^{\infty}c_nx^{n+r}$ into the differential equation and collecting terms, we obtain

$$\begin{aligned}x^2y''+xy'+\left(x^2-\frac{4}{9}\right)y&=\left(r^2-\frac{4}{9}\right)c_0x^r+\left(r^2+2r+\frac{5}{9}\right)c_1x^{r+1}\\&\quad+\sum_{k=2}^{\infty}\left[(k+r)(k+r-1)c_k+(k+r)c_k-\frac{4}{9}c_k+c_{k-2}\right]x^{k+r}\\&=0,\end{aligned}$$

which implies

$$r^2-\frac{4}{9}=\left(r+\frac{2}{3}\right)\left(r-\frac{2}{3}\right)=0,$$

$$\left(r^2+2r+\frac{5}{9}\right)c_1=0,$$

and

$$\left[(k+r)^2-\frac{4}{9}\right]c_k+c_{k-2}=0.$$

The indicial roots are $r=-2/3$ and $r=2/3$, so $c_1=0$. For $r=-2/3$ the recurrence relation is

$$c_k=-\frac{9c_{k-2}}{3k(3k-4)}, \quad k=2,3,4,\dots,$$

and

$$c_2 = -\frac{3}{4}c_0, \qquad c_3 = 0, \qquad c_4 = \frac{9}{128}c_0,$$

and so on. For $r = 2/3$ the recurrence relation is

$$c_k = -\frac{9c_{k-2}}{3k(3k+4)}, \quad k = 2, 3, 4, \ldots,$$

and

$$c_2 = -\frac{3}{20}c_0, \qquad c_3 = 0, \qquad c_4 = \frac{9}{1{,}280}c_0,$$

and so on. The general solution on $(0, \infty)$ is

$$y = C_1x^{-2/3}\left(1 - \frac{3}{4}x^2 + \frac{9}{128}x^4 + \cdots\right) + C_2x^{2/3}\left(1 - \frac{3}{20}x^2 + \frac{9}{1{,}280}x^4 + \cdots\right).$$

**23.** Substituting $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ into the differential equation and collecting terms, we obtain

$$\begin{aligned} 9x^2y'' + 9x^2y' + 2y &= \left(9r^2 - 9r + 2\right)c_0x^r \\ &\qquad + \sum_{k=1}^{\infty}[9(k+r)(k+r-1)c_k + 2c_k + 9(k+r-1)c_{k-1}]x^{k+r} \\ &= 0, \end{aligned}$$

which implies

$$9r^2 - 9r + 2 = (3r-1)(3r-2) = 0$$

and

$$[9(k+r)(k+r-1) + 2]c_k + 9(k+r-1)c_{k-1} = 0.$$

The indicial roots are $r = 1/3$ and $r = 2/3$. For $r = 1/3$ the recurrence relation is

$$c_k = -\frac{(3k-2)c_{k-1}}{k(3k-1)}, \quad k = 1, 2, 3, \ldots,$$

and

$$c_1 = -\frac{1}{2}c_0, \qquad c_2 = \frac{1}{5}c_0, \qquad c_3 = -\frac{7}{120}c_0,$$

and so on. For $r = 2/3$ the recurrence relation is

$$c_k = -\frac{(3k-1)c_{k-1}}{k(3k+1)}, \quad k = 1, 2, 3, \ldots,$$

and

$$c_1 = -\frac{1}{2}c_0, \qquad c_2 = \frac{5}{28}c_0, \qquad c_3 = -\frac{1}{21}c_0,$$

and so on. The general solution on $(0, \infty)$ is

$$y = C_1x^{1/3}\left(1 - \frac{1}{2}x + \frac{1}{5}x^2 - \frac{7}{120}x^3 + \cdots\right) + C_2x^{2/3}\left(1 - \frac{1}{2}x + \frac{5}{28}x^2 - \frac{1}{21}x^3 + \cdots\right).$$

**24.** Substituting $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ into the differential equation and collecting terms, we obtain

$$\begin{aligned} 2x^2y'' + 3xy' + (2x-1)y &= \left(2r^2 + r - 1\right) c_0 x^r \\ &\quad + \sum_{k=1}^{\infty} [2(k+r)(k+r-1)c_k + 3(k+r)c_k - c_k + 2c_{k-1}]x^{k+r} \\ &= 0, \end{aligned}$$

which implies

$$2r^2 + r - 1 = (2r-1)(r+1) = 0$$

and

$$[(k+r)(2k+2r+1) - 1]c_k + 2c_{k-1} = 0.$$

The indicial roots are $r = -1$ and $r = 1/2$. For $r = -1$ the recurrence relation is

$$c_k = -\frac{2c_{k-1}}{k(2k-3)}, \quad k = 1, 2, 3, \ldots,$$

and

$$c_1 = 2c_0, \qquad c_2 = -2c_0, \qquad c_3 = \frac{4}{9}c_0,$$

and so on. For $r = 1/2$ the recurrence relation is

$$c_k = -\frac{2c_{k-1}}{k(2k+3)}, \quad k = 1, 2, 3, \ldots,$$

and

$$c_1 = -\frac{2}{5}c_0, \qquad c_2 = \frac{2}{35}c_0, \qquad c_3 = -\frac{4}{945}c_0,$$

and so on. The general solution on $(0, \infty)$ is

$$y = C_1 x^{-1}\left(1 + 2x - 2x^2 + \frac{4}{9}x^3 + \cdots\right) + C_2 x^{1/2}\left(1 - \frac{2}{5}x + \frac{2}{35}x^2 - \frac{4}{945}x^3 + \cdots\right).$$

**25.** Substituting $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ into the differential equation and collecting terms, we obtain

$$\begin{aligned} xy'' + 2y' - xy &= \left(r^2 + r\right) c_0 x^{r-1} + \left(r^2 + 3r + 2\right) c_1 x^r \\ &\quad + \sum_{k=2}^{\infty} [(k+r)(k+r-1)c_k + 2(k+r)c_k - c_{k-2}]x^{k+r-1} \\ &= 0, \end{aligned}$$

which implies

$$\begin{gathered} r^2 + r = r(r+1) = 0, \\ \left(r^2 + 3r + 2\right) c_1 = 0, \end{gathered}$$

and

$$(k+r)(k+r+1)c_k - c_{k-2} = 0.$$

The indicial roots are $r_1 = 0$ and $r_2 = -1$, so $c_1 = 0$. For $r_1 = 0$ the recurrence relation is

$$c_k = \frac{c_{k-2}}{k(k+1)}, \quad k = 2, 3, 4, \dots,$$

and

$$c_2 = \frac{1}{3!}c_0 \qquad c_3 = c_5 = c_7 = \cdots = 0 \qquad c_4 = \frac{1}{5!}c_0 \qquad c_{2n} = \frac{1}{(2n+1)!}c_0.$$

For $r_2 = -1$ the recurrence relation is

$$c_k = \frac{c_{k-2}}{k(k-1)}, \quad k = 2, 3, 4, \dots,$$

and

$$c_2 = \frac{1}{2!}c_0 \qquad c_3 = c_5 = c_7 = \cdots = 0 \qquad c_4 = \frac{1}{4!}c_0 \qquad c_{2n} = \frac{1}{(2n)!}c_0.$$

The general solution on $(0, \infty)$ is

$$\begin{aligned} y &= C_1 \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}x^{2n} + C_2 x^{-1} \sum_{n=0}^{\infty} \frac{1}{(2n)!}x^{2n} \\ &= \frac{1}{x}\left[C_1 \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}x^{2n+1} + C_2 \sum_{n=0}^{\infty} \frac{1}{(2n)!}x^{2n}\right] \\ &= \frac{1}{x}\left[C_1 \sinh x + C_2 \cosh x\right]. \end{aligned}$$

**26.** Substituting $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ into the differential equation and collecting terms, we obtain

$$\begin{aligned} x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y &= \left(r^2 - \frac{1}{4}\right)c_0x^r + \left(r^2 + 2r + \frac{3}{4}\right)c_1x^{r+1} \\ &\qquad + \sum_{k=2}^{\infty}\left[(k+r)(k+r-1)c_k + (k+r)c_k - \frac{1}{4}c_k + c_{k-2}\right]x^{k+r} \\ &= 0, \end{aligned}$$

which implies

$$r^2 - \frac{1}{4} = \left(r - \frac{1}{2}\right)\left(r + \frac{1}{2}\right) = 0, \left(r^2 + 2r + \frac{3}{4}\right)c_1 = 0,$$

and

$$\left[(k+r)^2 - \frac{1}{4}\right]c_k + c_{k-2} = 0.$$

The indicial roots are $r_1 = 1/2$ and $r_2 = -1/2$, so $c_1 = 0$. For $r_1 = 1/2$ the recurrence relation is

$$c_k = -\frac{c_{k-2}}{k(k+1)}, \quad k = 2, 3, 4, \ldots,$$

and

$$c_2 = -\frac{1}{3!}c_0 \qquad c_3 = c_5 = c_7 = \cdots = 0 \qquad c_4 = \frac{1}{5!}c_0 \qquad c_{2n} = \frac{(-1)^n}{(2n+1)!}c_0.$$

For $r_2 = -1/2$ the recurrence relation is

$$c_k = -\frac{c_{k-2}}{k(k-1)}, \quad k = 2, 3, 4, \ldots,$$

and

$$c_2 = -\frac{1}{2!}c_0 \qquad c_3 = c_5 = c_7 = \cdots = 0 \qquad c_4 = \frac{1}{4!}c_0 \qquad c_{2n} = \frac{(-1)^n}{(2n)!}c_0.$$

The general solution on $(0, \infty)$ is

$$\begin{aligned} y &= C_1 x^{1/2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n} + C_2 x^{-1/2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \\ &= C_1 x^{-1/2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} + C_2 x^{-1/2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \\ &= x^{-1/2}[C_1 \sin x + C_2 \cos x]. \end{aligned}$$

**27.** Substituting $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ into the differential equation and collecting terms, we obtain

$$xy'' - xy' + y = \left(r^2 - r\right) c_0 x^{r-1} + \sum_{k=0}^{\infty} [(k+r+1)(k+r)c_{k+1} - (k+r)c_k + c_k] x^{k+r} = 0$$

which implies

$$r^2 - r = r(r-1) = 0$$

and

$$(k+r+1)(k+r)c_{k+1} - (k+r-1)c_k = 0.$$

The indicial roots are $r_1 = 1$ and $r_2 = 0$. For $r_1 = 1$ the recurrence relation is

$$c_{k+1} = \frac{kc_k}{(k+2)(k+1)}, \quad k = 0, 1, 2, \ldots,$$

and one solution is $y_1 = c_0 x$. A second solution is

$$\begin{aligned} y_2 &= x \int \frac{e^{-\int -1\,dx}}{x^2}\,dx = x \int \frac{e^x}{x^2}\,dx = x \int \frac{1}{x^2}\left(1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \cdots\right) dx \\ &= x \int \left(\frac{1}{x^2} + \frac{1}{x} + \frac{1}{2} + \frac{1}{3!}x + \frac{1}{4!}x^2 + \cdots\right) dx = x\left[-\frac{1}{x} + \ln x + \frac{1}{2}x + \frac{1}{12}x^2 + \frac{1}{72}x^3 + \cdots\right] \\ &= x \ln x - 1 + \frac{1}{2}x^2 + \frac{1}{12}x^3 + \frac{1}{72}x^4 + \cdots. \end{aligned}$$

The general solution on $(0,\infty)$ is

$$y = C_1x + C_2y_2(x).$$

**28.** Substituting $y=\sum_{n=0}^{\infty}c_nx^{n+r}$ into the differential equation and collecting terms, we obtain

$$\begin{aligned}
y'' + \frac{3}{x}y' - 2y &= \left(r^2+2r\right)c_0x^{r-2} + \left(r^2+4r+3\right)c_1x^{r-1} \\
&\qquad + \sum_{k=2}^{\infty}\left[(k+r)(k+r-1)c_k + 3(k+r)c_k - 2c_{k-2}\right]x^{k+r-2} \\
&= 0,
\end{aligned}$$

which implies

$$r^2+2r = r(r+2) = 0$$

$$\left(r^2+4r+3\right)c_1 = 0$$
$$(k+r)(k+r+2)c_k - 2c_{k-2} = 0.$$

The indicial roots are $r_1=0$ and $r_2=-2$, so $c_1=0$. For $r_1=0$ the recurrence relation is

$$c_k = \frac{2c_{k-2}}{k(k+2)}, \qquad k = 2,3,4,\dots,$$

and

$$c_2 = \frac{1}{4}c_0 \qquad c_3=c_5=c_7=\cdots=0 \qquad c_4=\frac{1}{48}c_0 \qquad c_6=\frac{1}{1{,}152}c_0.$$

The result is

$$y_1 = c_0\left(1+\frac{1}{4}x^2+\frac{1}{48}x^4+\frac{1}{1{,}152}x_6+\cdots\right).$$

A second solution is

$$\begin{aligned}
y_2 &= y_1\int\frac{e^{-\int(3/x)dx}}{y_1^2}\,dx = y_1\int\frac{dx}{x^3\left(1+\frac{1}{4}x^2+\frac{1}{48}x^4+\cdots\right)^2} \\
&= y_1\int\frac{dx}{x^3\left(1+\frac{1}{2}x^2+\frac{5}{48}x^4+\frac{7}{576}x^6+\cdots\right)} = y_1\int\frac{1}{x^3}\left(1-\frac{1}{2}x^2+\frac{7}{48}x^4+\frac{19}{576}x^6+\cdots\right)dx \\
&= y_1\int\left(\frac{1}{x^3}-\frac{1}{2x}+\frac{7}{48}x-\frac{19}{576}x^3+\cdots\right)dx = y_1\left[-\frac{1}{2x^2}-\frac{1}{2}\ln x+\frac{7}{96}x^2-\frac{19}{2{,}304}x^4+\cdots\right] \\
&= -\frac{1}{2}y_1\ln x + y\left[-\frac{1}{2x^2}+\frac{7}{96}x^2-\frac{19}{2{,}304}x^4+\cdots\right].
\end{aligned}$$

The general solution on $(0,\infty)$ is

$$y = C_1y_1(x) + C_2y_2(x).$$

**29.** Substituting $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ into the differential equation and collecting terms, we obtain

$$xy'' + (1-x)y' - y = r^2 c_0 x^{r-1} + \sum_{k=1}^{\infty} [(k+r)(k+r-1)c_k + (k+r)c_k - (k+r)c_{k-1}]x^{k+r-1} = 0,$$

which implies $r^2 = 0$ and

$$(k+r)^2 c_k - (k+r)c_{k-1} = 0.$$

The indicial roots are $r_1 = r_2 = 0$ and the recurrence relation is

$$c_k = \frac{c_{k-1}}{k}, \quad k = 1, 2, 3, \ldots .$$

One solution is

$$y_1 = c_0 \left( 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \cdots \right) = c_0 e^x.$$

A second solution is

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{-\int (1/x - 1)\,dx}}{e^{2x}}\,dx = e^x \int \frac{e^x/x}{e^{2x}}\,dx = e^x \int \frac{1}{x} e^{-x}\,dx \\ &= e^x \int \frac{1}{x} \left( 1 - x + \frac{1}{2}x^2 - \frac{1}{3!}x^3 + \cdots \right) dx = e^x \int \left( \frac{1}{x} - 1 + \frac{1}{2}x - \frac{1}{3!}x^2 + \cdots \right) dx \\ &= e^x \left[ \ln x - x + \frac{1}{2 \cdot 2}x^2 - \frac{1}{3 \cdot 3!}x^3 + \cdots \right] = e^x \ln x - e^x \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n \cdot n!} x^n. \end{aligned}$$

The general solution on $(0, \infty)$ is

$$y = C_1 e^x + C_2 e^x \left( \ln x - \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n \cdot n!} x^n \right).$$

**30.** Substituting $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ into the differential equation and collecting terms, we obtain

$$xy'' + y' + y = r^2 c_0 x^{r-1} + \sum_{k=1}^{\infty} [(k+r)(k+r-1)c_k + (k+r)c_k + c_{k-1}]x^{k+r-1} = 0$$

which implies $r^2 = 0$ and

$$(k+r)^2 c_k + c_{k-1} = 0.$$

The indicial roots are $r_1 = r_2 = 0$ and the recurrence relation is

$$c_k = -\frac{c_{k-1}}{k^2}, \quad k = 1, 2, 3, \ldots .$$

One solution is

$$y_1 = c_0 \left( 1 - x + \frac{1}{2^2}x^2 - \frac{1}{(3!)^2}x^3 + \frac{1}{(4!)^2}x^4 - \cdots \right) = c_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} x^n.$$

A second solution is

$$\begin{aligned}
y_2 = y_1\int \frac{e^{-\int(1/x)\,dx}}{y_1^2}\,dx &= y_1\int \frac{dx}{x\left(1 - x + \frac{1}{4}x^2 - \frac{1}{36}x^3 + \cdots\right)^2}\\
&= y_1\int \frac{dx}{x\left(1 - 2x + \frac{3}{2}x^2 - \frac{5}{9}x^3 + \frac{35}{288}x^4 - \cdots\right)}\\
&= y_1\int \frac{1}{x}\left(1 + 2x + \frac{5}{2}x^2 + \frac{23}{9}x^3 + \frac{677}{288}x^4 + \cdots\right)dx\\
&= y_1\int \left(\frac{1}{x} + 2 + \frac{5}{2}x + \frac{23}{9}x^2 + \frac{677}{288}x^3 + \cdots\right)dx\\
&= y_1\left[\ln x + 2x + \frac{5}{4}x^2 + \frac{23}{27}x^3 + \frac{677}{1{,}152}x^4 + \cdots\right]
\end{aligned}$$

The general solution on $(0,\infty)$ is

$$y = C_1y_1(x) + C_2y_2(x).$$

**31.** Substituting $y = \sum_{n=0}^{\infty} c_nx^{n+r}$ into the differential equation and collecting terms, we obtain

$$\begin{aligned}
xy'' + (x-6)y' - 3y = (r^2 - 7r)c_0x^{r-1} + \sum_{k=1}^{\infty}\big[(k+r)(k+r-1)c_k + (k+r-1)c_{k-1}\\
-6(k+r)c_k - 3c_{k-1}\big]\,x^{k+r-1} = 0,
\end{aligned}$$

which implies

$$r^2 - 7r = r(r-7) = 0$$

and

$$(k+r)(k+r-7)c_k + (k+r-4)c_{k-1} = 0.$$

The indicial roots are $r_1 = 7$ and $r_2 = 0$. For $r_1 = 7$ the recurrence relation is

$$(k+7)kc_k + (k+3)c_{k-1} = 0, \quad k = 1, 2, 3, \ldots\,,$$

or

$$c_k = -\frac{k+3}{k(k+7)}\,c_{k-1}\,, \quad k = 1, 2, 3, \ldots\,.$$

Taking $c_0 \neq 0$ we obtain

$$c_1 = -\frac{1}{2}c_0 \qquad c_2 = \frac{5}{18}c_0 \qquad c_3 = -\frac{1}{6}c_0,$$

and so on. Thus, the indicial root $r_1 = 7$ yields a single solution. Now, for $r_2 = 0$ the recurrence relation is

$$k(k-7)c_k + (k-4)c_{k-1} = 0, \quad k = 1, 2, 3, \ldots\,.$$

Then

$$-6c_1 - 3c_0 = 0 \qquad -10c_2 - 2c_1 = 0 \qquad -12c_3 - c_2 = 0$$

and

$$\begin{aligned} -12c_4 + 0c_3 &= 0 & -10c_5 + c_4 &= 0 & -c_6 + 2c_5 &= 0 & 0c_7 + 3c_6 &= 0 \\ c_4 &= 0 & c_5 &= 0 & c_6 &= 0 & c_7 &\text{ is arbitrary} \end{aligned}$$

and

$$c_k = -\frac{k-4}{k(k-7)}\, c_{k-1}, \quad k = 8, 9, 10, \ldots .$$

Taking $c_0 \neq 0$ and $c_7 = 0$ we obtain

$$c_1 = -\frac{1}{2}c_0 \qquad c_2 = \frac{1}{10}c_0 \qquad c_3 = -\frac{1}{120}c_0 \qquad c_4 = c_5 = c_6 = \cdots = 0.$$

Taking $c_0 = 0$ and $c_7 \neq 0$ we obtain

$$c_1 = c_2 = c_3 = c_4 = c_5 = c_6 = 0 \qquad c_8 = -\frac{1}{2}c_7 \qquad c_9 = \frac{5}{36}c_7 \qquad c_{10} = -\frac{1}{36}c_7,$$

and so on. In this case we obtain the two solutions

$$y_1 = 1 - \frac{1}{2}x + \frac{1}{10}x^2 - \frac{1}{120}x^3 \qquad \text{and} \qquad y_2 = x^7 - \frac{1}{2}x^8 + \frac{5}{36}x^9 - \frac{1}{36}x^{10} + \cdots .$$

**32.** Substituting $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ into the differential equation and collecting terms, we obtain

$$\begin{aligned} x(x-1)y'' + 3y' - 2y &= \left(4r - r^2\right) c_0 x^{r-1} \\ &\qquad + \sum_{k=1}^{\infty} \big[(k+r-1)(k+r-12)c_{k-1} - (k+r)(k+r-1)c_k \\ &\qquad\qquad + 3(k+r)c_k - 2c_{k-1}\big]\, x^{k+r-1} \\ &= 0, \end{aligned}$$

which implies

$$4r - r^2 = r(4-r) = 0$$

and

$$-(k+r)(k+r-4)c_k + [(k+r-1)(k+r-2) - 2]c_{k-1} = 0.$$

The indicial roots are $r_1 = 4$ and $r_2 = 0$. For $r_1 = 4$ the recurrence relation is

$$-(k+4)kc_k + [(k+3)(k+2) - 2]c_{k-1} = 0$$

or

$$c_k = \frac{k+1}{k}\, c_{k-1}, \quad k = 1, 2, 3, \ldots .$$

Taking $c_0 \neq 0$ we obtain

$$c_1 = 2c_0 \qquad c_2 = 3c_0 \qquad c_3 = 4c_0,$$

and so on. Thus, the indicial root $r_1 = 4$ yields a single solution. For $r_2 = 0$ the recurrence relation is

$$-k(k-4)c_k + k(k-3)c_{k-1} = 0, \quad k = 1, 2, 3, \ldots,$$

or

$$-(k-4)c_k + (k-3)c_{k-1} = 0, \quad k = 1, 2, 3, \ldots .$$

Then

$$\begin{array}{cccc} 3c_1 - 2c_0 = 0 & 2c_2 - c_1 = 0 & c_3 + 0c_2 = 0 & 0c_4 + c_3 = 0 \\ & & c_3 = 0 & c_4 \text{ arbitrary} \end{array}$$

and

$$c_k = \frac{(k-3)c_{k-1}}{k-4}, \quad k = 5, 6, 7, \ldots .$$

Taking $c_0 \neq 0$ and $c_4 = 0$ we obtain

$$c_1 = \frac{2}{3}c_0 \qquad c_2 = \frac{1}{3}c_0 \qquad c_3 = c_4 = c_5 = \cdots = 0.$$

Taking $c_0 = 0$ and $c_4 \neq 0$ we obtain

$$c_1 = c_2 = c_3 = 0 \qquad c_5 = 2c_4 \qquad c_6 = 3c_4 \qquad c_7 = 4c_4,$$

and so on. In this case we obtain the two solutions

$$y_1 = 1 + \frac{2}{3}x + \frac{1}{3}x^2 \qquad \text{and} \qquad y_2 = x^4 + 2x^5 + 3x^6 + 4x^7 + \cdots .$$

**33. (a)** From $t = 1/x$ we have $dt/dx = -1/x^2 = -t^2$. Then

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = -t^2\frac{dy}{dt}$$

and

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(-t^2\frac{dy}{dt}\right) = -t^2\frac{d^2y}{dt^2}\frac{dt}{dx} - \frac{dy}{dt}\left(2t\frac{dt}{dx}\right) = t^4\frac{d^2y}{dt^2} + 2t^3\frac{dy}{dt}.$$

Now

$$x^4\frac{d^2y}{dx^2} + \lambda y = \frac{1}{t^4}\left(t^4\frac{d^2y}{dt^2} + 2t^3\frac{dy}{dt}\right) + \lambda y = \frac{d^2y}{dt^2} + \frac{2}{t}\frac{dy}{dt} + \lambda y = 0$$

becomes

$$t\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + \lambda t y = 0.$$

**(b)** Substituting $y = \sum_{n=0}^{\infty} c_n t^{n+r}$ into the differential equation and collecting terms, we obtain

$$\begin{aligned} t\,\frac{d^2y}{dt^2} + 2\,\frac{dy}{dt} + \lambda t y &= (r^2+r)c_0 t^{r-1} + (r^2+3r+2)c_1 t^r \\ &\qquad + \sum_{k=2}^{\infty} [(k+r)(k+r-1)c_k + 2(k+r)c_k + \lambda c_{k-2}] t^{k+r-1} \\ &= 0, \end{aligned}$$

which implies

$$\begin{aligned} r^2 + r = r(r+1) &= 0, \\ \left(r^2+3r+2\right) c_1 &= 0, \end{aligned}$$

and

$$(k+r)(k+r+1)c_k + \lambda c_{k-2} = 0.$$

The indicial roots are $r_1 = 0$ and $r_2 = -1$, so $c_1 = 0$. For $r_1 = 0$ the recurrence relation is

$$c_k = -\frac{\lambda c_{k-2}}{k(k+1)}, \quad k = 2, 3, 4, \dots,$$

and

$$c_2 = -\frac{\lambda}{3!} c_0 \qquad c_3 = c_5 = c_7 = \cdots = 0 \qquad c_4 = \frac{\lambda^2}{5!} c_0 \qquad c_{2n} = (-1)^n \frac{\lambda^n}{(2n+1)!} c_0.$$

For $r_2 = -1$ the recurrence relation is

$$c_k = -\frac{\lambda c_{k-2}}{k(k-1)}, \quad k = 2, 3, 4, \dots,$$

and

$$c_2 = -\frac{\lambda}{2!} c_0 \qquad c_3 = c_5 = c_7 = \cdots = 0 \qquad c_4 = \frac{\lambda^2}{4!} c_0 \qquad c_{2n} = (-1)^n \frac{\lambda^n}{(2n)!} c_0.$$

The general solution on $(0, \infty)$ is

$$\begin{aligned} y(t) &= c_1 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (\sqrt{\lambda}\, t)^{2n} + c_2 t^{-1} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (\sqrt{\lambda}\, t)^{2n} \\ &= \frac{1}{t} \left[ C_1 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (\sqrt{\lambda}\, t)^{2n+1} + C_2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (\sqrt{\lambda}\, t)^{2n} \right] \\ &= \frac{1}{t} \left[ C_1 \sin \sqrt{\lambda}\, t + C_2 \cos \sqrt{\lambda}\, t \right]. \end{aligned}$$

**(c)** Using $t = 1/x$, the solution of the original equation is

$$y(x) = C_1 x \sin \frac{\sqrt{\lambda}}{x} + C_2 x \cos \frac{\sqrt{\lambda}}{x}.$$

**34.** **(a)** From the boundary conditions $y(a)=0$, $y(b)=0$ we find

$$C_1\sin\frac{\sqrt{\lambda}}{a}+C_2\cos\frac{\sqrt{\lambda}}{a}=0$$

$$C_1\sin\frac{\sqrt{\lambda}}{b}+C_2\cos\frac{\sqrt{\lambda}}{b}=0.$$

Since this is a homogeneous system of linear equations, it will have nontrivial solutions for $C_1$ and $C_2$ if

$$\begin{vmatrix}\sin\dfrac{\sqrt{\lambda}}{a} & \cos\dfrac{\sqrt{\lambda}}{a}\\ \sin\dfrac{\sqrt{\lambda}}{b} & \cos\dfrac{\sqrt{\lambda}}{b}\end{vmatrix}=\sin\frac{\sqrt{\lambda}}{a}\cos\frac{\sqrt{\lambda}}{b}-\cos\frac{\sqrt{\lambda}}{a}\sin\frac{\sqrt{\lambda}}{b}$$

$$=\sin\left(\frac{\sqrt{\lambda}}{a}-\frac{\sqrt{\lambda}}{b}\right)=\sin\left(\sqrt{\lambda}\,\frac{b-a}{ab}\right)=0.$$

This will be the case if

$$\sqrt{\lambda}\left(\frac{b-a}{ab}\right)=n\pi \qquad \text{or} \qquad \sqrt{\lambda}=\frac{n\pi ab}{b-a}=\frac{n\pi ab}{L},\quad n=1,\,2,\,\ldots\,,$$

or, if

$$\lambda_n=\frac{n^2\pi^2a^2b^2}{L^2}=\frac{P_nb^4}{EI}.$$

The critical loads are then $P_n=n^2\pi^2(a/b)^2EI_0/L^2$. Using $C_2=-C_1\sin(\sqrt{\lambda}/a)/\cos(\sqrt{\lambda}/a)$ we have

$$y=C_1x\left[\sin\frac{\sqrt{\lambda}}{x}-\frac{\sin(\sqrt{\lambda}/a)}{\cos(\sqrt{\lambda}/a)}\cos\frac{\sqrt{\lambda}}{x}\right]$$

$$=C_3x\left[\sin\frac{\sqrt{\lambda}}{x}\cos\frac{\sqrt{\lambda}}{a}-\cos\frac{\sqrt{\lambda}}{x}\sin\frac{\sqrt{\lambda}}{a}\right]$$

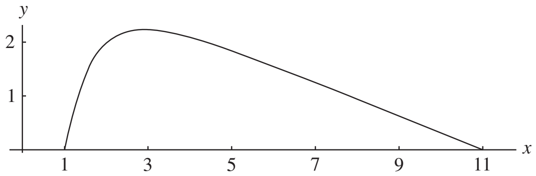
$$=C_3x\sin\sqrt{\lambda}\left(\frac{1}{x}-\frac{1}{a}\right),$$

and

$$y_n(x)=C_3x\sin\frac{n\pi ab}{L}\left(\frac{1}{x}-\frac{1}{a}\right)=C_3x\sin\frac{n\pi ab}{La}\left(\frac{a}{x}-1\right)=C_4x\sin\frac{n\pi b}{L}\left(1-\frac{a}{x}\right).$$

**(b)** When $n=1$, $b=11$, and $a=1$, we have, for $C_4=1$,

$$y_1(x)=x\sin 1.1\pi\left(1-\frac{1}{x}\right).$$



**35.** Express the differential equation in standard form:

$$y''' + P(x)y'' + Q(x)y' + R(x)y = 0.$$

Suppose $x_0$ is a singular point of the differential equation. Then we say that $x_0$ is a regular singular point if $(x - x_0)P(x)$, $(x - x_0)^2Q(x)$, and $(x - x_0)^3R(x)$ are analytic at $x = x_0$.

**36.** Substituting $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ into the first differential equation and collecting terms, we obtain

$$x^3y'' + y = c_0x^r + \sum_{k=1}^{\infty} [c_k + (k + r - 1)(k + r - 2)c_{k-1}]\, x^{k+r} = 0.$$

It follows that $c_0 = 0$ and

$$c_k = -(k + r - 1)(k + r - 2)c_{k-1}.$$

The only solution we obtain is $y(x) = 0$.

Substituting $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ into the second differential equation and collecting terms, we obtain

$$x^2y'' + (3x - 1)y' + y = -rc_0 + \sum_{k=0}^{\infty} \left[(k + r + 1)^2c_k - (k + r + 1)c_{k+1}\right] x^{k+r} = 0,$$

which implies

$$-rc_0 = 0$$
$$(k + r + 1)^2c_k - (k + r + 1)c_{k+1} = 0.$$

If $c_0 = 0$, then the solution of the differential equation is $y = 0$. Thus, we take $r = 0$, from which we obtain

$$c_{k+1} = (k + 1)c_k, \quad k = 0, 1, 2, \ldots .$$

Letting $c_0 = 1$ we get $c_1 = 2$, $c_2 = 3!$, $c_3 = 4!$, and so on. The solution of the differential equation is then $y = \sum_{n=0}^{\infty}(n + 1)!x^n$, which converges only at $x = 0$.

**37.** We write the differential equation in the form $x^2y'' + (b/a)xy' + (c/a)y = 0$ and identify $a_0 = b/a$ and $b_0 = c/a$ as in (14) in the text. Then the indicial equation is

$$r(r - 1) + \frac{b}{a}r + \frac{c}{a} = 0 \qquad \text{or} \qquad ar^2 + (b - a)r + c = 0,$$

which is also the auxiliary equation of $ax^2y'' + bxy' + cy = 0$.

## 6.4 Special Functions

**1.** Since $\nu^2 = 1/9$ the general solution is $y = c_1 J_{1/3}(x) + c_2 J_{-1/3}(x)$.

**2.** Since $\nu^2 = 1$ the general solution is $y = c_1 J_1(x) + c_2 Y_1(x)$.

**3.** Since $\nu^2 = 25/4$ the general solution is $y = c_1 J_{5/2}(x) + c_2 J_{-5/2}(x)$.

**4.** Since $\nu^2 = 1/16$ the general solution is $y = c_1 J_{1/4}(x) + c_2 J_{-1/4}(x)$.

**5.** Since $\nu^2 = 0$ the general solution is $y = c_1 J_0(x) + c_2 Y_0(x)$.

**6.** Since $\nu^2 = 4$ the general solution is $y = c_1 J_2(x) + c_2 Y_2(x)$.

**7.** We identify $\alpha = 3$ and $\nu = 2$. Then the general solution is $y = c_1 J_2(3x) + c_2 Y_2(3x)$.

**8.** We identify $\alpha = 6$ and $\nu = \frac{1}{2}$. Then the general solution is $y = c_1 J_{1/2}(6x) + c_2 J_{-1/2}(6x)$.

**9.** We identify $\alpha = 4$ and $\nu = 2/3$. Then the general solution is $y = c_1 I_{2/3}(4x) + c_2 K_{2/3}(4x)$

**10.** We identify $\alpha = \sqrt{2}$ and $\nu = 8$. Then the general solution is $y = c_1 I_8(\sqrt{2}\,x) + c_2 K_8(\sqrt{2}\,x)$.

**11.** If $y = x^{-1/2}v(x)$ then

$$\begin{aligned} y' &= x^{-1/2}v'(x) - \frac{1}{2}x^{-3/2}v(x),\\ y'' &= x^{-1/2}v''(x) - x^{-3/2}v'(x) + \frac{3}{4}x^{-5/2}v(x), \end{aligned}$$

and

$$x^2y'' + 2xy' + \alpha^2x^2y = x^{3/2}v''(x) + x^{1/2}v'(x) + \left(\alpha^2x^{3/2} - \frac{1}{4}x^{-1/2}\right)v(x) = 0.$$

Multiplying by $x^{1/2}$ we obtain

$$x^2v''(x) + xv'(x) + \left(\alpha^2x^2 - \frac{1}{4}\right)v(x) = 0,$$

whose solution is $v = c_1 J_{1/2}(\alpha x) + c_2 J_{-1/2}(\alpha x)$.
Then $y = c_1 x^{-1/2} J_{1/2}(\alpha x) + c_2 x^{-1/2} J_{-1/2}(\alpha x)$.

**12.** If $y = \sqrt{x}\, v(x)$ then

$$y' = x^{1/2}v'(x) + \frac{1}{2}x^{-1/2}v(x)$$

$$y'' = x^{1/2}v''(x) + x^{-1/2}v'(x) - \frac{1}{4}x^{-3/2}v(x)$$

and

$$\begin{aligned} x^2y'' + \left(\alpha^2x^2 - \nu^2 + \frac{1}{4}\right)y &= x^{5/2}v''(x) + x^{3/2}v'(x) - \frac{1}{4}x^{1/2}v(x) + \left(\alpha^2x^2 - \nu^2 + \frac{1}{4}\right)x^{1/2}v(x) \\ &= x^{5/2}v''(x) + x^{3/2}v'(x) + (\alpha^2x^{5/2} - \nu^2x^{1/2})v(x) = 0. \end{aligned}$$

Multiplying by $x^{-1/2}$ we obtain

$$x^2v''(x) + xv'(x) + (\alpha^2x^2 - \nu^2)v(x) = 0,$$

whose solution is $v(x) = c_1J_\nu(\alpha x) + c_2Y_\nu(\alpha x)$. Then $y = c_1\sqrt{x}\, J_\nu(\alpha x) + c_2\sqrt{x}\, Y_\nu(\alpha x)$.

**13.** Write the differential equation in the form $y'' + (2/x)y' + (4/x)y = 0$. This is the form of (18) in the text with $a = -\frac{1}{2}$, $c = \frac{1}{2}$, $b = 4$, and $p = 1$, so, by (19) in the text, the general solution is

$$y = x^{-1/2}[c_1J_1(4x^{1/2}) + c_2Y_1(4x^{1/2})].$$

**14.** Write the differential equation in the form $y'' + (3/x)y' + y = 0$. This is the form of (18) in the text with $a = -1$, $c = 1$, $b = 1$, and $p = 1$, so, by (19) in the text, the general solution is

$$y = x^{-1}[c_1J_1(x) + c_2Y_1(x)].$$

**15.** Write the differential equation in the form $y'' - (1/x)y' + y = 0$. This is the form of (18) in the text with $a = 1$, $c = 1$, $b = 1$, and $p = 1$, so, by (19) in the text, the general solution is

$$y = x[c_1J_1(x) + c_2Y_1(x)].$$

**16.** Write the differential equation in the form $y'' - (5/x)y' + y = 0$. This is the form of (18) in the text with $a = 3$, $c = 1$, $b = 1$, and $p = 2$, so, by (19) in the text, the general solution is

$$y = x^3[c_1J_3(x) + c_2Y_3(x)].$$

**17.** Write the differential equation in the form $y'' + (1 - 2/x^2)y = 0$. This is the form of (18) in the text with $a = \frac{1}{2}$, $c = 1$, $b = 1$, and $p = \frac{3}{2}$, so, by (19) in the text, the general solution is

$$y = x^{1/2}[c_1J_{3/2}(x) + c_2Y_{3/2}(x)] = x^{1/2}[C_1J_{3/2}(x) + C_2J_{-3/2}(x)].$$

**18.** Write the differential equation in the form $y'' + (4 + 1/4x^2)y = 0$. This is the form of (18) in the text with $a = \frac{1}{2}$, $c = 1$, $b = 2$, and $p = 0$, so, by (19) in the text, the general solution is

$$y = x^{1/2}[c_1J_0(2x) + c_2Y_0(2x)].$$

**19.** Write the differential equation in the form $y'' + (3/x)y' + x^2y = 0$. This is the form of (18) in the text with $a = -1$, $c = 2$, $b = \frac{1}{2}$, and $p = \frac{1}{2}$, so, by (19) in the text, the general solution is

$$y = x^{-1}\left[c_1 J_{1/2}\left(\frac{1}{2}x^2\right) + c_2 Y_{1/2}\left(\frac{1}{2}x^2\right)\right]$$

or

$$y = x^{-1}\left[C_1 J_{1/2}\left(\frac{1}{2}x^2\right) + C_2 J_{-1/2}\left(\frac{1}{2}x^2\right)\right].$$

**20.** Write the differential equation in the form $y'' + (1/x)y' + (\frac{1}{9}x^4 - 4/x^2)y = 0$. This is the form of (18) in the text with $a = 0$, $c = 3$, $b = \frac{1}{9}$, and $p = \frac{2}{3}$, so, by (19) in the text, the general solution is

$$y = c_1 J_{2/3}\left(\frac{1}{9}x^3\right) + c_2 Y_{2/3}\left(\frac{1}{9}x^3\right)$$

or

$$y = C_1 J_{2/3}\left(\frac{1}{9}x^3\right) + C_2 J_{-2/3}\left(\frac{1}{9}x^3\right).$$

**21.** Using the fact that $i^2 = -1$, along with the definition of $J_\nu(x)$ in (7) in the text, we have

$$\begin{aligned}
I_\nu(x) = i^{-\nu} J_\nu(ix) &= i^{-\nu}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(1+\nu+n)}\left(\frac{ix}{2}\right)^{2n+\nu} \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(1+\nu+n)} i^{2n+\nu-\nu}\left(\frac{x}{2}\right)^{2n+\nu} \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(1+\nu+n)} (i^2)^n\left(\frac{x}{2}\right)^{2n+\nu} \\
&= \sum_{n=0}^{\infty}\frac{(-1)^{2n}}{n!\Gamma(1+\nu+n)}\left(\frac{x}{2}\right)^{2n+\nu} \\
&= \sum_{n=0}^{\infty}\frac{1}{n!\Gamma(1+\nu+n)}\left(\frac{x}{2}\right)^{2n+\nu},
\end{aligned}$$

which is a real function.

**22. (a)** The differential equation has the form of (20) in the text with

$$\begin{array}{cccc}
1 - 2a = 0 & 2c - 2 = 2 & b^2c^2 = -\beta^2c^2 = -1 & a^2 - p^2c^2 = 0 \\
a = \dfrac{1}{2} & c = 2 & \beta = \dfrac{1}{2}, \quad b = \dfrac{1}{2}i & p = \dfrac{1}{4}
\end{array}$$

Then, by (21) in the text,

$$y = x^{1/2}\left[c_1 J_{1/4}\left(\frac{1}{2}ix^2\right) + c_2 J_{-1/4}\left(\frac{1}{2}ix^2\right)\right].$$

In terms of real functions the general solution can be written

$$y = x^{1/2}\left[C_1 I_{1/4}\left(\frac{1}{2}x^2\right) + C_2 K_{1/4}\left(\frac{1}{2}x^2\right)\right].$$

**(b)** Write the differential equation in the form $y'' + (1/x)y' - 7x^2y = 0$. This is the form of (18) in the text with

$$\begin{array}{cccc} 1-2a=1 & 2c-2=2 & b^2c^2=-\beta^2c^2=-7 & a^2-p^2c^2=0 \\ a=0 & c=2 & \beta=\frac{1}{2}\sqrt{7},\quad b=\frac{1}{2}\sqrt{7}\,i & p=0. \end{array}$$

Then, by (19) in the text,

$$y = c_1J_0\left(\frac{1}{2}\sqrt{7}\,ix^2\right) + c_2Y_0\left(\frac{1}{2}\sqrt{7}\,ix^2\right).$$

In terms of real functions the general solution can be written

$$y = C_1I_0\left(\frac{1}{2}\sqrt{7}\,x^2\right) + C_2K_0\left(\frac{1}{2}\sqrt{7}\,x^2\right).$$

**23.** The differential equation has the form of (20) in the text with

$$\begin{array}{cccc} 1-2a=0 & 2c-2=0 & b^2c^2=1 & a^2-p^2c^2=0 \\ a=\frac{1}{2} & c=1 & b=1 & p=\frac{1}{2}. \end{array}$$

Then, by (21) in the text,

$$y = x^{1/2}[c_1J_{1/2}(x) + c_2J_{-1/2}(x)] = x^{1/2}\left[c_1\sqrt{\frac{2}{\pi x}}\sin x + c_2\sqrt{\frac{2}{\pi x}}\cos x\right] = C_1\sin x + C_2\cos x.$$

**24.** Write the differential equation in the form $y'' + (4/x)y' + (1 + 2/x^2)y = 0$. This is the form of (20) in the text with

$$\begin{array}{cccc} 1-2a=4 & 2c-2=0 & b^2c^2=1 & a^2-p^2c^2=2 \\ a=-\frac{3}{2} & c=1 & b=1 & p=\frac{1}{2}. \end{array}$$

Then, by (21), (26), and (27) in the text,

$$\begin{aligned} y &= x^{-3/2}[c_1J_{1/2}(x) + c_2J_{-1/2}(x)] = x^{-3/2}\left[c_1\sqrt{\frac{2}{\pi x}}\sin x + c_2\sqrt{\frac{2}{\pi x}}\cos x\right] \\ &= C_1\frac{1}{x^2}\sin x + C_2\frac{1}{x^2}\cos x. \end{aligned}$$

**25.** Write the differential equation in the form $y'' + (2/x)y' + (\frac{1}{16}x^2 - 3/4x^2)y = 0$. This is the form of (20) in the text with

$$\begin{array}{cccc} 1-2a=2 & 2c-2=2 & b^2c^2=\frac{1}{16} & a^2-p^2c^2=-\frac{3}{4} \\ a=-\frac{1}{2} & c=2 & b=\frac{1}{8} & p=\frac{1}{2}. \end{array}$$

Then, by (21) in the text,

$$\begin{aligned} y &= x^{-1/2}\left[c_1 J_{1/2}\left(\frac{1}{8}x^2\right) + c_2 J_{-1/2}\left(\frac{1}{8}x^2\right)\right] \\ &= x^{-1/2}\left[c_1\sqrt{\frac{16}{\pi x^2}}\sin\left(\frac{1}{8}x^2\right) + c_2\sqrt{\frac{16}{\pi x^2}}\cos\left(\frac{1}{8}x^2\right)\right] \\ &= C_1 x^{-3/2}\sin\left(\frac{1}{8}x^2\right) + C_2 x^{-3/2}\cos\left(\frac{1}{8}x^2\right). \end{aligned}$$

**26.** Write the differential equation in the form $y'' - (1/x)y' + (4 + 3/4x^2)y = 0$. This is the form of (20) in the text with

$$1 - 2a = -1 \qquad 2c - 2 = 0 \qquad b^2c^2 = 4 \qquad a^2 - p^2c^2 = \frac{3}{4}$$

$$a = 1 \qquad c = 1 \qquad b = 2 \qquad p = \frac{1}{2}.$$

Then, by (21) in the text,

$$\begin{aligned} y &= x\left[c_1 J_{1/2}(2x) + c_2 J_{-1/2}(2x)\right] = x\left[c_1\sqrt{\frac{2}{\pi 2x}}\sin 2x + c_2\sqrt{\frac{2}{\pi 2x}}\cos 2x\right] \\ &= C_1 x^{1/2}\sin 2x + C_2 x^{1/2}\cos 2x. \end{aligned}$$

**27.** **(a)** The recurrence relation follows from

$$\begin{aligned} -\nu J_\nu(x) + xJ_{\nu-1}(x) &= -\sum_{n=0}^{\infty}\frac{(-1)^n \nu}{n!\Gamma(1+\nu+n)}\left(\frac{x}{2}\right)^{2n+\nu} + x\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(\nu+n)}\left(\frac{x}{2}\right)^{2n+\nu-1} \\ &= -\sum_{n=0}^{\infty}\frac{(-1)^n \nu}{n!\Gamma(1+\nu+n)}\left(\frac{x}{2}\right)^{2n+\nu} + \sum_{n=0}^{\infty}\frac{(-1)^n(\nu+n)}{n!\Gamma(1+\nu+n)}\cdot 2\left(\frac{x}{2}\right)\left(\frac{x}{2}\right)^{2n+\nu-1} \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n(2n+\nu)}{n!\Gamma(1+\nu+n)}\left(\frac{x}{2}\right)^{2n+\nu} = xJ'_\nu(x). \end{aligned}$$

**(b)** The formula in part (a) is a linear first-order differential equation in $J_\nu(x)$. An integrating factor for this equation is $x^\nu$, so

$$\frac{d}{dx}\left[x^\nu J_\nu(x)\right] = x^\nu J_{\nu-1}(x).$$

**28.** Subtracting the formula in part (a) of Problem 27 from the formula in Example 5 we obtain

$$0 = 2\nu J_\nu(x) - xJ_{\nu+1}(x) - xJ_{\nu-1}(x) \qquad \text{or} \qquad 2\nu J_\nu(x) = xJ_{\nu+1}(x) + xJ_{\nu-1}(x).$$

**29.** Letting $\nu = 1$ in (21) in the text we have

$$xJ_0(x) = \frac{d}{dx}[xJ_1(x)] \qquad \text{so} \qquad \int_0^x rJ_0(r)\,dr = rJ_1(r)\Big|_{r=0}^{r=x} = xJ_1(x).$$

**30.** From (20) we obtain $J_0'(x) = -J_1(x)$, and from (23) we obtain $J_0'(x) = J_{-1}(x)$. Thus $J_0'(x) = J_{-1}(x) = -J_1(x)$.

**31.** Using $\Gamma\left(1+\frac{1}{2}+n\right) = \frac{(2n+1)!}{2^{2n+1}n!}\sqrt{\pi}$ we get the following

$$\begin{aligned}
\Gamma\left(1+\frac{1}{2}+n\right) &= \frac{(2n+1)!}{2^{2n+1}n!}\sqrt{\pi} \\
\left(\frac{1}{2}+n\right)\Gamma\left(\frac{1}{2}+n\right) &= \frac{(2n+1)!}{2^{2n+1}n!}\sqrt{\pi} \\
\left(1-\frac{1}{2}+n\right)\Gamma\left(1-\frac{1}{2}+n\right) &= \frac{(2n+1)!}{2^{2n+1}n!}\sqrt{\pi} \\
\Gamma\left(1-\frac{1}{2}+n\right) &= \frac{(2n+1)!}{\left(1-\frac{1}{2}+n\right)2^{2n+1}n!}\sqrt{\pi} \\
\Gamma\left(1-\frac{1}{2}+n\right) &= \frac{(2n+1)!}{\frac{1}{2}(1+2n)\,2^{2n+1}n!}\sqrt{\pi} \\
\Gamma\left(1-\frac{1}{2}+n\right) &= \frac{(2n)!}{2^{2n}n!}\sqrt{\pi}
\end{aligned}$$

From the last result we obtain

$$\begin{aligned}
J_{-1/2}(x) &= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(1-\frac{1}{2}+n)}\left(\frac{x}{2}\right)^{2n-1/2} = \sum_{n=0}^{\infty}\left(\frac{(-1)^n}{n!\frac{(2n)!}{2^{2n}n!}\sqrt{\pi}}\right)\left(\frac{x}{2}\right)^{2n}\left(\frac{x}{2}\right)^{-1/2} \\
&= \sqrt{\frac{2}{\pi x}}\cdot\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\,x^{2n}
\end{aligned}$$

The last series is the Maclaurin series for the cosine therefore

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cdot\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\,x^{2n} = \sqrt{\frac{2}{\pi x}}\cos x$$

**32.** From Problem 28, $2\nu J_\nu(x) = xJ_{\nu+1}(x) + xJ_{\nu-1}(x)$ and so with $\nu = 1/2$ we get

$$\begin{aligned}
J_{1/2}(x) &= xJ_{3/2}(x) + xJ_{-1/2}(x) \\
J_{3/2}(x) &= \frac{1}{x}J_{1/2}(x) - J_{-1/2}(x) \\
&= \frac{1}{x}\sqrt{\frac{2}{\pi x}}\sin x - \sqrt{\frac{2}{\pi x}}\cos x \\
&= \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right)
\end{aligned}$$

From the last result and using $\nu = 3/2$ we obtain

$$\begin{aligned}3J_{3/2}(x) &= xJ_{5/2}(x) + xJ_{1/2}(x)\\ J_{5/2}(x) &= \frac{3}{x}J_{3/2}(x) - J_{1/2}(x)\\ &= \frac{3}{x}\sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right) - \sqrt{\frac{2}{\pi x}}\sin x\\ &= \sqrt{\frac{2}{\pi x}}\left[\left(\frac{3}{x^2} - 1\right)\sin x - \frac{3\cos x}{x}\right]\end{aligned}$$

From the last result and using $\nu = 5/2$ we obtain

$$\begin{aligned}5J_{5/2}(x) &= xJ_{7/2}(x) + xJ_{3/2}(x)\\ J_{7/2}(x) &= \frac{5}{x}J_{5/2}(x) - J_{3/2}(x)\\ &= \frac{5}{x}\sqrt{\frac{2}{\pi x}}\left(\frac{3\sin x}{x^2} - \frac{3\cos x}{x} - \sin x\right) - \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right)\\ &= \sqrt{\frac{2}{\pi x}}\left[\left(\frac{15}{x^3} - \frac{6}{x}\right)\sin x - \left(\frac{15}{x^2} - 1\right)\cos x\right]\end{aligned}$$

**33. (a)** To find the spherical Bessel functions $j_1(x)$ and $j_2(x)$ we use the first formula in (30),

$$j_n(x) = \sqrt{\frac{\pi}{2x}}\,J_{n+1/2}$$

with $n = 1$ and $n = 2$,

$$j_1(x) = \sqrt{\frac{\pi}{2x}\,J_{3/2}(x)} \qquad \text{and} \qquad j_2(x) = \sqrt{\frac{\pi}{2x}}\,J_{5/2}(x)\,.$$

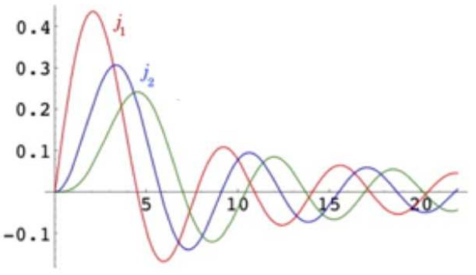
Then from Problem 32 we have

$$J_{3/2}(x) = \sqrt{2}\pi x\left(\frac{\sin x}{x} - \cos x\right) \qquad \text{so} \qquad j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x}$$

and

$$J_{5/2}(x) = \sqrt{2}\pi x\left(\frac{3\sin x}{x^2} - \frac{3\cos x}{x} - \sin x\right) \qquad \text{so} \qquad j_2(x) = \left(\frac{3}{x^3} - \frac{1}{x}\right)\sin x - \frac{3\cos x}{x^2}$$

**(b)** Using a graphing utility to plot the graphs of $j_1(x)$ and $j_2(x)$, we get the red and blue graphcs in the figure to the right.



**34. (a)** To find the spherical Bessel functions $y_1(x)$ and $y_2(x)$, we use the second formula in (30),

$$y_n(x) = \sqrt{\frac{\pi}{2x}}\,Y_{n+1/2}(x)$$

with $n = 1$ and $n = 2$,

$$y_1(x) = \sqrt{\frac{\pi}{2x}}\,Y_{3/2}(x) \qquad \text{and} \qquad y_2(x) = \sqrt{\frac{\pi}{2x}}\,Y_{5/2}(X)\,.$$

From the formula $Y_{n+1/2}(x) = (-1)^{n+1}J_{-(n+1/2)}(x)$ we get

$$Y_{3/2}(x) = J_{-3/2}(x), \qquad \text{and} \qquad Y_{5/2}(x) = -J_{-5/2}(x).$$

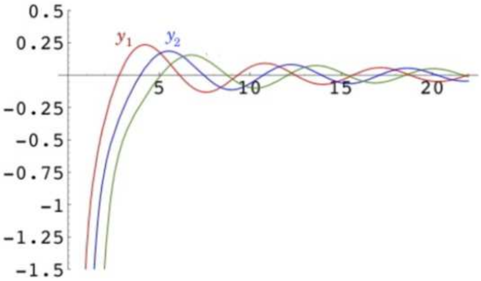
Then from Problem 32 we have

$$J_{-3/2}(x) = -\sqrt{\frac{2}{\pi x}}\left(\frac{\cos x}{x} + \sin x\right) \qquad \text{so} \qquad y_1(x) = -\frac{\cos x}{x^2} - \frac{\sin x}{x}$$

and

$$J_{-5/2}(x) = -\sqrt{\frac{2}{\pi x}}\left(\frac{3\cos x}{x^2} + \frac{3\sin x}{x} - \cos x\right) \qquad \text{so} \qquad y_2(x) = \left(-\frac{3}{x^3} + \frac{1}{x}\right)\cos x - \frac{3\sin x}{x^2}$$

**(b)** Using a graphing utility to plot the graphs of $y_1(x)$ and $y_2(x)$, we get the red and blue graphcs in the figure to the right.



**35.** Letting

$$s = \frac{2}{\alpha}\sqrt{\frac{k}{m}}\,e^{-\alpha t/2},$$

we have

$$\frac{dx}{dt} = \frac{dx}{ds}\frac{ds}{dt} = \frac{dx}{dt}\left[\frac{2}{\alpha}\sqrt{\frac{k}{m}}\left(-\frac{\alpha}{2}\right)e^{-\alpha t/2}\right] = \frac{dx}{ds}\left(-\sqrt{\frac{k}{m}}\,e^{-\alpha t/2}\right)$$

and

$$\begin{aligned}\frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{dt}\right) &= \frac{dx}{ds}\left(\frac{\alpha}{2}\sqrt{\frac{k}{m}}\,e^{-\alpha t/2}\right) + \frac{d}{dt}\left(\frac{dx}{ds}\right)\left(-\sqrt{\frac{k}{m}}\,e^{-\alpha t/2}\right)\\ &= \frac{dx}{ds}\left(\frac{\alpha}{2}\sqrt{\frac{k}{m}}\,e^{-\alpha t/2}\right) + \frac{d^2x}{ds^2}\frac{ds}{dt}\left(-\sqrt{\frac{k}{m}}\,e^{-\alpha t/2}\right)\\ &= \frac{dx}{ds}\left(\frac{\alpha}{2}\sqrt{\frac{k}{m}}\,e^{-\alpha t/2}\right) + \frac{d^2x}{ds^2}\left(\frac{k}{m}\,e^{-\alpha t}\right).\end{aligned}$$

Then

$$m\frac{d^2x}{dt^2}+ke^{-\alpha t}x=ke^{-\alpha t}\frac{d^2x}{ds^2}+\frac{m\alpha}{2}\sqrt{\frac{k}{m}}\,e^{-\alpha t/2}\frac{dx}{ds}+ke^{-\alpha t}x=0.$$

Multiplying by $2^2/\alpha^2 m$ we have

$$\frac{2^2}{\alpha^2}\frac{k}{m}e^{-\alpha t}\frac{d^2x}{ds^2}+\frac{2}{\alpha}\sqrt{\frac{k}{m}}\,e^{-\alpha t/2}\frac{dx}{ds}+\frac{2^2}{\alpha^2}\frac{k}{m}e^{-\alpha t}x=0$$

or, since $s=(2/\alpha)\sqrt{k/m}\,e^{-\alpha t/2}$,

$$s^2\frac{d^2x}{ds^2}+s\frac{dx}{ds}+s^2x=0.$$

**36. (a)** Differentiating $y=x^{1/2}w\left(\frac{2}{3}\alpha x^{3/2}\right)$ with respect to $\frac{2}{3}\alpha x^{3/2}$ we obtain

$$y'=x^{1/2}w'\left(\frac{2}{3}\alpha x^{3/2}\right)\alpha x^{1/2}+\frac{1}{2}x^{-1/2}w\left(\frac{2}{3}\alpha x^{3/2}\right)$$

and

$$y''=\alpha x w''\left(\frac{2}{3}\alpha x^{3/2}\right)\alpha x^{1/2}+\alpha w'\left(\frac{2}{3}\alpha x^{3/2}\right)$$
$$+\frac{1}{2}\alpha w'\left(\frac{2}{3}\alpha x^{3/2}\right)-\frac{1}{4}x^{-3/2}w\left(\frac{2}{3}\alpha x^{3/2}\right).$$

Then, after combining terms and simplifying, we have

$$y''+\alpha^2xy=\alpha\left[\alpha x^{3/2}w''+\frac{3}{2}w'+\left(\alpha x^{3/2}-\frac{1}{4\alpha x^{3/2}}\right)w\right]=0.$$

Letting $t=\frac{2}{3}\alpha x^{3/2}$ or $\alpha x^{3/2}=\frac{3}{2}t$ this differential equation becomes

$$\frac{3}{2}\frac{\alpha}{t}\left[t^2w''(t)+tw'(t)+\left(t^2-\frac{1}{9}\right)w(t)\right]=0,\quad t>0.$$

**(b)** Using the same substitution $y=x^{1/2}w\left(\frac{2}{3}\alpha x^{3/2}\right)$ the differential equation is now $y''-\alpha^2xy=0$:

$$y'=x^{1/2}w'\left(\frac{2}{3}\alpha x^{3/2}\right)\alpha x^{1/2}+\frac{1}{2}x^{-1/2}w\left(\frac{2}{3}\alpha x^{3/2}\right)$$
$$=\alpha x w'\left(\frac{2}{3}\alpha x^{3/2}\right)+\frac{1}{2}x^{-1/2}w\left(\frac{2}{3}\alpha x^{3/2}\right)$$
$$y''=\alpha x w''\left(\frac{2}{3}\alpha x^{3/2}\right)\alpha x^{1/2}+\alpha w'\left(\frac{2}{3}\alpha x^{3/2}\right)+\frac{1}{2}\alpha w'\left(\frac{2}{3}\alpha x^{3/2}\right)-\frac{1}{4}x^{-3/2}w\left(\frac{2}{3}\alpha x^{3/2}\right)$$
$$=\alpha^2x^{3/2}w''\left(\frac{2}{3}\alpha x^{3/2}\right)+\frac{3}{2}\alpha w'\left(\frac{2}{3}\alpha x^{3/2}\right)-\frac{1}{4}x^{-3/2}w\left(\frac{2}{3}\alpha x^{3/2}\right)$$

Then

$$y'' - \alpha^2 xy = \left[\alpha^2 x^{3/2} w''\left(\frac{2}{3}\alpha x^{3/2}\right) + \frac{3}{2}\alpha w'\left(\frac{2}{3}\alpha x^{3/2}\right) - \frac{1}{4}x^{-3/2} w\left(\frac{2}{3}\alpha x^{3/2}\right)\right]$$

$$- \alpha^2 x^{3/2} w\left(\frac{2}{3}\alpha x^{3/2}\right)$$

$$= \alpha^2 x^{3/2} w'' + \frac{3}{2}\alpha w' - \left(\alpha^2 x^{3/2} + \frac{1}{4}x^{-3/2}\right) w$$

$$= \alpha^2 x^{3/2} w'' + \frac{3}{2}\alpha w' - \left(\alpha^2 x^{3/2} + \frac{1}{4x^{3/2}}\right) w$$

$$= \alpha\left[\alpha x^{3/2} w'' + \frac{3}{2} w' = \left(\alpha x^{3/2} + \frac{1}{4\alpha x^{3/2}}\right) w\right]$$

$$= \alpha\left[\frac{3}{2} t w'' + \frac{3}{2} w' - \left(\frac{3}{2} t + \frac{1}{6} t^{-1}\right) w\right] \qquad \longleftarrow \qquad t = \frac{2}{3}\alpha x^{3/2}$$

$$= \frac{3\alpha}{2t}\left[t^2 w'' + t w' - \left(t^2 + \frac{1}{9}\right) w\right] = 0$$

when $w$ is a solution of $t^2 w'' + t w' - \left(t^2 + \frac{1}{9}\right) w = 0$.

**37.** **(a)** By part (a) of Problem 34, a solution of Airy's equation is $y = x^{1/2} w(\frac{2}{3}\alpha x^{3/2})$, where

$$w(t) = c_1 J_{1/3}(t) + c_2 J_{-1/3}(t)$$

is a solution of Bessel's equation of order $\frac{1}{3}$. Thus, the general solution of Airy's equation for $x > 0$ is

$$y = x^{1/2} w\left(\frac{2}{3}\alpha x^{3/2}\right) = c_1 x^{1/2} J_{1/3}\left(\frac{2}{3}\alpha x^{3/2}\right) + c_2 x^{1/2} J_{-1/3}\left(\frac{2}{3}\alpha x^{3/2}\right).$$

**(b)** By part (b) of Problem 34, a solution of Airy's equation is $y = x^{1/2} w(\frac{2}{3}\alpha x^{3/2})$, where

$$w(t) = c_1 I_{1/3}(t) + c_2 I_{-1/3}(t)$$

is a solution of modified Bessel's equation of order $\frac{1}{3}$. Thus, the general solution of Airy's equation for $x > 0$ is

$$y = x^{1/2} w\left(\frac{2}{3}\alpha x^{3/2}\right) = c_1 x^{1/2} I_{1/3}\left(\frac{2}{3}\alpha x^{3/2}\right) + c_2 x^{1/2} I_{-1/3}\left(\frac{2}{3}\alpha x^{3/2}\right).$$

**38.** The general solution of the differential equation is

$$y(x) = c_1 J_0(\alpha x) + c_2 Y_0(\alpha x).$$

In order to satisfy the conditions that $\lim\limits_{x\to 0^+} y(x)$ and $\lim\limits_{x\to 0^+} y'(x)$ are finite we are forced to define $c_2 = 0$. Thus, $y(x) = c_1 J_0(\alpha x)$. The second boundary condition, $y(2) = 0$, implies

$c_1 = 0$ or $J_0(2\alpha) = 0$. In order to have a nontrivial solution we require that $J_0(2\alpha) = 0$. From Table 5.3.1, the first three positive zeros of $J_0$ are found to be

$$2\alpha_1 = 2.4048, \quad 2\alpha_2 = 5.5201, \quad 2\alpha_3 = 8.6537$$

and so $\alpha_1 = 1.2024$, $\alpha_2 = 2.7601$, $\alpha_3 = 4.3269$. The eigenfunctions corresponding to the eigenvalues $\lambda_1 = \alpha_1^2$, $\lambda_2 = \alpha_2^2$, $\lambda_3 = \alpha_3^2$ are $J_0(1.2024x)$, $J_0(2.7601x)$, and $J_0(4.3269x)$.

**39. (a)** The differential equation $y'' + (\lambda/x)y = 0$ has the form of (20) in the text with

$$1 - 2a = 0 \qquad 2c - 2 = -1 \qquad b^2c^2 = \lambda \qquad a^2 - p^2c^2 = 0$$

$$a = \frac{1}{2} \qquad c = \frac{1}{2} \qquad b = 2\sqrt{\lambda} \qquad p = 1.$$

Then, by (21) in the text,

$$y = x^{1/2}\left[c_1J_1(2\sqrt{\lambda x}\,) + c_2Y_1(2\sqrt{\lambda x}\,)\right].$$

**(b)** We first note that $y = J_1(t)$ is a solution of Bessel's equation, $t^2y'' + ty' + (t^2 - 1)y = 0$, with $\nu = 1$. That is,

$$t^2J_1''(t) + tJ_1'(t) + (t^2 - 1)J_1(t) = 0,$$

or, letting $t = 2\sqrt{x}\,$,

$$4xJ_1''\left(2\sqrt{x}\,\right) + 2\sqrt{x}J_1'\left(2\sqrt{x}\,\right) + (4x - 1)J_1\left(2\sqrt{x}\,\right) = 0.$$

Now, if $y = \sqrt{x}J_1(2\sqrt{x}\,)$, we have

$$y' = \sqrt{x}\,J_1'\left(2\sqrt{x}\,\right)\frac{1}{\sqrt{x}} + \frac{1}{2\sqrt{x}}\,J_1\left(2\sqrt{x}\,\right) = J_1'\left(2\sqrt{x}\,\right) + \frac{1}{2}x^{-1/2}J_1\left(2\sqrt{x}\,\right)$$

and

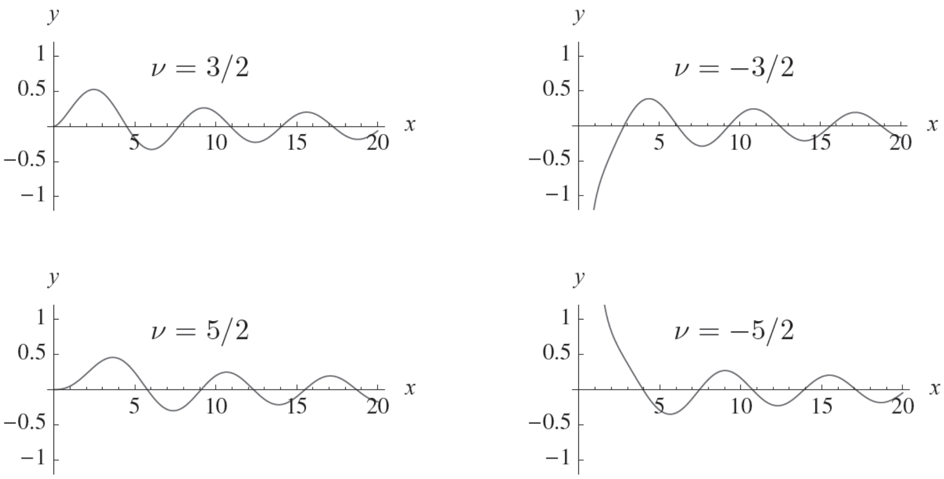
$$y'' = x^{-1/2}J_1''\left(2\sqrt{x}\,\right) + \frac{1}{2x}\,J_1'\left(2\sqrt{x}\,\right) - \frac{1}{4}x^{-3/2}J_1\left(2\sqrt{x}\,\right).$$

Then

$$\begin{aligned} xy'' + y &= \sqrt{x}\,J_1''\,2\sqrt{x} + \frac{1}{2}J_1'\left(2\sqrt{x}\,\right) - \frac{1}{4}x^{-1/2}J_1\left(2\sqrt{x}\,\right) + \sqrt{x}\,J\left(2\sqrt{x}\,\right) \\ &= \frac{1}{4\sqrt{x}}\left[4xJ_1''\left(2\sqrt{x}\,\right) + 2\sqrt{x}\,J_1'\left(2\sqrt{x}\,\right) - J_1\left(2\sqrt{x}\,\right) + 4xJ\left(2\sqrt{x}\,\right)\right] \\ &= 0, \end{aligned}$$

and $y = \sqrt{x}\,J_1\left(2\sqrt{x}\,\right)$ is a solution of Airy's differential equation.

**40.**



**41. (a)** We identify $m = 4$, $k = 1$, and $\alpha = 0.1$. Then

$$x(t) = c_1 J_0(10e^{-0.05t}) + c_2 Y_0(10e^{-0.05t})$$

and

$$x'(t) = -0.5c_1 J_0'(10e^{-0.05t}) - 0.5c_2 Y_0'(10e^{-0.05t}).$$

Now $x(0) = 1$ and $x'(0) = -1/2$ imply

$$\begin{aligned} c_1 J_0(10) + c_2 Y_0(10) &= 1 \\ c_1 J_0'(10) + c_2 Y_0'(10) &= 1. \end{aligned}$$

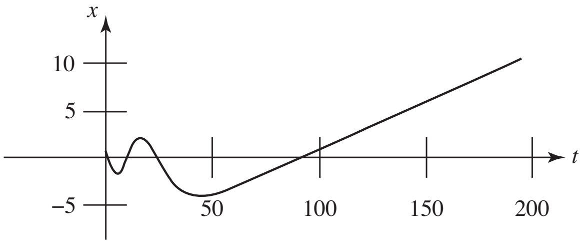
Using Cramer's rule we obtain

$$c_1 = \frac{Y_0'(10) - Y_0(10)}{J_0(10)Y_0'(10) - J_0'(10)Y_0(10)} \qquad \text{and} \qquad c_2 = \frac{J_0(10) - J_0'(10)}{J_0(10)Y_0'(10) - J_0'(10)Y_0(10)}.$$

Using $Y_0' = -Y_1$ and $J_0' = -J_1$ and Table 5.2 we find $c_1 = -4.7860$ and $c_2 = -3.1803$. Thus

$$x(t) = -4.7860 J_0(10e^{-0.05t}) - 3.1803 Y_0(10e^{-0.05t}).$$

**(b)**

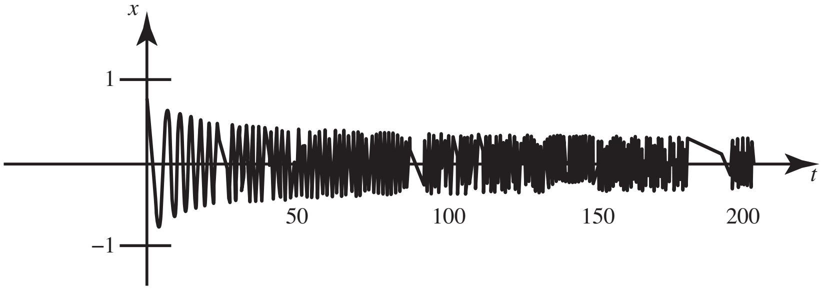


**42. (a)** Identifying $\alpha = \frac{1}{2}$, the general solution of $x'' + \frac{1}{4}tx = 0$ is

$$x(t) = c_1 x^{1/2} J_{1/3}\left(\frac{1}{3}x^{3/2}\right) + c_2 x^{1/2} J_{-1/3}\left(\frac{1}{3}x^{3/2}\right).$$

Solving the system $x(0.1) = 1$, $x'(0.1) = -\frac{1}{2}$ we find $c_1 = -0.809264$ and $c_2 = 0.782397$.

**(b)**



**43. (a)** Letting $t = L - x$, the boundary-value problem becomes

$$\frac{d^2\theta}{dt^2} + \alpha^2 t\theta = 0, \qquad \theta'(0) = 0, \quad \theta(L) = 0,$$

where $\alpha^2 = \delta g/EI$. This is Airy's differential equation, so by Problem 35 its solution is

$$y = c_1 t^{1/2} J_{1/3}\left(\frac{2}{3}\alpha t^{3/2}\right) + c_2 t^{1/2} J_{-1/3}\left(\frac{2}{3}\alpha t^{3/2}\right) = c_1\theta_1(t) + c_2\theta_2(t).$$

**(b)** Looking at the series forms of $\theta_1$ and $\theta_2$ we see that $\theta_1'(0) \neq 0$, while $\theta_2'(0) = 0$. Thus, the boundary condition $\theta'(0) = 0$ implies $c_1 = 0$, and so

$$\theta(t) = c_2\sqrt{t}\, J_{-1/3}\left(\frac{2}{3}\alpha t^{3/2}\right).$$

From $\theta(L) = 0$ we have

$$c_2\sqrt{L}\, J_{-1/3}\left(\frac{2}{3}\alpha L^{3/2}\right) = 0,$$

so either $c_2 = 0$, in which case $\theta(t) = 0$, or $J_{-1/3}(\frac{2}{3}\alpha L^{3/2}) = 0$. The column will just start to bend when $L$ is the length corresponding to the smallest positive zero of $J_{-1/3}$.

**(c)** Using *Mathematica*, the first positive root of $J_{-1/3}(x)$ is $x_1 \approx 1.86635$. Thus $\frac{2}{3}\alpha L^{3/2} = 1.86635$ implies

$$L = \left(\frac{3(1.86635)}{2\alpha}\right)^{2/3} = \left[\frac{9EI}{4\delta g}(1.86635)^2\right]^{1/3}$$

$$= \left[\frac{9(2.6\times 10^7)\pi(0.05)^4/4}{4(0.28)\pi(0.05)^2}(1.86635)^2\right]^{1/3} \approx 76.9 \text{ in.}$$

**44. (a)** Writing the differential equation in the form $xy'' + (PL/M)y = 0$, we identify $\lambda = PL/M$. From Problem 37 the solution of this differential equation is

$$y = c_1\sqrt{x}\, J_1\left(2\sqrt{PLx/M}\right) + c_2\sqrt{x}\, Y_1\left(2\sqrt{PLx/M}\right).$$

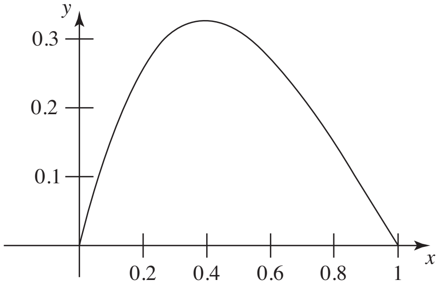
Now $J_1(0) = 0$, so $y(0) = 0$ implies $c_2 = 0$ and

$$y = c_1\sqrt{x}\, J_1\left(2\sqrt{PLx/M}\right).$$

**(b)** From $y(L)=0$ we have $y=J_1(2L\sqrt{PM}\,)=0$. The first positive zero of $J_1$ is 3.8317 so, solving $2L\sqrt{P_1/M}=3.8317$, we find $P_1=3.6705M/L^2$. Therefore,

$$y_1(x)=c_1\sqrt{x}\,J_1\left(2\sqrt{\frac{3.6705x}{L}}\right)=c_1\sqrt{x}\,J_1\left(\frac{3.8317}{\sqrt{L}}\sqrt{x}\right).$$

**(c)** For $c_1=1$ and $L=1$ the graph of $y_1=\sqrt{x}\,J_1(3.8317\sqrt{x}\,)$ is shown.



**45. (a)** Since $l'=v$, we integrate to obtain $l(t)=vt+c$. Now $l(0)=l_0$ implies $c=l_0$, so $l(t)=vt+l_0$. Using $\sin\theta\approx\theta$ in $l\,d^2\theta/dt^2+2l'\,d\theta/dt+g\sin\theta=0$ gives

$$(l_0+vt)\frac{d^2\theta}{dt^2}+2v\frac{d\theta}{dt}+g\theta=0.$$

**(b)** Dividing by $v$, the differential equation in part (a) becomes

$$\frac{l_0+vt}{v}\frac{d^2\theta}{dt^2}+2\frac{d\theta}{dt}+\frac{g}{v}\theta=0.$$

Letting $x=(l_0+vt)/v=t+l_0/v$ we have $dx/dt=1$, so

$$\frac{d\theta}{dt}=\frac{d\theta}{dx}\frac{dx}{dt}=\frac{d\theta}{dx}$$

and

$$\frac{d^2\theta}{dt^2}=\frac{d(d\theta/dt)}{dt}=\frac{d(d\theta/dx)}{dx}\frac{dx}{dt}=\frac{d^2\theta}{dx^2}.$$

Thus, the differential equation becomes

$$x\frac{d^2\theta}{dx^2}+2\frac{d\theta}{dx}+\frac{g}{v}\theta=0 \qquad \text{or} \qquad \frac{d^2\theta}{dx^2}+\frac{2}{x}\frac{d\theta}{dx}+\frac{g}{vx}\theta=0.$$

**(c)** The differential equation in part (b) has the form of (20) in the text with

$$1-2a=2 \qquad 2c-2=-1 \qquad b^2c^2=\frac{g}{v} \qquad a^2-p^2c^2=0$$

$$a=-\frac{1}{2} \qquad c=\frac{1}{2} \qquad b=2\sqrt{\frac{g}{v}} \qquad p=1.$$

Then, by (21) in the text,

$$\theta(x)=x^{-1/2}\left[c_1J_1\left(2\sqrt{\frac{g}{v}}\,x^{1/2}\right)+c_2Y_1\left(2\sqrt{\frac{g}{v}}\,x^{1/2}\right)\right]$$

or

$$\theta(t)=\sqrt{\frac{v}{l_0+vt}}\left[c_1J_1\left(\frac{2}{v}\sqrt{g(l_0+vt)}\right)+c_2Y_1\left(\frac{2}{v}\sqrt{g(l_0+vt)}\right)\right].$$

**(d)** To simplify calculations, let

$$u = \frac{2}{v}\sqrt{g(l_0 + vt)} = 2\sqrt{\frac{g}{v}}\, x^{1/2},$$

and at $t = 0$ let $u_0 = 2\sqrt{gl_0}/v$. The general solution for $\theta(t)$ can then be written

$$\theta = C_1 u^{-1} J_1(u) + C_2 u^{-1} Y_1(u). \tag{1}$$

Before applying the initial conditions, note that

$$\frac{d\theta}{dt} = \frac{d\theta}{du}\,\frac{du}{dt}$$

so when $d\theta/dt = 0$ at $t = 0$ we have $d\theta/du = 0$ at $u = u_0$. Also,

$$\frac{d\theta}{du} = C_1 \frac{d}{du}[u^{-1}J_1(u)] + C_2 \frac{d}{du}[u^{-1}Y_1(u)]$$

which, in view of (20) in the text, is the same as

$$\frac{d\theta}{du} = -C_1 u^{-1} J_2(u) - C_2 u^{-1} Y_2(u). \tag{2}$$

Now at $t = 0$, or $u = u_0$, (1) and (2) give the system

$$C_1 u_0^{-1} J_1(u_0) + C_2 u_0^{-1} Y_1(u_0) = \theta_0$$

$$C_1 u_0^{-1} J_2(u_0) + C_2 u_0^{-1} Y_2(u_0) = 0$$

whose solution is easily obtained using Cramer's rule:

$$C_1 = \frac{u_0\theta_0 Y_2(u_0)}{J_1(u_0)Y_2(u_0) - J_2(u_0)Y_1(u_0)}, \qquad C_2 = \frac{-u_0\theta_0 J_2(u_0)}{J_1(u_0)Y_2(u_0) - J_2(u_0)Y_1(u_0)}.$$

In view of the given identity these results simplify to

$$C_1 = -\frac{\pi}{2}\, u_0^2\, \theta_0\, Y_2(u_0) \qquad \text{and} \qquad C_2 = \frac{\pi}{2}\, u_0^2\, \theta_0\, J_2(u_0).$$

The solution is then

$$\theta = \frac{\pi}{2}\, u_0^2\, \theta_0 \left[ -Y_2(u_0)\frac{J_1(u)}{u} + J_2(u_0)\frac{Y_1(u)}{u} \right].$$

Returning to $u = (2/v)\sqrt{g(l_0 + vt)}$ and $u_0 = (2/v)\sqrt{gl_0}$, we have

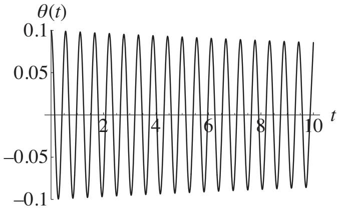
$$\theta(t) = \frac{\pi\sqrt{gl_0}\,\theta_0}{v} \left[ -Y_2\left(\frac{2}{v}\sqrt{gl_0}\right) \frac{J_1\left(\frac{2}{v}\sqrt{g(l_0 + vt)}\right)}{\sqrt{l_0 + vt}} \right.$$

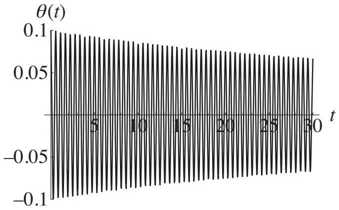
$$\left. + J_2\left(\frac{2}{v}\sqrt{gl_0}\right) \frac{Y_1\left(\frac{2}{v}\sqrt{g(l_0 + vt)}\right)}{\sqrt{l_0 + vt}} \right].$$

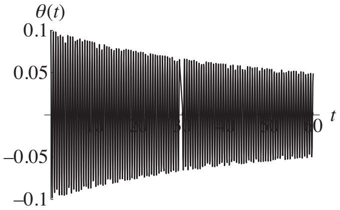
**(e)** When $l_0 = 1$ ft, $\theta_0 = \frac{1}{10}$ radian, and $v = \frac{1}{60}$ ft/s, the above function is

$$\theta(t) = -1.69045\,\frac{J_1(480\sqrt{2}(1+t/60))}{\sqrt{1+t/60}} - 2.79381\,\frac{Y_1(480\sqrt{2}\,(1+t/60))}{\sqrt{1+t/60}}\,.$$

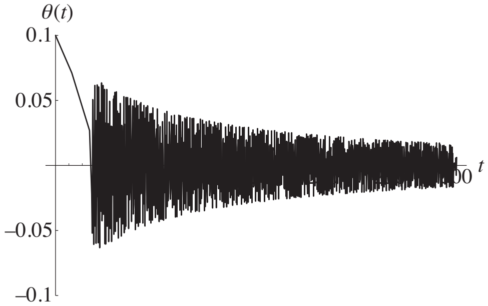
The plots of $\theta(t)$ on $[0,10]$, $[0,30]$, and $[0,60]$ are







**(f)** The graphs indicate that $\theta(t)$ decreases as $l$ increases. The graph of $\theta(t)$ on $[0,300]$ is shown.



**46. (a)** From (32) in the text, we have

$$P_6(x) = c_0\left(1 - \frac{6\cdot 7}{2!}\,x^2 + \frac{4\cdot 6\cdot 7\cdot 9}{4!}\,x^4 - \frac{2\cdot 4\cdot 6\cdot 7\cdot 9\cdot 11}{6!}\,x^6\right),$$

where

$$c_0 = (-1)^3\,\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6} = -\frac{5}{16}\,.$$

Thus,

$$P_6(x) = -\frac{5}{16}\left(1 - 21x^2 + 63x^4 - \frac{231}{5}\,x^6\right) = \frac{1}{16}(231x^6 - 315x^4 + 105x^2 - 5).$$

Also, from (26) in the text we have

$$P_7(x) = c_1\left(x - \frac{6\cdot 9}{3!}\,x^3 + \frac{4\cdot 6\cdot 9\cdot 11}{5!}\,x^5 - \frac{2\cdot 4\cdot 6\cdot 9\cdot 11\cdot 13}{7!}\,x^7\right)$$

where

$$c_1 = (-1)^3\,\frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6} = -\frac{35}{16}\,.$$

Thus

$$P_7(x) = -\frac{35}{16}\left(x - 9x^3 + \frac{99}{5}x^5 - \frac{429}{35}x^7\right) = \frac{1}{16}(429x^7 - 693x^5 + 315x^3 - 35x).$$

**(b)** $P_6(x)$ satisfies $\left(1-x^2\right)y'' - 2xy' + 42y = 0$ and $P_7(x)$ satisfies $\left(1-x^2\right)y'' - 2xy' + 56y = 0$.

**47.** The recurrence relation can be written

$$P_{k+1}(x)=\frac{2k+1}{k+1}\,xP_k(x)-\frac{k}{k+1}\,P_{k-1}(x),\qquad k=2,\ 3,\ 4,\ \ldots\ .$$

$$k=1:\quad P_2(x)=\frac{3}{2}x^2-\frac{1}{2}$$

$$k=2:\quad P_3(x)=\frac{5}{3}x\left(\frac{3}{2}x^2-\frac{1}{2}\right)-\frac{2}{3}x=\frac{5}{2}x^3-\frac{3}{2}x$$

$$k=3:\quad P_4(x)=\frac{7}{4}x\left(\frac{5}{2}x^3-\frac{3}{2}x\right)-\frac{3}{4}\left(\frac{3}{2}x^2-\frac{1}{2}\right)=\frac{35}{8}x^4-\frac{30}{8}x^2+\frac{3}{8}$$

$$k=4:\quad P_5(x)=\frac{9}{5}x\left(\frac{35}{8}x^4-\frac{30}{8}x^2+\frac{3}{8}\right)-\frac{4}{5}\left(\frac{5}{2}x^3-\frac{3}{2}x\right)=\frac{63}{8}x^5-\frac{35}{4}x^3+\frac{15}{8}x$$

$$k=5:\quad P_6(x)=\frac{11}{6}x\left(\frac{63}{8}x^5-\frac{35}{4}x^3+\frac{15}{8}x\right)-\frac{5}{6}\left(\frac{35}{8}x^4-\frac{30}{8}x^2+\frac{3}{8}\right)$$

$$=\frac{231}{16}x^6-\frac{315}{16}x^4+\frac{105}{16}x^2-\frac{5}{16}$$

$$k=6:\quad P_7(x)=\frac{13}{7}x\left(\frac{231}{16}x^6-\frac{315}{16}x^4+\frac{105}{16}x^2-\frac{5}{16}\right)-\frac{6}{7}\left(\frac{63}{8}x^5-\frac{35}{4}x^3+\frac{15}{8}x\right)$$

$$=\frac{429}{16}x^7-\frac{693}{16}x^5+\frac{315}{16}x^3-\frac{35}{16}x$$

**48.** If $x=\cos\theta$ then

$$\frac{dy}{d\theta}=-\sin\theta\,\frac{dy}{dx}\,,$$

$$\frac{d^2y}{d\theta^2}=\sin^2\theta\,\frac{d^2y}{dx^2}-\cos\theta\,\frac{dy}{dx}\,,$$

and

$$\sin\theta\,\frac{d^2y}{d\theta^2}+\cos\theta\,\frac{dy}{d\theta}+n(n+1)(\sin\theta)y=\sin\theta\left[\left(1-\cos^2\theta\right)\frac{d^2y}{dx^2}-2\cos\theta\,\frac{dy}{dx}+n(n+1)y\right]=0.$$

That is,

$$\left(1-x^2\right)\frac{d^2y}{dx^2}-2x\,\frac{dy}{dx}+n(n+1)y=0.$$

**49.** The only solutions bounded on $[-1,1]$ are $y=cP_n(x)$, $c$ a constant and $n=0,1,2,\ldots$. By (*iv*) of the properties of the Legendre polynomials, $y(0)=0$ or $P_n(0)=0$ implies $n$ must be odd. Thus the first three positive eigenvalues correspond to $n=1$, 3, and 5 or $\lambda_1=1\cdot2$, $\lambda_2=3\cdot4=12$, and $\lambda_3=5\cdot6=30$. We can take the eigenfunctions to be $y_1=P_1(x)$, $y_2=P_3(x)$, and $y_3=P_5(x)$.

**50.** Using a CAS we find

$$P_1(x)=\frac{1}{2}\,\frac{d}{dx}\,(x^2-1)^1=x\qquad P_2(x)=\frac{1}{2^22!}\,\frac{d^2}{dx^2}\,(x^2-1)^2=\frac{1}{2}(3x^2-1)$$

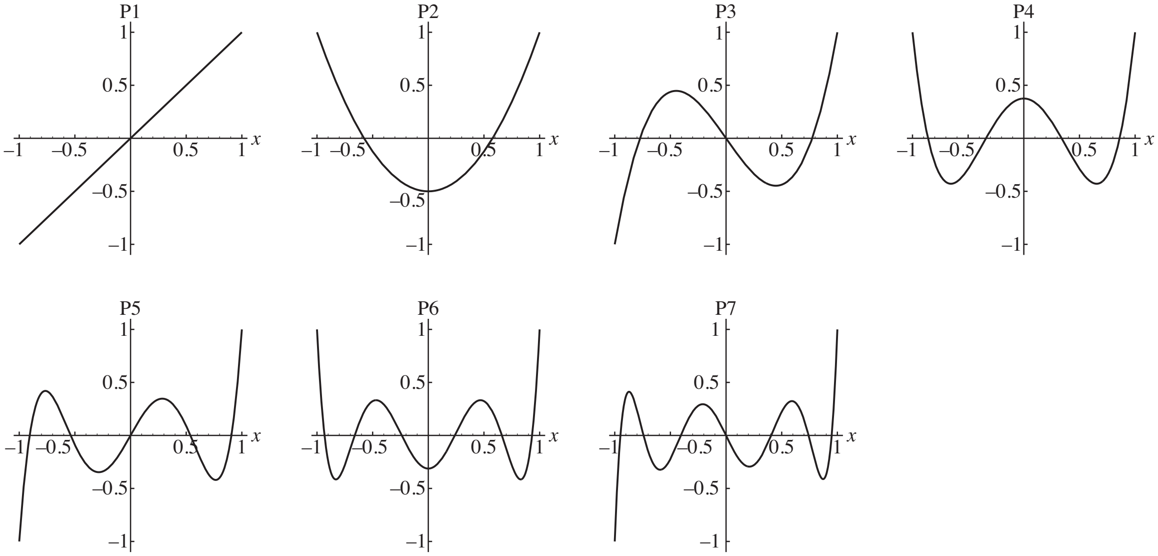
$$P_3(x)=\frac{1}{2^33!}\,\frac{d^3}{dx^3}\,(x^2-1)^3=\frac{1}{2}(5x^3-3x)\qquad P_4(x)=\frac{1}{2^44!}\,\frac{d^4}{dx^4}\,(x^2-1)^4=\frac{1}{8}(35x^4-30x^2+3)$$

$$P_5(x) = \frac{1}{2^5 5!}\,\frac{d^5}{dx^5}\,(x^2-1)^5 = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$

$$P_6(x) = \frac{1}{2^6 6!}\,\frac{d^6}{dx^6}\,(x^2-1)^6 = \frac{1}{16}(231x^6 - 315x^4 + 105x^2 - 5)$$

$$P_7(x) = \frac{1}{2^7 7!}\,\frac{d^7}{dx^7}\,(x^2-1)^7 = \frac{1}{16}(429x^7 - 693x^5 + 315x^3 - 35x)$$

**51.**



**52.** Zeros of Legendre polynomials for $n \geq 1$ are

$$P_1(x): 0$$
$$P_2(x): \pm 0.57735$$
$$P_3(x): 0,\ \pm 0.77460$$
$$P_4(x): \pm 0.33998,\ \pm 0.86115$$
$$P_5(x): 0,\ \pm 0.53847,\ \pm 0.90618$$
$$P_6(x): \pm 0.23862,\ \pm 0.66121,\ \pm 0.93247$$
$$P_7(x): 0,\ \pm 0.40585,\ \pm 0.74153\ ,\pm 0.94911$$
$$P_{10}(x): \pm 0.14887,\ \pm 0.43340,\ \pm 0.67941,\ \pm 0.86506,\ \pm 0.097391$$

The zeros of any Legendre polynomial are in the interval $(-1,1)$ and are symmetric with respect to 0.

**53.** Letting $y = \sum_{k=0}^{\infty} c_k x^k$ we have

$$y' = \sum_{k=1}^{\infty} nc_n x^{n-1} \qquad \text{and} \qquad y'' = \sum_{n=2}^{\infty} n(n+1)c_n x^{n-2}$$

Then, with appropriate substitutions, we have

$$\sum_{k=0}^{\infty}[(k+2)(k+1)c_{k+2}-2kc_k+2\alpha c_k]x^k=0$$

This leads to the recurrence relation

$$c_{k+2}=-\frac{(2\alpha-2k)}{(k+2)(k+1)}c_k, \qquad \text{for } k=0,1,2,3,\dots$$

Thus

$$\begin{aligned}
c_2&=-\frac{(2\alpha)}{(2)(1)}c_0=-\frac{2\alpha}{2!}c_0\\
c_3&=-\frac{(2\alpha-2)}{(3)(2)}c_1=-\frac{2(\alpha-1)}{3!}c_1\\
c_4&=-\frac{(2\alpha-4)}{(4)(3)}c_2=-\frac{2(\alpha-2)}{4\cdot 3}\cdot\left(-\frac{2\alpha}{2!}c_0\right)=\frac{2^2\alpha(\alpha-2)}{4!}c_0\\
c_5&=-\frac{(2\alpha-6)}{(5)(4)}c_3=-\frac{2(\alpha-3)}{5\cdot 4}\cdot\left(-\frac{2(\alpha-1)}{3!}c_1\right)=\frac{2^2(\alpha-1)(\alpha-3)}{5!}c_1\\
&\vdots
\end{aligned}$$

Then two solutions are

$$y_1(x)=1-\frac{2\alpha}{2!}x^2+\frac{2^2\alpha(\alpha-2)}{4!}x^4-\cdots=1+\sum_{k=1}^{\infty}\frac{(-1)^k2^k\alpha(\alpha-2)(\alpha-4)\dots(\alpha-(2k-2))}{(2k)!}x^{2k}$$

and

$$\begin{aligned}
y_2(x)&=x-\frac{2\alpha-1}{3!}x^3+\frac{2^2(\alpha-1)(\alpha-3)}{5!}x^5-\cdots\\
&=1+\sum_{k=1}^{\infty}\frac{(-1)^k2^k(\alpha-1)(\alpha-3)\dots(\alpha-(2k-1))}{(2k+1)!}x^{2k+1}
\end{aligned}$$

and the general solution is $y(x)=c_0y_1(x)+c_1y_2(x)$.

**54. (a)** From Problem 53 we know that

$$\begin{aligned}
y_1(x)&=1-\frac{2\alpha}{2!}x^2+\frac{2^2\alpha(\alpha-2)}{4!}x^4-\frac{2^3\alpha(\alpha-2)(\alpha-4)}{6!}x^6+\dots\\
y_2(x)&=x-\frac{2(\alpha-1)}{3!}x^3+\frac{2^2(\alpha-1)(\alpha-3)}{5!}x^5+\dots
\end{aligned}$$

Therefore we have the following

$$\begin{aligned}
&\text{When } \alpha=n=0: \quad y_1(x)=1-0x^2+0x^4-0x^6+\dots=1\\
&\text{When } \alpha=n=2: \quad y_1(x)=1-\frac{4}{2!}x^2+0x^4-0x^6+\dots=1-2x^2\\
&\text{When } \alpha=n=4: \quad y_1(x)=1-\frac{8}{2!}x^2+\frac{32}{4!}x^4-0x^6+\dots=1-4x^2+\frac{4}{3}x^4
\end{aligned}$$

Also

$$\begin{aligned}
&\text{When } \alpha = n = 1: \quad y_2(x) = x - 0x^3 + 0x^5 + \ldots = x \\
&\text{When } \alpha = n = 3: \quad y_2(x) = x - \frac{4}{3!}x^3 + 0x^5 + \ldots = x - \frac{2}{3}x^3 \\
&\text{When } \alpha = n = 5: \quad y_2(x) = x - \frac{8}{3!}x^3 + \frac{32}{5!}x^5 + 0x^7 + \ldots = x - \frac{4}{3}x^3 + \frac{4}{15}x^5
\end{aligned}$$

**(b)** Using the results of part (a) we get

$$\begin{aligned}
&H_0(x) = 2^0 \cdot (1) = 1 \\
&H_1(x) = 2^1 \cdot (x) = 2x \\
&H_2(x) = -2^1 \cdot (1 - 2x^2) = 2^2x^2 - 2 = 4x^2 - 2 \\
&H_3(x) = -2^2 \cdot 3\left(x - \frac{2}{3}x^3\right) = 2^3x^3 - 2^2 \cdot 3x = 8x^3 - 12x \\
&H_4(x) = 2^2 \cdot 3\left(1 - 4x^2 + \frac{4}{3}x^4\right) = 2^4x^4 - 4 \cdot 2^2 \cdot 3x^2 + 2^2 \cdot 3 = 16x^4 - 48x^2 + 12 \\
&H_5(x) = 2^3 \cdot 15\left(x - \frac{4}{3}x^3 + \frac{4}{15}x^5\right) = 2^5x^5 - 2^3 \cdot 5 \cdot 4x^3 + 2^3 \cdot 15x = 32x^5 - 160x^3 + 120x
\end{aligned}$$

**55.** Substitute the assumed solution $y = \sum_{k=0}^{\infty} c_k x^k$ into the equation to get

$$(1 - x^2) \cdot \sum_{k=2}^{\infty} k(k-1)c_k x^{k-2} - x \cdot \sum_{k=1}^{\infty} k c_k x^{k-1} + \alpha^2 \sum_{k=0}^{\infty} c_k x^k = 0$$

From this we get

$$(2c_2 + \alpha^2 c_0) + (6c_3 - c_1 + \alpha^2 c_1)x + \sum_{k=2}^{\infty} [(k+2)(k+1)c_{k+2} - k(k-1)c_k - kc_k + \alpha^2 c_k]x^k = 0$$

Therefore we have

$$c_2 = -\frac{\alpha^2}{2!}c_0, \qquad c_3 = \frac{1 - \alpha^2}{3!}c_1, \qquad \text{and} \qquad c_{k+2} = \frac{k^2 - \alpha^2}{(k+2)(k+1)}c_k \text{ for } k = 2,\, 3,\, 4,\, \ldots$$

Therefore we get

$$\begin{aligned}
y &= \sum_{k=0}^{\infty} c_k x^k = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + c_6 x^6 + \cdots \\
&= c_0 + c_1 x - \frac{\alpha^2}{2!} c_0 x^2 + \frac{1-\alpha^2}{3!} c_1 x^3 - \frac{(4-\alpha^2)\alpha^2}{4!} c_0 x^4 + \frac{(9-\alpha^2)(1-\alpha^2)}{5!} c_1 x^5 \\
&\qquad - \frac{(16-\alpha^2)(4-\alpha^2)\alpha^2}{6!} c_0 x^6 + \cdots \\
&= c_0 \left[ 1 - \frac{\alpha^2}{2!} x^2 - \frac{\alpha^2(4-\alpha^2)}{4!} x^4 - \frac{(16-\alpha^2)(4-\alpha^2)\alpha^2}{6!} x^6 + \cdots \right] \\
&\qquad + c_1 \left[ x + \frac{1-\alpha^2}{3!} x^3 + \frac{(9-\alpha^2)(1-\alpha^2)}{5!} x^5 + \cdots \right] \\
&= c_0 y_1(x) + c_1 y_2(x)
\end{aligned}$$

where we take

$$\begin{aligned}
y_1(x) &= 1 - \frac{\alpha^2}{2!} x^2 - \frac{(4-\alpha^2)\alpha^2}{4!} x^4 - \frac{(16-\alpha^2)(4-\alpha^2)\alpha^2}{6!} x^6 + \ldots \\
y_2(x) &= x + \frac{1-\alpha^2}{3!} x^3 + \frac{(9-\alpha^2)(1-\alpha^2)}{5!} x^5 + \ldots
\end{aligned}$$

When $\alpha = n$ is a nonnegative even integer $y_1(x)$ terminates at $x^n$, and when $\alpha = n$ is a positive odd integer $y_2(x)$ terminates at $x^n$. With $\alpha = n = 5$, $y_2(x)$ yields the fifth degree polynomial solution $y = x - 4x^3 + \frac{16}{5} x^5$.

**56.** With $R(x) = (\alpha x)^{-1/2} Z(z)$ the product rule gives:

$$R' = \alpha^{-1/2} \left( x^{-1/2} Z' - \frac{1}{2} x^{-3/2} Z \right)$$

and

$$\begin{aligned}
R'' &= \alpha^{-1/2} \left( x^{-1/2} Z'' - \frac{1}{2} x^{-3/2} Z' - \frac{1}{2} x^{-3/2} Z' + \frac{3}{4} x^{-5/2} Z \right) \\
&= \alpha^{-1/2} \left( x^{-1/2} Z'' - x^{-3/2} Z' + \frac{3}{4} x^{-5/2} Z \right).
\end{aligned}$$

The differential equation then becomes

$$\begin{aligned}
\alpha^{-1/2} x^2 \left( x^{-1/2} Z'' - x^{-3/2} Z' + \frac{3}{4} x^{-5/2} Z \right) + \alpha^{-1/2} 2x \left( x^{-1/2} Z' - \frac{1}{2} x^{-3/2} Z \right) \\
+ \left[ \alpha^2 x^2 - n(n+1) \right] \alpha^{-1/2} x^{-1/2} Z(x) = 0
\end{aligned}$$

or

$$x^2 Z'' + x Z' + \left[ \alpha^2 x^2 - \left( n^2 + n + \frac{1}{4} \right) \right] Z = 0.$$

This is equivalent to

$$x^2Z'' + xZ' + \left[\alpha^2x^2 - \left(n + \frac{1}{2}\right)^2\right] Z = 0.$$

which is the parametric Bessel equation, so

$$Z(x) = C_1 J_{n+1/2}(\alpha x) + C_2 Y_{n+1/2}(\alpha x),$$

and

$$R(x) = \alpha^{-1/2}x^{1/2}Z(x) = \alpha^{-1/2}x^{-1/2}\left[C_1 J_{n+1/2}(\alpha x) + C_2 Y_{n+1/2}(\alpha x)\right].$$

Renaming $C_1$ and $C_2$ this becomes

$$\begin{aligned} R(x) &= \left(c_1\sqrt{\frac{\pi}{2}}\right)\frac{J_{n+1/2}(\alpha x)}{\sqrt{\alpha x}} + \left(c_2\sqrt{\frac{\pi}{2}}\right)\frac{Y_{n+1/2}(\alpha x)}{\sqrt{\alpha x}} \\ &= c_1\sqrt{\frac{\pi}{2\alpha x}}\,J_{n+1/2}(\alpha x) + c_2\sqrt{\frac{\pi}{2\alpha x}}\,Y_{n+1/2}(\alpha x) \\ &= c_1 j_n(\alpha x) + c_2 y_n(\alpha x), \end{aligned}$$

where $j_n(\alpha x)$ and $y_n(\alpha x)$ are the spherical Bessel functions of the first and second kind defined in the text.

## Chapter 6 in Review

**1.** False; $J_1(x)$ and $J_{-1}(x)$ are not linearly independent when $\nu$ is a positive integer. (In this case $\nu = 1$). The general solution of $x^2y'' + xy' + (x^2 - 1)y = 0$ is $y = c_1J_1(x) + c_2Y_1(x)$.

**2.** False; $y = x$ is a solution that is analytic at $x = 0$.

**3.** $x = -1$ is the nearest singular point to the ordinary point $x = 0$. Theorem 5.1.1 guarantees the existence of two power series solutions $y = \sum_{n=1}^{\infty} c_n x^n$ of the differential equation that converge at least for $-1 < x < 1$. Since $-\frac{1}{2} \le x \le \frac{1}{2}$ is properly contained in $-1 < x < 1$, both power series must converge for all points contained in $-\frac{1}{2} \le x \le \frac{1}{2}$.

**4.** The easiest way to solve the system

$$\begin{aligned} 2c_2 + 2c_1 + c_0 &= 0 \\ 6c_3 + 4c_2 + c_1 &= 0 \\ 12c_4 + 6c_3 - \frac{1}{3}c_1 + c_2 &= 0 \\ 20c_5 + 8c_4 - \frac{2}{3}c_2 + c_3 &= 0 \end{aligned}$$

is to choose, in turn, $c_0 \neq 0$, $c_1 = 0$ and $c_0 = 0$, $c_1 \neq 0$. Assuming that $c_0 \neq 0$, $c_1 = 0$, we have

$$\begin{aligned} c_2 &= -\frac{1}{2}c_0 \\ c_3 &= -\frac{2}{3}c_2 = \frac{1}{3}c_0 \\ c_4 &= -\frac{1}{2}c_3 - \frac{1}{12}c_2 = -\frac{1}{8}c_0 \\ c_5 &= -\frac{2}{5}c_4 + \frac{1}{30}c_2 - \frac{1}{20}c_3 = \frac{1}{60}c_0; \end{aligned}$$

whereas the assumption that $c_0 = 0$, $c_1 \neq 0$ implies

$$\begin{aligned} c_2 &= -c_1 \\ c_3 &= -\frac{2}{3}c_2 - \frac{1}{6}c_1 = \frac{1}{2}c_1 \\ c_4 &= -\frac{1}{2}c_3 + \frac{1}{36}c_1 - \frac{1}{12}c_2 = -\frac{5}{36}c_1 \\ c_5 &= -\frac{2}{5}c_4 + \frac{1}{30}c_2 - \frac{1}{20}c_3 = -\frac{1}{360}c_1. \end{aligned}$$

five terms of two power series solutions are then

$$y_1(x) = c_0\left[1 - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{8}x^4 + \frac{1}{60}x^5 + \cdots\right]$$

and

$$y_2(x) = c_1\left[x - x^2 + \frac{1}{2}x^3 - \frac{5}{36}x^4 - \frac{1}{360}x^5 + \cdots\right].$$

**5.** The interval of convergence is centered at 4. Since the series converges at $-2$, it converges at least on the interval $[-2, 10)$. Since it diverges at 13, it converges at most on the interval $[-5, 13)$. Thus, at $-7$ it does not converge, at 0 and 7 it does converge, and at 10 and 11 it might converge.

**6.** We have

$$f(x) = \frac{\sin x}{\cos x} = \frac{x - \dfrac{x^3}{6} + \dfrac{x^5}{120} - \cdots}{1 - \dfrac{x^2}{2} + \dfrac{x^4}{24} - \cdots} = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots.$$

**7.** The differential equation $(x^3 - x^2)y'' + y' + y = 0$ has a regular singular point at $x = 1$ and an irregular singular point at $x = 0$.

**8.** The differential equation $(x-1)(x+3)y'' + y = 0$ has regular singular points at $x = 1$ and $x = -3$.

**9.** Substituting $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ into the differential equation we obtain

$$2xy'' + y' + y = \left(2r^2 - r\right) c_0 x^{r-1} + \sum_{k=1}^{\infty} [2(k+r)(k+r-1)c_k + (k+r)c_k + c_{k-1}]x^{k+r-1} = 0$$

which implies

$$2r^2 - r = r(2r-1) = 0$$

and

$$(k+r)(2k+2r-1)c_k + c_{k-1} = 0.$$

The indicial roots are $r = 0$ and $r = 1/2$. For $r = 0$ the recurrence relation is

$$c_k = -\frac{c_{k-1}}{k(2k-1)}, \quad k = 1, 2, 3, \dots,$$

so

$$c_1 = -c_0, \qquad c_2 = \frac{1}{6}c_0, \qquad c_3 = -\frac{1}{90}c_0.$$

For $r = 1/2$ the recurrence relation is

$$c_k = -\frac{c_{k-1}}{k(2k+1)}, \quad k = 1, 2, 3, \dots,$$

so

$$c_1 = -\frac{1}{3}c_0, \qquad c_2 = \frac{1}{30}c_0, \qquad c_3 = -\frac{1}{630}c_0.$$

Two linearly independent solutions are

$$y_1 = 1 - x + \frac{1}{6}x^2 - \frac{1}{90}x^3 + \cdots$$

and

$$y_2 = x^{1/2}\left(1 - \frac{1}{3}x + \frac{1}{30}x^2 - \frac{1}{630}x^3 + \cdots\right).$$

**10.** Substituting $y = \sum_{n=0}^{\infty} c_n x^n$ into the differential equation we have

$$y'' - xy' - y = \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{k=n-2} - \underbrace{\sum_{n=1}^{\infty} nc_n x^n}_{k=n} - \underbrace{\sum_{n=0}^{\infty} c_n x^n}_{k=n}$$

$$= \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k - \sum_{k=1}^{\infty} kc_k x^k - \sum_{k=0}^{\infty} c_k x^k$$

$$= 2c_2 - c_0 + \sum_{k=1}^{\infty}[(k+2)(k+1)c_{k+2} - (k+1)c_k]x^k = 0.$$

Thus

$$2c_2 - c_0 = 0(k+2)(k+1)c_{k+2} - (k+1)c_k = 0$$

and

$$c_2 = \frac{1}{2}c_0$$
$$c_{k+2} = \frac{1}{k+2}\,c_k, \quad k = 1, 2, 3, \dots .$$

Choosing $c_0 = 1$ and $c_1 = 0$ we find

$$c_2 = \frac{1}{2} \qquad c_3 = c_5 = c_7 = \cdots = 0 \qquad c_4 = \frac{1}{8} \qquad c_6 = \frac{1}{48}$$

and so on. For $c_0 = 0$ and $c_1 = 1$ we obtain

$$c_2 = c_4 = c_6 = \cdots = 0 \qquad c_3 = \frac{1}{3} \qquad c_5 = \frac{1}{15} \qquad c_7 = \frac{1}{105}$$

and so on. Thus, two solutions are

$$y_1 = 1 + \frac{1}{2}x^2 + \frac{1}{8}x^4 + \frac{1}{48}x^6 + \cdots$$

and

$$y_2 = x + \frac{1}{3}x^3 + \frac{1}{15}x^5 + \frac{1}{105}x^7 + \cdots .$$

**11.** Substituting $y = \sum_{n=0}^{\infty} c_n x^n$ into the differential equation we obtain

$$(x-1)y'' + 3y = (-2c_2 + 3c_0) + \sum_{k=1}^{\infty}[(k+1)kc_{k+1} - (k+2)(k+1)c_{k+2} + 3c_k]x^k = 0$$

which implies $c_2 = 3c_0/2$ and

$$c_{k+2} = \frac{(k+1)kc_{k+1} + 3c_k}{(k+2)(k+1)}, \quad k = 1, 2, 3, \dots .$$

Choosing $c_0 = 1$ and $c_1 = 0$ we find

$$c_2 = \frac{3}{2}, \qquad c_3 = \frac{1}{2}, \qquad c_4 = \frac{5}{8}$$

and so on. For $c_0 = 0$ and $c_1 = 1$ we obtain

$$c_2 = 0, \qquad c_3 = \frac{1}{2}, \qquad c_4 = \frac{1}{4}$$

and so on. Thus, two solutions are

$$y_1 = 1 + \frac{3}{2}x^2 + \frac{1}{2}x^3 + \frac{5}{8}x^4 + \cdots$$

and

$$y_2 = x + \frac{1}{2}x^3 + \frac{1}{4}x^4 + \cdots .$$

**12.** Substituting $y = \sum_{n=0}^{\infty} c_n x^n$ into the differential equation we obtain

$$y'' - x^2y' + xy = 2c_2 + (6c_3 + c_0)x + \sum_{k=1}^{\infty}[(k+3)(k+2)c_{k+3} - (k-1)c_k]x^{k+1} = 0$$

which implies $c_2 = 0$, $c_3 = -c_0/6$, and

$$c_{k+3} = \frac{k-1}{(k+3)(k+2)}c_k, \quad k = 1, 2, 3, \ldots .$$

Choosing $c_0 = 1$ and $c_1 = 0$ we find

$$c_3 = -\frac{1}{6} \qquad c_4 = c_7 = c_{10} = \cdots = 0 \qquad c_5 = c_8 = c_{11} = \cdots = 0 \qquad c_6 = -\frac{1}{90}$$

and so on. For $c_0 = 0$ and $c_1 = 1$ we obtain

$$c_3 = c_6 = c_9 = \cdots = 0 \qquad c_4 = c_7 = c_{10} = \cdots = 0 \qquad c_5 = c_8 = c_{11} = \cdots = 0$$

and so on. Thus, two solutions are

$$y_1 = 1 - \frac{1}{6}x^3 - \frac{1}{90}x^6 - \cdots \quad \text{and} \quad y_2 = x.$$

**13.** Substituting $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ into the differential equation, we obtain

$$xy'' - (x+2)y' + 2y = (r^2 - 3r)c_0x^{r-1} + \sum_{k=1}^{\infty}[(k+r)(k+r-3)c_k - (k+r-3)c_{k-1}]x^{k+r-1} = 0,$$

which implies

$$r^2 - 3r = r(r-3) = 0$$

and

$$(k+r)(k+r-3)c_k - (k+r-3)c_{k-1} = 0.$$

The indicial roots are $r_1 = 3$ and $r_2 = 0$. For $r_2 = 0$ the recurrence relation is

$$k(k-3)c_k - (k-3)c_{k-1} = 0, \qquad k = 1,\ 2,\ 3,\ \ldots .$$

Then

$$c_1 - c_0 = 0 \qquad 2c_2 - c_1 = 0 \qquad 0c_3 - 0c_2 = 0$$

Therefore $c_3$ is arbitrary and

$$c_k = \frac{1}{k}c_{k-1}, \qquad k = 4,\ 5,\ 6,\ \ldots .$$

Taking $c_0 \neq 0$ and $c_3 = 0$ we obtain

$$c_1 = c_0 \qquad c_2 = \frac{1}{2}c_0 \qquad c_3 = c_4 = c_5 = \cdots = 0.$$

Taking $c_0 = 0$ and $c_3 \neq 0$ we obtain

$$c_0 = c_1 = c_2 = 0 \qquad c_4 = \frac{1}{4}c_3 = \frac{6}{4!}c_3 \qquad c_5 = \frac{1}{5\cdot 4}c_3 = \frac{6}{5!}c_3 \qquad c_6 = \frac{1}{6\cdot 5\cdot 4}c_3 = \frac{6}{6!}c_3,$$

and so on. In this case we obtain the two solutions

$$y_1 = 1 + x + \frac{1}{2}x^2$$

and

$$y_2 = x^3 + \frac{6}{4!}x^4 + \frac{6}{5!}x^5 + \frac{6}{6!}x^6 + \cdots = 6e^x - 6\left(1 + x + \frac{1}{2}x^2\right).$$

**14.** Substituting $y = \sum_{n=0}^{\infty} c_n x^n$ into the differential equation we have

$$\begin{aligned}(\cos x)y'' + y &= \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \cdots\right)(2c_2 + 6c_3x + 12c_4x^2 + 20c_5x^3 \\ &\qquad + 30c_6x^4 + \cdots) + \sum_{n=0}^{\infty} c_n x^n \\ &= \left[2c_2 + 6c_3x + (12c_4 - c_2)x^2 + (20c_5 - 3c_3)x^3 + \left(30c_6 - 6c_4 + \frac{1}{12}c_2\right)x^4 + \cdots\right] \\ &\qquad + \left[c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \cdots\right] \\ &= (c_0 + 2c_2) + (c_1 + 6c_3)x + 12c_4x^2 + (20c_5 - 2c_3)x^3 + \left(30c_6 - 5c_4 + \frac{1}{12}c_2\right)x^4 + \cdots \\ &= 0.\end{aligned}$$

Thus

$$c_0 + 2c_2 = 0 \qquad c_1 + 6c_3 = 0 \qquad 12c_4 = 0 \qquad 20c_5 - 2c_3 = 0 \qquad 30c_6 - 5c_4 + \frac{1}{12}c_2 = 0$$

and

$$c_2 = -\frac{1}{2}c_0 \qquad c_3 = -\frac{1}{6}c_1 \qquad c_4 = 0 \qquad c_5 = \frac{1}{10}c_3 \qquad c_6 = \frac{1}{6}c_4 - \frac{1}{360}c_2.$$

Choosing $c_0 = 1$ and $c_1 = 0$ we find

$$c_2 = -\frac{1}{2}, \qquad c_3 = 0, \qquad c_4 = 0, \qquad c_5 = 0, \qquad c_6 = \frac{1}{720}$$

and so on. For $c_0 = 0$ and $c_1 = 1$ we find

$$c_2 = 0, \qquad c_3 = -\frac{1}{6}, \qquad c_4 = 0, \qquad c_5 = -\frac{1}{60}, \qquad c_6 = 0$$

and so on. Thus, two solutions are

$$y_1 = 1 - \frac{1}{2}x^2 + \frac{1}{720}x^6 + \cdots \qquad \text{and} \qquad y_2 = x - \frac{1}{6}x^3 - \frac{1}{60}x^5 + \cdots.$$

**15.** Substituting $y = \sum_{n=0}^{\infty} c_n x^n$ into the differential equation we have

$$y'' + xy' + 2y = \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{k=n-2} + \underbrace{\sum_{n=1}^{\infty} nc_n x^n}_{k=n} + 2\underbrace{\sum_{n=0}^{\infty} c_n x^n}_{k=n}$$

$$= \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k + \sum_{k=1}^{\infty} kc_k x^k + 2\sum_{k=0}^{\infty} c_k x^k$$

$$= 2c_2 + 2c_0 + \sum_{k=1}^{\infty}[(k+2)(k+1)c_{k+2} + (k+2)c_k]x^k = 0.$$

Thus

$$2c_2 + 2c_0 = 0$$
$$(k+2)(k+1)c_{k+2} + (k+2)c_k = 0$$

and

$$c_2 = -c_0$$
$$c_{k+2} = -\frac{1}{k+1}\,c_k, \quad k = 1, 2, 3, \dots .$$

Choosing $c_0 = 1$ and $c_1 = 0$ we find

$$c_2 = -1 \qquad c_3 = c_5 = c_7 = \cdots = 0 \qquad c_4 = \frac{1}{3} \qquad c_6 = -\frac{1}{15}$$

and so on. For $c_0 = 0$ and $c_1 = 1$ we obtain

$$c_2 = c_4 = c_6 = \cdots = 0 \qquad c_3 = -\frac{1}{2} \qquad c_5 = \frac{1}{8} \qquad c_7 = -\frac{1}{48}$$

and so on. Thus, the general solution is

$$y = C_0\left(1 - x^2 + \frac{1}{3}x^4 - \frac{1}{15}x^6 + \cdots\right) + C_1\left(x - \frac{1}{2}x^3 + \frac{1}{8}x^5 - \frac{1}{48}x^7 + \cdots\right)$$

and

$$y' = C_0\left(-2x + \frac{4}{3}x^3 - \frac{2}{5}x^5 + \cdots\right) + C_1\left(1 - \frac{3}{2}x^2 + \frac{5}{8}x^4 - \frac{7}{48}x^6 + \cdots\right).$$

Setting $y(0) = 3$ and $y'(0) = -2$ we find $c_0 = 3$ and $c_1 = -2$. Therefore, the solution of the initial-value problem is

$$y = 3 - 2x - 3x^2 + x^3 + x^4 - \frac{1}{4}x^5 - \frac{1}{5}x^6 + \frac{1}{24}x^7 + \cdots .$$

**16.** Substituting $y = \sum_{n=0}^{\infty} c_n x^n$ into the differential equation we have

$$\begin{aligned}
(x+2)y'' + 3y &= \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-1}}_{k=n-1} + 2\underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{k=n-2} + 3\underbrace{\sum_{n=0}^{\infty} c_n x^n}_{k=n} \\
&= \sum_{k=1}^{\infty}(k+1)kc_{k+1}x^k + 2\sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k + 3\sum_{k=0}^{\infty} c_k x^k \\
&= 4c_2 + 3c_0 + \sum_{k=1}^{\infty}[(k+1)kc_{k+1} + 2(k+2)(k+1)c_{k+2} + 3c_k]x^k = 0.
\end{aligned}$$

Thus

$$\begin{gathered}
4c_2 + 3c_0 = 0 \\
(k+1)kc_{k+1} + 2(k+2)(k+1)c_{k+2} + 3c_k = 0
\end{gathered}$$

and

$$\begin{aligned}
c_2 &= -\frac{3}{4}c_0 \\
c_{k+2} &= -\frac{k}{2(k+2)}c_{k+1} - \frac{3}{2(k+2)(k+1)}c_k, \qquad k = 1, 2, 3, \dots .
\end{aligned}$$

Choosing $c_0 = 1$ and $c_1 = 0$ we find

$$c_2 = -\frac{3}{4} \qquad c_3 = \frac{1}{8} \qquad c_4 = \frac{1}{16} \qquad c_5 = -\frac{9}{320}$$

and so on. For $c_0 = 0$ and $c_1 = 1$ we obtain

$$c_2 = 0 \qquad c_3 = -\frac{1}{4} \qquad c_4 = \frac{1}{16} \qquad c_5 = 0$$

and so on. Thus, the general solution is

$$y = C_0\left(1 - \frac{3}{4}x^2 + \frac{1}{8}x^3 + \frac{1}{16}x^4 - \frac{9}{320}x^5 + \cdots\right) + C_1\left(x - \frac{1}{4}x^3 + \frac{1}{16}x^4 + \cdots\right)$$

and

$$y' = C_0\left(-\frac{3}{2}x + \frac{3}{8}x^2 + \frac{1}{4}x^3 - \frac{9}{64}x^4 + \cdots\right) + C_1\left(1 - \frac{3}{4}x^2 + \frac{1}{4}x^3 + \cdots\right).$$

Setting $y(0) = 0$ and $y'(0) = 1$ we find $c_0 = 0$ and $c_1 = 1$. Therefore, the solution of the initial-value problem is

$$y = x - \frac{1}{4}x^3 + \frac{1}{16}x^4 + \cdots .$$

**17.** The singular point of $(1 - 2\sin x)y'' + xy = 0$ closest to $x = 0$ is $\pi/6$. Hence a lower bound is $\pi/6$.

**18.** While we can find two solutions of the form

$$y_1 = c_0[1 + \cdots] \quad \text{and} \quad y_2 = c_1[x + \cdots],$$

the initial conditions at $x = 1$ give solutions for $c_0$ and $c_1$ in terms of infinite series. Letting $t = x - 1$ the initial-value problem becomes

$$\frac{d^2y}{dt^2} + (t+1)\frac{dy}{dt} + y = 0, \qquad y(0) = -6,\ y'(0) = 3.$$

Substituting $y = \sum_{n=0}^{\infty} c_n t^n$ into the differential equation, we have

$$\begin{aligned}
\frac{d^2y}{dt^2} + (t+1)\frac{dy}{dt} + y &= \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n t^{n-2}}_{k=n-2} + \underbrace{\sum_{n=1}^{\infty} nc_n t^n}_{k=n} + \underbrace{\sum_{n=1}^{\infty} nc_n t^{n-1}}_{k=n-1} + \underbrace{\sum_{n=0}^{\infty} c_n t^n}_{k=n} \\
&= \sum_{k=0}^{\infty} (k+2)(k+1)c_{k+2}t^k + \sum_{k=1}^{\infty} kc_k t^k + \sum_{k=0}^{\infty} (k+1)c_{k+1}t^k + \sum_{k=0}^{\infty} c_k t^k \\
&= 2c_2 + c_1 + c_0 + \sum_{k=1}^{\infty} \left[(k+2)(k+1)c_{k+2} + (k+1)c_{k+1} + (k+1)c_k\right] t^k = 0.
\end{aligned}$$

Thus

$$\begin{gathered}
2c_2 + c_1 + c_0 = 0 \\
(k+2)(k+1)c_{k+2} + (k+1)c_{k+1} + (k+1)c_k = 0
\end{gathered}$$

and

$$\begin{aligned}
c_2 &= -\frac{c_1 + c_0}{2} \\
c_{k+2} &= -\frac{c_{k+1} + c_k}{k+2}, \quad k = 1, 2, 3, \ldots .
\end{aligned}$$

Choosing $c_0 = 1$ and $c_1 = 0$ we find

$$c_2 = -\frac{1}{2}, \qquad c_3 = \frac{1}{6}, \qquad c_4 = \frac{1}{12},$$

and so on. For $c_0 = 0$ and $c_1 = 1$ we find

$$c_2 = -\frac{1}{2}, \qquad c_3 = -\frac{1}{6}, \qquad c_4 = \frac{1}{6},$$

and so on. Thus, the general solution is

$$y = c_0\left[1 - \frac{1}{2}t^2 + \frac{1}{6}t^3 + \frac{1}{12}t^4 + \cdots\right] + c_1\left[t - \frac{1}{2}t^2 - \frac{1}{6}t^3 + \frac{1}{6}t^4 + \cdots\right].$$

The initial conditions then imply $c_0 = -6$ and $c_1 = 3$. Thus the solution of the initial-value problem is

$$\begin{aligned} y = -6 &\left[1 - \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 + \frac{1}{12}(x-1)^4 + \cdots\right] \\ &+ 3\left[(x-1) - \frac{1}{2}(x-1)^2 - \frac{1}{6}(x-1)^3 + \frac{1}{6}(x-1)^4 + \cdots\right]. \end{aligned}$$

**19.** Writing the differential equation in the form

$$y'' + \left(\frac{1-\cos x}{x}\right)y' + xy = 0,$$

and noting that

$$\frac{1-\cos x}{x} = \frac{x}{2} - \frac{x^3}{24} + \frac{x^5}{720} - \cdots$$

is analytic at $x = 0$, we conclude that $x = 0$ is an ordinary point of the differential equation.

**20.** Writing the differential equation in the form

$$y'' + \left(\frac{x}{e^x - 1 - x}\right)y = 0$$

and noting that

$$\frac{x}{e^x - 1 - x} = \frac{2}{x} - \frac{2}{3} + \frac{x}{18} + \frac{x^2}{270} - \cdots$$

we see that $x = 0$ is a singular point of the differential equation. Since

$$x^2\left(\frac{x}{e^x - 1 - x}\right) = 2x - \frac{2x^2}{3} + \frac{x^3}{18} + \frac{x^4}{270} - \cdots,$$

we conclude that $x = 0$ is a regular singular point.

**21.** Substituting $y = \sum_{n=0}^{\infty} c_n x^n$ into the differential equation we have

$$\begin{aligned} y'' + x^2y' + 2xy &= \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{k=n-2} + \underbrace{\sum_{n=1}^{\infty} nc_n x^{n+1}}_{k=n+1} + 2\underbrace{\sum_{n=0}^{\infty} c_n x^{n+1}}_{k=n+1} \\ &= \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k + \sum_{k=2}^{\infty}(k-1)c_{k-1}x^k + 2\sum_{k=1}^{\infty} c_{k-1}x^k \\ &= 2c_2 + (6c_3 + 2c_0)x + \sum_{k=2}^{\infty}[(k+2)(k+1)c_{k+2} + (k+1)c_{k-1}]x^k = 5 - 2x + 10x^3. \end{aligned}$$

Thus, equating coefficients of like powers of $x$ gives

$$2c_2 = 5 \qquad 6c_3 + 2c_0 = -2 \qquad 12c_4 + 3c_1 = 0 \qquad 20c_5 + 4c_2 = 10$$

and

$$(k+2)(k+1)c_{k+2} + (k+1)c_{k-1} = 0, \quad k = 4, 5, 6, \dots,$$

Therefore

$$c_2 = \frac{5}{2} \qquad c_3 = -\frac{1}{3}c_0 - \frac{1}{3} \qquad c_4 = -\frac{1}{4}c_1 \qquad c_5 = \frac{1}{2} - \frac{1}{5}c_2 = \frac{1}{2} - \frac{1}{5}\left(\frac{5}{2}\right) = 0$$

and

$$c_{k+2} = -\frac{1}{k+2}\,c_{k-1}.$$

Using the recurrence relation, we find

$$\begin{aligned}
c_6 &= -\frac{1}{6}c_3 = \frac{1}{3\cdot 6}(c_0+1) = \frac{1}{3^2\cdot 2!}c_0 + \frac{1}{3^2\cdot 2!}\\
c_7 &= -\frac{1}{7}c_4 = \frac{1}{4\cdot 7}c_1\\
c_8 &= c_{11} = c_{14} = \cdots = 0\\
c_9 &= -\frac{1}{9}c_6 = -\frac{1}{3^3\cdot 3!}c_0 - \frac{1}{3^3\cdot 3!}\\
c_{10} &= -\frac{1}{10}c_7 = -\frac{1}{4\cdot 7\cdot 10}c_1\\
c_{12} &= -\frac{1}{12}c_9 = \frac{1}{3^4\cdot 4!}c_0 + \frac{1}{3^4\cdot 4!}\\
c_{13} &= -\frac{1}{13}c_0 = \frac{1}{4\cdot 7\cdot 10\cdot 13}c_1
\end{aligned}$$

and so on. Thus

$$\begin{aligned}
y = c_0&\left[1 - \frac{1}{3}x^3 + \frac{1}{3^2\cdot 2!}x^6 - \frac{1}{3^3\cdot 3!}x^9 + \frac{1}{3^4\cdot 4!}x^{12} - \cdots\right]\\
&+ c_1\left[x - \frac{1}{4}x^4 + \frac{1}{4\cdot 7}x^7 - \frac{1}{4\cdot 7\cdot 10}x^{10} + \frac{1}{4\cdot 7\cdot 10\cdot 13}x^{13} - \cdots\right]\\
&+ \left[\frac{5}{2}x^2 - \frac{1}{3}x^3 + \frac{1}{3^2\cdot 2!}x^6 - \frac{1}{3^3\cdot 3!}x^9 + \frac{1}{3^4\cdot 4!}x^{12} - \cdots\right].
\end{aligned}$$

**22. (a)** From $y = -\dfrac{1}{u}\dfrac{du}{dx}$ we obtain

$$\frac{dy}{dx} = -\frac{1}{u}\frac{d^2u}{dx^2} + \frac{1}{u^2}\left(\frac{du}{dx}\right)^2.$$

Then $dy/dx = x^2 + y^2$ becomes

$$-\frac{1}{u}\frac{d^2u}{dx^2} + \frac{1}{u^2}\left(\frac{du}{dx}\right)^2 = x^2 + \frac{1}{u^2}\left(\frac{du}{dx}\right)^2,$$

so $\dfrac{d^2u}{dx^2} + x^2u = 0$.

**(b)** The differential equation $u'' + x^2 u = 0$ has the form (20) in the text with

$$\begin{array}{cccc} 1-2a=0 & 2c-2=2 & b^2c^2=1 & a^2-p^2c^2=0 \\ a=\dfrac{1}{2} & c=2 & b=\dfrac{1}{2} & p=\dfrac{1}{4} \end{array}$$

Then, by (19) in the text,

$$u = x^{1/2}\left[c_1 J_{1/4}\left(\frac{1}{2}x^2\right) + c_2 J_{-1/4}\left(\frac{1}{2}x^2\right)\right].$$

**(c)** Letting $t = \frac{1}{2}x^2$ and $w(t) = c_1 J_{1/4}(t) + c_2 J_{-1/4}(t)$, we have

$$y = -\frac{1}{u}\frac{du}{dx} = -\frac{1}{x^{1/2}w(t)}\frac{d}{dx}\,x^{1/2}w(t) = -\frac{1}{x^{1/2}w}\left[x^{1/2}\frac{dw}{dt}\frac{dt}{dx} + \frac{1}{2}x^{-1/2}w\right]$$

$$= -\frac{1}{x^{1/2}w}\left[x^{3/2}\frac{dw}{dt} + \frac{1}{2x^{1/2}}\,w\right] = -\frac{1}{2xw}\left[2x^2\frac{dw}{dt} + w\right] = -\frac{1}{2xw}\left[4t\,\frac{dw}{dt} + w\right].$$

Now

$$\begin{aligned} 4t\,\frac{dw}{dt} + w &= 4t\,\frac{d}{dt}[c_1 J_{1/4}(t) + c_2 J_{-1/4}(t)] + c_1 J_{1/4}(t) + c_2 J_{-1/4}(t) \\ &= 4t\left[c_1\left(J_{-3/4}(t) - \frac{1}{4t}J_{1/4}(t)\right) + c_2\left(-\frac{1}{4t}J_{-1/4}(t) - J_{3/4}(t)\right)\right] \\ &\qquad + c_1 J_{1/4}(t) + c_2 J_{-1/4}(t) \\ &= 4c_1 t J_{-3/4}(t) - 4c_2 t J_{3/4}(t) = 2c_1 x^2 J_{-3/4}\left(\frac{1}{2}x^2\right) - 2c_2 x^2 J_{3/4}\left(\frac{1}{2}x^2\right), \end{aligned}$$

so

$$y = -\frac{2c_1x^2J_{-3/4}(\frac{1}{2}x^2) - 2c_2x^2J_{3/4}(\frac{1}{2}x^2)}{2x[c_1J_{1/4}(\frac{1}{2}x^2) + c_2J_{-1/4}(\frac{1}{2}x^2)]} = x\,\frac{-c_1J_{-3/4}(\frac{1}{2}x^2) + c_2J_{3/4}(\frac{1}{2}x^2)}{c_1J_{1/4}(\frac{1}{2}x^2) + c_2J_{-1/4}(\frac{1}{2}x^2)}.$$

Letting $c = c_1/c_2$ we have

$$y = x\,\frac{J_{3/4}(\frac{1}{2}x^2) - cJ_{-3/4}(\frac{1}{2}x^2)}{cJ_{1/4}(\frac{1}{2}x^2) + J_{-1/4}(\frac{1}{2}x^2)}.$$

**23. (a)** From (10) of Section 6.4, with $n = \frac{3}{2}$, we have

$$Y_{3/2}(x) = \frac{-J_{-3/2}(x)}{-1} = J_{-3/2}(x).$$

Then from the solutions of Problems 28 and 32 in Section 6.4 we have

$$J_{-3/2}(x) = -\sqrt{\frac{2}{\pi x}}\left(\frac{\cos x}{x} + \sin x\right) \qquad \text{so} \qquad Y_{3/2}(x) = -\sqrt{\frac{2}{\pi}x}\left(\frac{\cos x}{x} + \sin x\right)$$

**(b)** From (15) of Section 6.4 in the text

$$I_{1/2}(x) = i^{-1/2}J_{1/2}(ix) \qquad \text{and} \qquad I_{-1/2}(x) = i^{1/2}J_{-1/2}(ix)$$

so

$$I_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\,x^{2n+1} = \sqrt{\frac{2}{\pi x}}\,\sinh x$$

and

$$I_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\sum_{n=0}^{\infty}\frac{1}{(2n)!}\,x^{2n} = \sqrt{\frac{2}{\pi x}}\,\cosh x.$$

**(c)** Equation (16) of Section 6.4 in the text and part (b) imply

$$\begin{aligned} K_{1/2}(x) &= \frac{\pi}{2}\,\frac{I_{-1/2}(x) - I_{1/2}(x)}{\sin\frac{\pi}{2}} = \frac{\pi}{2}\left[\sqrt{\frac{2}{\pi x}}\,\cosh x - \sqrt{\frac{2}{\pi x}}\,\sinh x\right] \\ &= \sqrt{\frac{\pi}{2x}}\left[\frac{e^x + e^{-x}}{2} - \frac{e^x - e^{-x}}{2}\right] = \sqrt{\frac{\pi}{2x}}\,e^{-x}. \end{aligned}$$

**24. (a)** Using formula (5) of Section 4.2 in the text, we find that a second solution of $(1 - x^2)y'' - 2xy' = 0$ is

$$\begin{aligned} y_2(x) &= 1 \cdot \int \frac{e^{\int 2x\,dx/(1-x^2)}}{1^2}\,dx = \int e^{-\ln(1-x^2)}\,dx \\ &= \int \frac{dx}{1-x^2} = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right), \end{aligned}$$

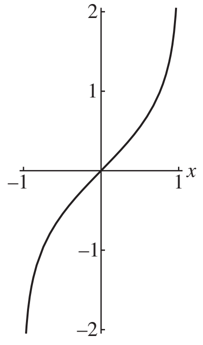
where partial fractions was used to obtain the last integral.

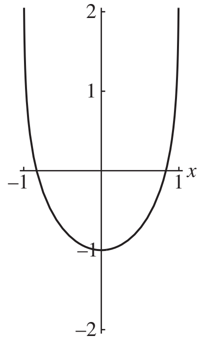
**(b)** Using formula (5) of Section 4.2 in the text, we find that a second solution of $(1 - x^2)y'' - 2xy' + 2y = 0$ is

$$\begin{aligned} y_2(x) &= x \cdot \int \frac{e^{\int 2x\,dx/(1-x^2)}}{x^2}\,dx = x\int \frac{e^{-\ln(1-x^2)}}{x^2}\,dx \\ &= x\int \frac{dx}{x^2(1-x^2)} = x\left[\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) - \frac{1}{x}\right] = \frac{x}{2}\ln\left(\frac{1+x}{1-x}\right) - 1, \end{aligned}$$

where partial fractions was used to obtain the last integral.

**(c)**





$$y_2(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \qquad\qquad y_2 = \frac{x}{2}\ln\left(\frac{1+x}{1-x}\right) - 1$$

**25. (a)** By the binomial theorem we have

$$\begin{aligned}
&\left[1+\left(t^2-2xt\right)\right]^{-1/2}\\
&\quad = 1 - \frac{1}{2}\left(t^2-2xt\right) + \frac{(-1/2)(-3/2)}{2!}\left(t^2-2xt\right)^2 + \frac{(-1/2)(-3/2)(-5/2)}{3!}\left(t^2-2xt\right)^3 + \cdots\\
&\quad = 1 - \frac{1}{2}(t^2-2xt) + \frac{3}{8}(t^2-2xt)^2 - \frac{5}{16}(t^2-2xt)^3 + \cdots\\
&\quad = 1 + xt + \frac{1}{2}(3x^2-1)t^2 + \frac{1}{2}(5x^3-3x)t^3 + \cdots = \sum_{n=0}^{\infty} P_n(x)t^n.
\end{aligned}$$

**(b)** Letting $x = 1$ in $\left(1-2xt+t^2\right)^{-1/2}$, we have

$$\begin{aligned}
\left(1-2t+t^2\right)^{-1/2} &= (1-t)^{-1} = \frac{1}{1-t} = 1 + t + t^2 + t^3 + \cdots \qquad (|t| < 1)\\
&= \sum_{n=0}^{\infty} t^n.
\end{aligned}$$

From part (a) we have

$$\sum_{n=0}^{\infty} P_n(1)t^n = \left(1-2t+t^2\right)^{-1/2} = \sum_{n=0}^{\infty} t^n.$$

Equating the coefficients of corresponding terms in the two series, we see that $P_n(1) = 1$. SImilarly, letting $x = -1$ we have

$$\begin{aligned}
\left(1+2t+t^2\right)^{-1/2} &= (1+t)^{-1} = \frac{1}{1+t} = 1 - t + t^2 - 3t^3 + \cdots \qquad (|t| < 1)\\
&= \sum_{n=0}^{\infty} (-1)^n t^n = \sum_{n=0}^{\infty} P_n(-1)t^n,
\end{aligned}$$

so that $P_n(-1) = (-1)^n$.

**26. (a)** Rewrite the equation as

$$x^2y'' + xy' + \left(16x^2 - \frac{25}{4}\right)y = 0.$$

The new equation has the form of (12) from Section 5.3 with $\alpha = 4$ and $\upsilon = \frac{5}{2}$. Thus the general solution is given by

$$y = c_1 J_{5/2}(4x) + c_2 Y_{5/2}(4x)$$

**(b)** This equation has the parametric form of the modified Bessel equation of order $\upsilon = 3$ with $\alpha = 3\sqrt{2}$. Therefore the general solution is

$$y = c_1 I_3(3\sqrt{2}\,x) + c_2 K_3(3\sqrt{2}\,x)$$

**27.** Using the Product rule gives

$$\begin{aligned} \frac{d}{dr}\left(r\,\frac{dT}{dr}\right) &= a^2 r\,(T - T_m) \\ r\,\frac{d^2T}{dr^2} + \frac{dT}{dr} - a^2 r\,(T - T_m) &= 0 \\ r^2\,\frac{d^2T}{dr^2} + r\,\frac{dT}{dr} - a^2 r^2\,(T - T_m) &= 0 \end{aligned}$$

The given boundary-value problem is

$$r^2\,\frac{d^2T}{dr^2} + r\,\frac{dT}{dr} - a^2 r^2\,(T - 70) = 0, \quad T(1) = 160, \quad \left.\frac{dT}{dr}\right|_{r=3} = 0$$

If we change the dependent variable by means of the substitution $w(r) = T(r) - 70$ the differential equation becomes

$$r^2\,\frac{d^2w}{dr^2} + r\,\frac{dw}{dr} - a^2 r^2 w = 0.$$

This differential equation is the parametric form of the modified Bessel's equation of order $\nu = 0$ so:

$$\begin{aligned} w(r) &= c_1 I_0(ar) + c_2 K_0(ar) \\ T(r) &= 70 + w(r) \\ T(r) &= 70 + c_1 I_0(ar) + c_2 K_0(ar) \end{aligned}$$

The derivatives $\dfrac{d}{dx} I_0(x) = I_1(x)$ and $\dfrac{d}{dx} K_0(x) = -K_1(x)$ and so the Chain rule gives

$$\frac{d}{dr} I_0\,\overbrace{(ar)}^{u} = \frac{dI_0}{du}\,\frac{du}{dr} = I_0'(ar)\,\frac{d}{dr}ar = aI_1(ar) \quad \text{and} \quad \frac{d}{dr} K_0(ar) = -aK_1(ar)$$

Therefore

$$T'(r) = c_1\,\frac{d}{dr} I_0(ar) + c_2\,\frac{d}{dr} K_0(ar) \quad \text{or} \quad T'(r) = c_1 a I_1(ar) - c_2 a K_1(ar)$$

The boundary conditions $T(1) = 160$ and $T'(3) = 0$ then give the linear system of equations

$$\begin{aligned} c_1 I_0(a) \;+\; c_2 K_0(a) \;&= 90 \\ c_1 I_1(3a) - c_2 K_1(3a) &= 0. \end{aligned}$$

By Cramer's Rule, the solution of the system is

$$c_1 = \frac{\begin{vmatrix} 90 & K_0(a) \\ 0 & -K_1(3a) \end{vmatrix}}{\begin{vmatrix} I_0(a) & K_0(a) \\ I_1(3a) & -K_1(3a) \end{vmatrix}} = \frac{90K_1(3a)}{I_0(a)K_1(3a) + I_1(3a)K_0(a)},$$

$$c_2 = \frac{\begin{vmatrix} I_0(a) & 90 \\ I_1(3a) & 0 \end{vmatrix}}{\begin{vmatrix} I_0(a) & K_0(a) \\ I_1(3a) & -K_1(3a) \end{vmatrix}} = \frac{90I_1(3a)}{I_0(a)K_1(3a) + I_1(3a)K_0(a)}.$$

Hence the solution of the boundary-value problem is

$$\begin{aligned} T(r) &= 70 + c_1 I_0(ar) + c_2 K_0(ar) \\ &= 70 + \frac{90K_1(3a)I_0(ar)}{I_0(a)K_1(3a) + I_1(3a)K_0(a)} + \frac{90I_1(3a)K_0(ar)}{I_0(a)K_1(3a) + I_1(3a)K_0(a)} \\ &= 70 + 90\frac{K_1(3a)I_0(ar) + I_1(3a)K_0(ar)}{K_1(3a)I_0(a) + I_1(3a)K_0(a)}. \end{aligned}$$

**28.** The differential equation is the same as in Problem 27 so the substitution $w(r) = T(r) - 70$ again yields

$$r^2\frac{d^2w}{dr^2} + r\frac{dw}{dr} - a^2r^2w = 0$$

and so

$$\begin{aligned} w(r) &= c_1 I_0(ar) + c_2 K_0(ar) \\ T(r) &= 70 + w(r) \\ T(r) &= 70 + c_1 I_0(ar) + c_2 K_0(ar) \end{aligned}$$

The boundary conditions $T(1) = 160$ and $T(3) = 90$ then give the linear system of equations

$$\begin{aligned} c_1 I_0(a) \; + \; c_2 K_0(a) \; &= 90 \\ c_1 I_0(3a) + c_2 K_0(3a) &= 20 \end{aligned}$$

By Cramer's Rule, the solution of the system is

$$c_1 = \frac{\begin{vmatrix} 90 & K_0(a) \\ 20 & K_0(3a) \end{vmatrix}}{\begin{vmatrix} I_0(a) & K_0(a) \\ I_0(3a) & K_0(3a) \end{vmatrix}} = \frac{90K_0(3a) - 20K_0(a)}{I_0(a)K_0(3a) - I_0(3a)K_0(a)}\,,$$

$$c_2 = \frac{\begin{vmatrix} I_0(a) & 90 \\ I_0(3a) & 20 \end{vmatrix}}{\begin{vmatrix} I_0(a) & K_0(a) \\ I_0(3a) & K_0(3a) \end{vmatrix}} = \frac{20I_0(a) - 90I_0(3a)}{I_0(a)K_0(3a) - I_0(3a)K_0(a)}\,.$$

Hence the solution of the boundary-value problem is

$$T(r) = 70 + \frac{90K_0(3a) - 20K_0(a)}{I_0(a)K_0(3a) - I_0(3a)K_0(a)}\, I_0(ar) + \frac{20I_0(a) - 90I_0(3a)}{I_0(a)K_0(3a) - I_0(3a)K_0(a)}\, K_0(ar)\,.$$

# Chapter 7

## The Laplace Transform

### 7.1 Definition of the Laplace Transform

**1.**
$$\begin{aligned}\mathscr{L}\{f(t)\} &= \int_0^1 \left(-e^{-st}\right) dt + \int_1^\infty e^{-st}\,dt = \frac{1}{s}e^{-st}\bigg|_0^1 - \frac{1}{s}e^{-st}\bigg|_1^\infty \\ &= \frac{1}{s}e^{-s} - \frac{1}{s} - \left(0 - \frac{1}{s}e^{-s}\right) = \frac{2}{s}e^{-s} - \frac{1}{s}, \quad s > 0\end{aligned}$$

**2.**
$$\mathscr{L}\{f(t)\} = \int_0^2 4e^{-st}\,dt = -\frac{4}{s}e^{-st}\bigg|_0^2 = -\frac{4}{s}\left(e^{-2s} - 1\right), \quad s > 0$$

**3.**
$$\begin{aligned}\mathscr{L}\{f(t)\} &= \int_0^1 te^{-st}\,dt + \int_1^\infty e^{-st}\,dt = \left(-\frac{1}{s}te^{-st} - \frac{1}{s^2}e^{-st}\right)\bigg|_0^1 - \frac{1}{s}e^{-st}\bigg|_1^\infty \\ &= \left(-\frac{1}{s}e^{-s} - \frac{1}{s^2}e^{-s}\right) - \left(0 - \frac{1}{s^2}\right) - \frac{1}{s}(0 - e^{-s}) = \frac{1}{s^2}(1 - e^{-s}), \quad s > 0\end{aligned}$$

**4.**
$$\begin{aligned}\mathscr{L}\{f(t)\} &= \int_0^1 (2t+1)e^{-st}\,dt = \left(-\frac{2}{s}te^{-st} - \frac{2}{s^2}e^{-st} - \frac{1}{s}e^{-st}\right)\bigg|_0^1 \\ &= \left(-\frac{2}{s}e^{-s} - \frac{2}{s^2}e^{-s} - \frac{1}{s}e^{-s}\right) - \left(0 - \frac{2}{s^2} - \frac{1}{s}\right) = \frac{1}{s}(1 - 3e^{-s}) + \frac{2}{s^2}(1 - e^{-s}),\ s > 0\end{aligned}$$

**5.**
$$\begin{aligned}\mathscr{L}\{f(t)\} &= \int_0^\pi (\sin t)e^{-st}\,dt = \left(-\frac{s}{s^2+1}e^{-st}\sin t - \frac{1}{s^2+1}e^{-st}\cos t\right)\bigg|_0^\pi \\ &= \left(0 + \frac{1}{s^2+1}e^{-\pi s}\right) - \left(0 - \frac{1}{s^2+1}\right) = \frac{1}{s^2+1}(e^{-\pi s} + 1), \quad s > 0\end{aligned}$$

**6.**
$$\begin{aligned}\mathscr{L}\{f(t)\} &= \int_{\pi/2}^\infty (\cos t)e^{-st}\,dt = \left(-\frac{s}{s^2+1}e^{-st}\cos t + \frac{1}{s^2+1}e^{-st}\sin t\right)\bigg|_{\pi/2}^\infty \\ &= 0 - \left(0 + \frac{1}{s^2+1}e^{-\pi s/2}\right) = -\frac{1}{s^2+1}e^{-\pi s/2}, \quad s > 0\end{aligned}$$

**7.** $f(t) = \begin{cases} 0, & 0 < t < 1 \\ t, & t > 1 \end{cases}$

$$\mathscr{L}\{f(t)\} = \int_1^\infty te^{-st}\,dt = \left(-\frac{1}{s}te^{-st} - \frac{1}{s^2}\,e^{-st}\right)\bigg|_1^\infty = \frac{1}{s}\,e^{-s} + \frac{1}{s^2}\,e^{-s}, \quad s > 0$$

**8.** $f(t) = \begin{cases} 0, & 0 < t < 1 \\ 2t-2, & t > 1 \end{cases}$

$$\mathscr{L}\{f(t)\} = 2\int_1^\infty (t-1)e^{-st}\,dt = 2\left(-\frac{1}{s}(t-1)e^{-st} - \frac{1}{s^2}\,e^{-st}\right)\bigg|_1^\infty = \frac{2}{s^2}\,e^{-s}, \quad s > 0$$

**9.** $f(t) = \begin{cases} 1-t, & 0 < t < 1 \\ 0, & t > 1 \end{cases}$

$$\mathscr{L}\{f(t)\} = \int_0^1 (1-t)e^{-st}\,dt + \int_1^\infty 0e^{-st}\,dt = \int_0^1 (1-t)e^{-st}\,dt$$

$$= \left(-\frac{1}{s}(1-t)e^{-st} + \frac{1}{s^2}\,e^{-st}\right)\bigg|_0^1 = \frac{1}{s^2}\,e^{-s} + \frac{1}{s} - \frac{1}{s^2}\,, \quad s > 0$$

**10.** $f(t) = \begin{cases} 0, & 0 < t < a \\ c, & a < t < b \end{cases}$

$$\mathscr{L}\{f(t)\} = \int_a^b ce^{-st}\,dt = -\frac{c}{s}\,e^{-st}\bigg|_a^b = \frac{c}{s}(e^{-sa} - e^{-sb}), \quad s > 0$$

**11.** $\mathscr{L}\{f(t)\} = \int_0^\infty e^{t+7}e^{-st}\,dt = e^7\int_0^\infty e^{(1-s)t}\,dt$

$$= \frac{e^7}{1-s}e^{(1-s)t}\bigg|_0^\infty = 0 - \frac{e^7}{1-s} = \frac{e^7}{s-1}\,, \quad s > 1$$

**12.** $\mathscr{L}\{f(t)\} = \int_0^\infty e^{-2t-5}e^{-st}\,dt = e^{-5}\int_0^\infty e^{-(s+2)t}\,dt$

$$= -\frac{e^{-5}}{s+2}e^{-(s+2)t}\bigg|_0^\infty = \frac{e^{-5}}{s+2}\,, \quad s > -2$$

**13.** $\mathscr{L}\{f(t)\} = \int_0^\infty te^{4t}e^{-st}\,dt = \int_0^\infty te^{(4-s)t}\,dt$

$$= \left(\frac{1}{4-s}te^{(4-s)t} - \frac{1}{(4-s)^2}e^{(4-s)t}\right)\bigg|_0^\infty = \frac{1}{(4-s)^2}\,, \quad s > 4$$

**14.** $$\mathscr{L}\{f(t)\} = \int_0^\infty t^2 e^{-2t} e^{-st}\,dt = \int_0^\infty t^2 e^{-(s+2)t}\,dt$$
$$= \left(-\frac{1}{s+2}t^2e^{-(s+2)t} - \frac{2}{(s+2)^2}te^{-(s+2)t} - \frac{2}{(s+2)^3}e^{-(s+2)t}\right)\Bigg|_0^\infty$$
$$= \frac{2}{(s+2)^3}, \quad s > -2$$

**15.** $$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-t}(\sin t)e^{-st}\,dt = \int_0^\infty (\sin t)e^{-(s+1)t}\,dt$$
$$= \left(\frac{-(s+1)}{(s+1)^2+1}e^{-(s+1)t}\sin t - \frac{1}{(s+1)^2+1}e^{-(s+1)t}\cos t\right)\Bigg|_0^\infty$$
$$= \frac{1}{(s+1)^2+1} = \frac{1}{s^2+2s+2}, \quad s > -1$$

**16.** $$\mathscr{L}\{f(t)\} = \int_0^\infty e^{t}(\cos t)e^{-st}\,dt = \int_0^\infty (\cos t)e^{(1-s)t}\,dt$$
$$= \left(\frac{1-s}{(1-s)^2+1}e^{(1-s)t}\cos t + \frac{1}{(1-s)^2+1}e^{(1-s)t}\sin t\right)\Bigg|_0^\infty$$
$$= -\frac{1-s}{(1-s)^2+1} = \frac{s-1}{s^2-2s+2}, \quad s > 1$$

**17.** $$\mathscr{L}\{f(t)\} = \int_0^\infty t(\cos t)e^{-st}\,dt$$
$$= \left[\left(-\frac{st}{s^2+1} - \frac{s^2-1}{(s^2+1)^2}\right)(\cos t)e^{-st} + \left(\frac{t}{s^2+1} + \frac{2s}{(s^2+1)^2}\right)(\sin t)e^{-st}\right]_0^\infty$$
$$= \frac{s^2-1}{(s^2+1)^2}, \quad s > 0$$

**18.** $$\mathscr{L}\{f(t)\} = \int_0^\infty t(\sin t)e^{-st}\,dt$$
$$= \left[\left(-\frac{t}{s^2+1} - \frac{2s}{(s^2+1)^2}\right)(\cos t)e^{-st} - \left(\frac{st}{s^2+1} + \frac{s^2-1}{(s^2+1)^2}\right)(\sin t)e^{-st}\right]_0^\infty$$
$$= \frac{2s}{(s^2+1)^2}, \quad s > 0$$

**19.** $\mathscr{L}\{2t^4\} = 2\frac{4!}{s^5} = \frac{48}{s^5}$

**20.** $\mathscr{L}\{t^5\} = \frac{5!}{s^6} = \frac{120}{s^6}$

**21.** $\mathscr{L}\{4t-10\} = \frac{4}{s^2} - \frac{10}{s}$

**22.** $\mathscr{L}\{7t+3\} = \frac{7}{s^2} + \frac{3}{s}$

**23.** $\mathscr{L}\{t^2+6t-3\} = \frac{2}{s^3} + \frac{6}{s^2} - \frac{3}{s}$

**24.** $\mathscr{L}\{-4t^2+16t+9\} = -\frac{8}{s^3} + \frac{16}{s^2} + \frac{9}{s}$

**25.** $\mathscr{L}\{t^3+3t^2+3t+1\}=\dfrac{6}{s^4}+\dfrac{6}{s^3}+\dfrac{3}{s^2}+\dfrac{1}{s}$

**26.** $\mathscr{L}\{8t^3-12t^2+6t-1\}=\dfrac{48}{s^4}-\dfrac{24}{s^3}+\dfrac{6}{s^2}-\dfrac{1}{s}$

**27.** $\mathscr{L}\{1+e^{4t}\}=\dfrac{1}{s}+\dfrac{1}{s-4}$

**28.** $\mathscr{L}\{t^2-e^{-9t}+5\}=\dfrac{2}{s^3}-\dfrac{1}{s+9}+\dfrac{5}{s}$

**29.** $\mathscr{L}\{1+2e^{2t}+e^{4t}\}=\dfrac{1}{s}+\dfrac{2}{s-2}+\dfrac{1}{s-4}$

**30.** $\mathscr{L}\{e^{2t}-2+e^{-2t}\}=\dfrac{1}{s-2}-\dfrac{2}{s}+\dfrac{1}{s+2}$

**31.** $\mathscr{L}\{4t^2-5\sin 3t\}=\dfrac{8}{s^3}-\dfrac{15}{s^2+9}$

**32.** $\mathscr{L}\{\cos 5t+\sin 2t\}=\dfrac{s}{s^2+25}+\dfrac{2}{s^2+4}$

**33.** $\mathscr{L}\{\sinh kt\}=\dfrac{1}{2}\mathscr{L}\{e^{kt}-e^{-kt}\}=\dfrac{1}{2}\left[\dfrac{1}{s-k}-\dfrac{1}{s+k}\right]=\dfrac{k}{s^2-k^2}$

**34.** $\mathscr{L}\{\cosh kt\}=\dfrac{1}{2}\mathscr{L}\{e^{kt}+e^{-kt}\}=\dfrac{s}{s^2-k^2}$

**35.** $\mathscr{L}\{e^t\sinh t\}=\mathscr{L}\left\{e^t\,\dfrac{e^t-e^{-t}}{2}\right\}=\mathscr{L}\left\{\dfrac{1}{2}e^{2t}-\dfrac{1}{2}\right\}=\dfrac{1}{2(s-2)}-\dfrac{1}{2s}$

**36.** $\mathscr{L}\{e^{-t}\cosh t\}=\mathscr{L}\left\{e^{-t}\,\dfrac{e^t+e^{-t}}{2}\right\}=\mathscr{L}\left\{\dfrac{1}{2}+\dfrac{1}{2}e^{-2t}\right\}=\dfrac{1}{2s}+\dfrac{1}{2(s+2)}$

**37.** $\mathscr{L}\{\sin 2t\cos 2t\}=\mathscr{L}\left\{\dfrac{1}{2}\sin 4t\right\}=\dfrac{2}{s^2+16}$

**38.** $\mathscr{L}\{\cos^2 t\}=\mathscr{L}\left\{\dfrac{1}{2}+\dfrac{1}{2}\cos 2t\right\}=\dfrac{1}{2s}+\dfrac{1}{2}\,\dfrac{s}{s^2+4}$

**39.** From the addition formula for the sine function, $\sin(4t+5)=\sin 4t\cos 5+\cos 4t\sin 5$ so

$$\begin{aligned}\mathscr{L}\{\sin(4t+5)\}&=(\cos 5)\,\mathscr{L}\{\sin 4t\}+(\sin 5)\,\mathscr{L}\{\cos 4t\}\\&=(\cos 5)\,\frac{4}{s^2+16}+(\sin 5)\,\frac{s}{s^2+16}=\frac{4\cos 5+(\sin 5)s}{s^2+16}.\end{aligned}$$

**40.** From the addition formula for the cosine function,

$$\cos\left(t-\frac{\pi}{6}\right)=\cos t\cos\frac{\pi}{6}+\sin t\sin\frac{\pi}{6}=\frac{\sqrt{3}}{2}\cos t+\frac{1}{2}\sin t$$

so

$$\begin{aligned}\mathscr{L}\left\{10\cos\left(t-\frac{\pi}{6}\right)\right\}&=10\mathscr{L}\left\{\cos\left(t-\frac{\pi}{6}\right)\right\}=10\cdot\frac{\sqrt{3}}{2}\mathscr{L}\{\cos t\}+10\cdot\frac{1}{2}\mathscr{L}\{\sin t\}\\&=5\sqrt{3}\,\frac{s}{s^2+1}+5\,\frac{1}{s^2+1}=5\cdot\frac{\sqrt{3}\,s+1}{s^2+1}\,.\end{aligned}$$

**41.** Use integration by parts for $\alpha>0$ to get

$$\Gamma(\alpha+1)=\int_0^\infty t^\alpha e^{-t}\,dt=-t^\alpha e^{-t}\Big|_0^\infty+\alpha\int_0^\infty t^{\alpha-1}e^{-t}\,dt=\alpha\Gamma(\alpha)$$

**42.** Let $u = st$ so that $du = s\,dt$ and we get

$$\mathscr{L}\{t^\alpha\} = \int_0^\infty t^\alpha e^{-st}\,dt = \int_0^\infty \left(\frac{u}{s}\right)^\alpha e^{-u}\frac{1}{s}\,du = \int_0^\infty \frac{1}{s^{\alpha+1}}u^\alpha e^{-u}\,du = \frac{1}{s^{\alpha+1}}\Gamma(\alpha+1)$$

**43.** $\mathscr{L}\left\{t^{-1/2}\right\} = \dfrac{1}{s^{-1/2+1}}\Gamma\left(-\dfrac{1}{2}+1\right) = \dfrac{\Gamma\left(\frac{1}{2}\right)}{s^{1/2}} = \sqrt{\dfrac{\pi}{s}}$

**44.** $\mathscr{L}\left\{t^{1/2}\right\} = \dfrac{1}{s^{1/2+1}}\Gamma\left(\dfrac{1}{2}+1\right) = \dfrac{\Gamma\left(\frac{3}{2}\right)}{s^{3/2}} = \dfrac{\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{s^{3/2}} = \dfrac{\sqrt{\pi}}{2s^{3/2}}$

**45.** $\mathscr{L}\left\{t^{3/2}\right\} = \dfrac{1}{s^{3/2+1}}\Gamma\left(\dfrac{3}{2}+1\right) = \dfrac{\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right)}{s^{5/2}} = \dfrac{3}{2s^{5/2}}\Gamma\left(\dfrac{1}{2}+1\right) = \dfrac{3}{4s^{5/2}}\Gamma\left(\dfrac{1}{2}\right) = \dfrac{3\sqrt{\pi}}{4s^{5/2}}$

**46.** 
$$\begin{aligned}\mathscr{L}\{f(t)\} &= 2\mathscr{L}\left\{t^{1/2}\right\} + 8\mathscr{L}\left\{t^{5/2}\right\} = 2\cdot\frac{\Gamma\left(\frac{1}{2}+1\right)}{s^{3/2}} + 8\cdot\frac{\Gamma\left(\frac{5}{2}+1\right)}{s^{7/2}} \\ &= \frac{\sqrt{\pi}}{s^{3/2}} + \frac{8}{s^{7/2}}\cdot\frac{5}{2}\Gamma\left(\frac{5}{2}\right) = \frac{\sqrt{\pi}}{s^{3/2}} + \frac{8}{s^{7/2}}\cdot\frac{5}{2}\cdot\frac{3}{2}\Gamma\left(\frac{3}{2}\right) \\ &= \frac{\sqrt{\pi}}{s^{3/2}} + \frac{8}{s^{7/2}}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{s^{3/2}} + \frac{8}{s^{7/2}}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi} \\ &= \frac{\sqrt{\pi}}{s^{3/2}} + \frac{15\sqrt{\pi}}{s^{7/2}}\end{aligned}$$

**47.** The relation will be valid when $s$ is greater than the maximum of $c_1$ and $c_2$.

**48.** Since $e^t$ is an increasing function and $t^2 > \ln M + ct$ for $M > 0$ we have $e^{t^2} > e^{\ln M + ct} = Me^{ct}$ for $t$ sufficiently large and for any $c$. Thus, $e^{t^2}$ is not of exponential order.

**49.** Assuming that (c) of Theorem 7.1.1 is applicable with a complex exponent, we have

$$\mathscr{L}\{e^{(a+ib)t}\} = \frac{1}{s-(a+ib)} = \frac{1}{(s-a)-ib}\,\frac{(s-a)+ib}{(s-a)+ib} = \frac{s-a+ib}{(s-a)^2+b^2}.$$

By Euler's formula, $e^{i\theta} = \cos\theta + i\sin\theta$, so

$$\begin{aligned}\mathscr{L}\{e^{(a+ib)t}\} &= \mathscr{L}\{e^{at}e^{ibt}\} = \mathscr{L}\{e^{at}(\cos bt + i\sin bt)\} \\ &= \mathscr{L}\{e^{at}\cos bt\} + i\mathscr{L}\{e^{at}\sin bt\} \\ &= \frac{s-a}{(s-a)^2+b^2} + i\,\frac{b}{(s-a)^2+b^2}.\end{aligned}$$

Equating real and imaginary parts we get

$$\mathscr{L}\{e^{at}\cos bt\} = \frac{s-a}{(s-a)^2+b^2} \quad\text{and}\quad \mathscr{L}\{e^{at}\sin bt\} = \frac{b}{(s-a)^2+b^2}.$$

**50.** We want $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$ or

$$m(\alpha x + \beta y) + b = \alpha(mx + b) + \beta(my + b) = m(\alpha x + \beta y) + (\alpha + \beta)b$$

for all real numbers $\alpha$ and $\beta$. Taking $\alpha = \beta = 1$ we see that $b = 2b$, so $b = 0$. Thus, $f(x) = mx + b$ will be a linear transformation when $b = 0$.

**51.** As written, the function is not defined at $x = 5$. As should be written, $f(x) = 1/(t - 5)$, $t \geq 5$, the function $f$ is not bounded on, say, the interval $[4, 6]$ because it has an infinite discontiunity at $x = 5$.

**52.** $\mathscr{L}\left\{\dfrac{1}{t^2}\right\} = \displaystyle\int_0^\infty \frac{e^{-st}}{t^2}\,dt = \int_0^1 \frac{e^{-st}}{t^2}\,dt + \int_1^\infty \frac{e^{-st}}{t^2}\,dt = I_1 + I_2.$

Then we use the comparison test in three cases: $s > 0$, $s = 0$, and $s < 0$. For $s > 0$,

$$I_1 = \int_0^1 \frac{e^{-st}}{t^2}\,dt > e^{-s}\int_0^1 \frac{1}{t^2}\,dt\,.$$

Because $\int_0^1 \frac{1}{t^2}\,dt$ diverges, the integral $\int_0^1 \frac{e^{-st}}{t^2}\,dt$ diverges.

If $s = 0$, then the integral $I_1 = \int_0^1 \frac{1}{t^2}\,dt$ diverges. If $s < 0$, then

$$I_1 = \int_0^1 \frac{e^{-st}}{t^2}\,dt > \int_0^1 \frac{1}{t^2}\,dt\,.$$

Again, because $\int_0^1 \frac{1}{t^2}\,dt$ diverges, the integral $\int_0^1 \frac{e^{-st}}{t^2}\,dt$ diverges.

**53.** Applying the definition of the Laplace transform gives us

$$\begin{aligned}
\mathscr{L}\left\{2te^{t^2}\cos e^{t^2}\right\} &= \int_0^\infty e^{-st}\,2te^{t^2}\cos e^{t^2}\,dt \\
&= e^{-st}\sin e^{t^2}\Big|_0^\infty + s\int_0^\infty e^{-st}\sin e^{t^2}\,dt \\
&= -\sin 1 + s\int_0^\infty e^{-st}\sin e^{t^2}\,dt
\end{aligned}$$

The last integral exists for $s > 0$ since it's piecewise continuous on $(0, \infty)$ and of exponential order. Alternatively, for $s > 0$

$$\int_0^\infty \left|e^{-st}\sin e^{t^2}\right|\,dt \leq 1 \cdot \int_0^\infty e^{-st}\,dt = \frac{1}{s}$$

The absolute convergence of the integral $\int_0^\infty e^{-st}\sin e^{t^2}\,dt$ implies the convergence of the integral.

**54.** Working backwards we get

$$\begin{aligned}\frac{1}{a}F\left(\frac{s}{a}\right) &= \frac{1}{a}\cdot\int_0^\infty e^{-(s/a)t}f(t)\,dt \\ &= \frac{1}{a}\cdot a\int_0^\infty e^{-su}f(au)\,du \qquad \longleftarrow \quad t = at, \quad dt = a\,du \\ &= \mathscr{L}\{f(at)\}\end{aligned}$$

**55.** $\mathscr{L}\{e^{at}\} = \dfrac{1}{a}\cdot\dfrac{1}{(s/a-1)} = \dfrac{1}{s-a}$

**56.** $\mathscr{L}\{\sin kt\} = \dfrac{1}{k}\cdot\dfrac{1}{(s/k)^2+1} = \dfrac{1}{k}\cdot\dfrac{k^2}{s^2+k^2} = \dfrac{k}{s^2+k^2}$

**57.** $\mathscr{L}\{1-\cos kt\} = \dfrac{1}{k}\cdot\dfrac{1}{(s/k)\left[(s/k)^2+1\right]} = \dfrac{1}{k}\cdot\dfrac{k^3}{s\left(s^2+k^2\right)} = \dfrac{k^2}{s\left(s^2+k^2\right)}$

**58.** $\mathscr{L}\{\sin kt\,\sinh kt\} = \dfrac{1}{k}\cdot\dfrac{2(s/k)}{(s/k)^4+4} = \dfrac{1}{k}\cdot\dfrac{2sk^3}{s^4+4k^4} = \dfrac{2k^3s}{s^4+4k^4}$

## 7.2 The Inverse Transform and Transforms of Derivatives

**1.** $\mathscr{L}^{-1}\left\{\dfrac{1}{s^3}\right\} = \dfrac{1}{2}\mathscr{L}^{-1}\left\{\dfrac{2}{s^3}\right\} = \dfrac{1}{2}t^2$

**2.** $\mathscr{L}^{-1}\left\{\dfrac{1}{s^4}\right\} = \dfrac{1}{6}\mathscr{L}^{-1}\left\{\dfrac{3!}{s^4}\right\} = \dfrac{1}{6}t^3$

**3.** $\mathscr{L}^{-1}\left\{\dfrac{1}{s^2}-\dfrac{48}{s^5}\right\} = \mathscr{L}^{-1}\left\{\dfrac{1}{s^2}-\dfrac{48}{24}\cdot\dfrac{4!}{s^5}\right\} = t-2t^4$

**4.** $\mathscr{L}^{-1}\left\{\left(\dfrac{2}{s}-\dfrac{1}{s^3}\right)^2\right\} = \mathscr{L}^{-1}\left\{4\cdot\dfrac{1}{s^2}-\dfrac{4}{6}\cdot\dfrac{3!}{s^4}+\dfrac{1}{120}\cdot\dfrac{5!}{s^6}\right\} = 4t-\dfrac{2}{3}t^3+\dfrac{1}{120}t^5$

**5.** $\mathscr{L}^{-1}\left\{\dfrac{(s+1)^3}{s^4}\right\} = \mathscr{L}^{-1}\left\{\dfrac{1}{s}+3\cdot\dfrac{1}{s^2}+\dfrac{3}{2}\cdot\dfrac{2}{s^3}+\dfrac{1}{6}\cdot\dfrac{3!}{s^4}\right\} = 1+3t+\dfrac{3}{2}t^2+\dfrac{1}{6}t^3$

**6.** $\mathscr{L}^{-1}\left\{\dfrac{(s+2)^2}{s^3}\right\} = \mathscr{L}^{-1}\left\{\dfrac{1}{s}+4\cdot\dfrac{1}{s^2}+2\cdot\dfrac{2}{s^3}\right\} = 1+4t+2t^2$

**7.** $\mathscr{L}^{-1}\left\{\dfrac{1}{s^2}-\dfrac{1}{s}+\dfrac{1}{s-2}\right\} = t-1+e^{2t}$

**8.** $\mathscr{L}^{-1}\left\{\dfrac{4}{s}+\dfrac{6}{s^5}-\dfrac{1}{s+8}\right\} = \mathscr{L}^{-1}\left\{4\cdot\dfrac{1}{s}+\dfrac{1}{4}\cdot\dfrac{4!}{s^5}-\dfrac{1}{s+8}\right\} = 4+\dfrac{1}{4}t^4-e^{-8t}$

**9.** $\mathscr{L}^{-1}\left\{\dfrac{1}{4s+1}\right\} = \dfrac{1}{4}\mathscr{L}^{-1}\left\{\dfrac{1}{s+1/4}\right\} = \dfrac{1}{4}e^{-t/4}$

**10.** $\mathscr{L}^{-1}\left\{\frac{1}{5s-2}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{5}\cdot\frac{1}{s-2/5}\right\} = \frac{1}{5}e^{2t/5}$

**11.** $\mathscr{L}^{-1}\left\{\frac{5}{s^2+49}\right\} = \mathscr{L}^{-1}\left\{\frac{5}{7}\cdot\frac{7}{s^2+49}\right\} = \frac{5}{7}\sin 7t$

**12.** $\mathscr{L}^{-1}\left\{\frac{10s}{s^2+16}\right\} = 10\cos 4t$

**13.** $\mathscr{L}^{-1}\left\{\frac{4s}{4s^2+1}\right\} = \mathscr{L}^{-1}\left\{\frac{s}{s^2+1/4}\right\} = \cos\frac{1}{2}t$

**14.** $\mathscr{L}^{-1}\left\{\frac{1}{4s^2+1}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{2}\cdot\frac{1/2}{s^2+1/4}\right\} = \frac{1}{2}\sin\frac{1}{2}t$

**15.** $\mathscr{L}^{-1}\left\{\frac{2s-6}{s^2+9}\right\} = \mathscr{L}^{-1}\left\{2\cdot\frac{s}{s^2+9} - 2\cdot\frac{3}{s^2+9}\right\} = 2\cos 3t - 2\sin 3t$

**16.** $\mathscr{L}^{-1}\{fracs+1s^2+2\} = \mathscr{L}^{-1}\left\{\frac{s}{s^2+2} + \frac{1}{\sqrt{2}}\,\frac{\sqrt{2}}{s^2+2}\right\} = \cos\sqrt{2}\,t + \frac{\sqrt{2}}{2}\sin\sqrt{2}\,t$

**17.** $\mathscr{L}^{-1}\left\{\frac{1}{s^2+3s}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{3}\cdot\frac{1}{s} - \frac{1}{3}\cdot\frac{1}{s+3}\right\} = \frac{1}{3} - \frac{1}{3}e^{-3t}$

**18.** $\mathscr{L}^{-1}\left\{\frac{s+1}{s^2-4s}\right\} = \mathscr{L}^{-1}\left\{-\frac{1}{4}\cdot\frac{1}{s} + \frac{5}{4}\cdot\frac{1}{s-4}\right\} = -\frac{1}{4} + \frac{5}{4}e^{4t}$

**19.** $\mathscr{L}^{-1}\left\{\frac{s}{s^2+2s-3}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{4}\cdot\frac{1}{s-1} + \frac{3}{4}\cdot\frac{1}{s+3}\right\} = \frac{1}{4}e^{t} + \frac{3}{4}e^{-3t}$

**20.** $\mathscr{L}^{-1}\left\{\frac{1}{s^2+s-20}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{9}\cdot\frac{1}{s-4} - \frac{1}{9}\cdot\frac{1}{s+5}\right\} = \frac{1}{9}e^{4t} - \frac{1}{9}e^{-5t}$

**21.** $\mathscr{L}^{-1}\left\{\frac{0.9s}{(s-0.1)(s+0.2)}\right\} = \mathscr{L}^{-1}\left\{(0.3)\cdot\frac{1}{s-0.1} + (0.6)\cdot\frac{1}{s+0.2}\right\} = 0.3e^{0.1t} + 0.6e^{-0.2t}$

**22.** $\mathscr{L}^{-1}\left\{\frac{s-3}{(s-\sqrt{3}\,)(s+\sqrt{3}\,)}\right\} = \mathscr{L}^{-1}\left\{\frac{s}{s^2-3} - \sqrt{3}\cdot\frac{\sqrt{3}}{s^2-3}\right\} = \cosh\sqrt{3}\,t - \sqrt{3}\sinh\sqrt{3}\,t$

**23.** $\mathscr{L}^{-1}\left\{\frac{s}{(s-2)(s-3)(s-6)}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{2}\cdot\frac{1}{s-2} - \frac{1}{s-3} + \frac{1}{2}\cdot\frac{1}{s-6}\right\} = \frac{1}{2}\,e^{2t} - e^{3t} + \frac{1}{2}\,e^{6t}$

**24.** $\mathscr{L}^{-1}\left\{\frac{s^2+1}{s(s-1)(s+1)(s-2)}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{2}\cdot\frac{1}{s} - \frac{1}{s-1} - \frac{1}{3}\cdot\frac{1}{s+1} + \frac{5}{6}\cdot\frac{1}{s-2}\right\}$

$$= \frac{1}{2} - e^{t} - \frac{1}{3}e^{-t} + \frac{5}{6}e^{2t}$$

**25.** $\mathscr{L}^{-1}\left\{\frac{1}{s^3+5s}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{s(s^2+5)}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{5}\cdot\frac{1}{s} - \frac{1}{5}\,\frac{s}{s^2+5}\right\} = \frac{1}{5} - \frac{1}{5}\cos\sqrt{5}\,t$

**26.** $$\mathscr{L}^{-1}\left\{\frac{s}{(s^2+4)(s+2)}\right\}=\mathscr{L}^{-1}\left\{\frac{1}{4}\cdot\frac{s}{s^2+4}+\frac{1}{4}\cdot\frac{2}{s^2+4}-\frac{1}{4}\cdot\frac{1}{s+2}\right\}$$
$$=\frac{1}{4}\cos 2t+\frac{1}{4}\sin 2t-\frac{1}{4}e^{-2t}$$

**27.** $$\mathscr{L}^{-1}\left\{\frac{2s-4}{(s^2+s)(s^2+1)}\right\}=\mathscr{L}^{-1}\left\{\frac{2s-4}{s(s+1)(s^2+1)}\right\}$$
$$=\mathscr{L}^{-1}\left\{-\frac{4}{s}+\frac{3}{s+1}+\frac{s}{s^2+1}+\frac{3}{s^2+1}\right\}$$
$$=-4+3e^{-t}+\cos t+3\sin t$$

**28.** $$\mathscr{L}^{-1}\left\{\frac{1}{s^4-9}\right\}=\mathscr{L}^{-1}\left\{\frac{1}{6\sqrt{3}}\cdot\frac{\sqrt{3}}{s^2-3}-\frac{1}{6\sqrt{3}}\cdot\frac{\sqrt{3}}{s^2+3}\right\}=\frac{1}{6\sqrt{3}}\sinh\sqrt{3}\,t-\frac{1}{6\sqrt{3}}\sin\sqrt{3}\,t$$

**29.** $$\mathscr{L}^{-1}\left\{\frac{1}{(s^2+1)(s^2+4)}\right\}=\mathscr{L}^{-1}\left\{\frac{1}{3}\cdot\frac{1}{s^2+1}-\frac{1}{3}\cdot\frac{1}{s^2+4}\right\}$$
$$=\mathscr{L}^{-1}\left\{\frac{1}{3}\cdot\frac{1}{s^2+1}-\frac{1}{6}\cdot\frac{2}{s^2+4}\right\}=\frac{1}{3}\sin t-\frac{1}{6}\sin 2t$$

**30.** $$\mathscr{L}^{-1}\left\{\frac{6s+3}{(s^2+1)(s^2+4)}\right\}=\mathscr{L}^{-1}\left\{2\cdot\frac{s}{s^2+1}+\frac{1}{s^2+1}-2\cdot\frac{s}{s^2+4}-\frac{1}{2}\cdot\frac{2}{s^2+4}\right\}$$
$$=2\cos t+\sin t-2\cos 2t-\frac{1}{2}\sin 2t$$

**31.** Since $f(t)=e^{at}\sinh bt=e^{at}\frac{1}{2}\left(e^{bt}-e^{-bt}\right)=\frac{1}{2}e^{(1+b)t}-\frac{1}{2}e^{(a-b)t}$, then by linearity and part (c) of Theorem 7.1.1

$$\mathscr{L}\left\{\frac{1}{2}e^{(a+b)t}-\frac{1}{2}e^{(a-b)t}\right\}=\frac{1}{2}\frac{1}{s-(a+b)}-\frac{1}{2}\frac{1}{s-(a-b)}=\frac{b}{((s-a)-b)\,((s-a)+b)}$$
$$=\frac{b}{(s-a)^2-b^2}$$
$$\mathscr{L}^{-1}\left\{\frac{1}{(s-a)^2-b^2}\right\}=\frac{1}{b}e^{at}\sinh bt$$

**32.** By linearity and part (d) of Theorem 7.1.1:

$$\mathscr{L}\left\{at-\sin at\right\}=\frac{a}{s^2}-\frac{a}{s^2+b^2}=\frac{a^3}{s^2\left(s^2+a^2\right)}$$
$$\mathscr{L}^{-1}\left\{\frac{1}{s^2\left(s^2+a^2\right)}\right\}=\frac{at-\sin at}{a^3}$$

**33.** By linearity and part (d) of Theorem 7.1.1:

$$\begin{aligned}\mathscr{L}\left\{a\sin bt-b\sin at\right\}&=a\,\frac{b}{s^2+b^2}-b\,\frac{a}{s^2+a^2}=\frac{a^3b-ab^3}{\left(s^2+a^2\right)\left(s^2+b^2\right)}\\&=\frac{ab\left(a^2-b^2\right)}{\left(s^2+a^2\right)\left(s^2+b^2\right)}\\\mathscr{L}^{-1}\left\{\frac{1}{\left(s^2+a^2\right)\left(s^2+b^2\right)}\right\}&=\frac{a\sin bt-b\sin at}{ab\left(a^2-b^2\right)}\end{aligned}$$

**34.** By linearity and part (e) of Theorem 7.1.1:

$$\begin{aligned}\mathscr{L}\left\{\cos bt-\cos at\right\}&=\frac{s}{s^2+b^2}-\frac{s}{s^2+a^2}=\frac{s\left(a^2-b^2\right)}{\left(s^2+a^2\right)\left(s^2+b^2\right)}\\\mathscr{L}^{-1}\left\{\frac{s}{\left(s^2+a^2\right)\left(s^2+b^2\right)}\right\}&=\frac{\cos bt-\cos at}{a^2-b^2}\end{aligned}$$

**35.** The Laplace transform of the differential equation is

$$s\mathscr{L}\{y\}-y(0)-\mathscr{L}\{y\}=\frac{1}{s}\,.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\}=-\frac{1}{s}+\frac{1}{s-1}\,.$$

Thus

$$y=-1+e^t.$$

**36.** The Laplace transform of the differential equation is

$$2s\mathscr{L}\{y\}-2y(0)+\mathscr{L}\{y\}=0.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\}=\frac{-6}{2s+1}=\frac{-3}{s+1/2}\,.$$

Thus

$$y=-3e^{-t/2}.$$

**37.** The Laplace transform of the differential equation is

$$s\mathscr{L}\{y\}-y(0)+6\mathscr{L}\{y\}=\frac{1}{s-4}\,.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\}=\frac{1}{(s-4)(s+6)}+\frac{2}{s+6}=\frac{1}{10}\cdot\frac{1}{s-4}+\frac{19}{10}\cdot\frac{1}{s+6}\,.$$

Thus

$$y=\frac{1}{10}e^{4t}+\frac{19}{10}e^{-6t}.$$

**38.** The Laplace transform of the differential equation is

$$s\mathscr{L}\{y\} - \mathscr{L}\{y\} = \frac{2s}{s^2+25}\,.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\} = \frac{2s}{(s-1)(s^2+25)} = \frac{1}{13}\cdot\frac{1}{s-1} - \frac{1}{13}\,\frac{s}{s^2+25} + \frac{5}{13}\cdot\frac{5}{s^2+25}\,.$$

Thus

$$y = \frac{1}{13}e^t - \frac{1}{13}\cos 5t + \frac{5}{13}\sin 5t.$$

**39.** The Laplace transform of the differential equation is

$$s^2\mathscr{L}\{y\} - sy(0) - y'(0) + 5\left[s\mathscr{L}\{y\} - y(0)\right] + 4\mathscr{L}\{y\} = 0.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\} = \frac{s+5}{s^2+5s+4} = \frac{4}{3}\,\frac{1}{s+1} - \frac{1}{3}\,\frac{1}{s+4}\,.$$

Thus

$$y = \frac{4}{3}e^{-t} - \frac{1}{3}e^{-4t}.$$

**40.** The Laplace transform of the differential equation is

$$s^2\mathscr{L}\{y\} - sy(0) - y'(0) - 4\left[s\mathscr{L}\{y\} - y(0)\right] = \frac{6}{s-3} - \frac{3}{s+1}\,.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\begin{aligned}\mathscr{L}\{y\} &= \frac{6}{(s-3)(s^2-4s)} - \frac{3}{(s+1)(s^2-4s)} + \frac{s-5}{s^2-4s}\\ &= \frac{5}{2}\cdot\frac{1}{s} - \frac{2}{s-3} - \frac{3}{5}\cdot\frac{1}{s+1} + \frac{11}{10}\cdot\frac{1}{s-4}\,.\end{aligned}$$

Thus

$$y = \frac{5}{2} - 2e^{3t} - \frac{3}{5}e^{-t} + \frac{11}{10}e^{4t}\,.$$

**41.** The Laplace transform of the differential equation is

$$s^2\mathscr{L}\{y\} - sy(0) - y'(0) + \mathscr{L}\{y\} = \frac{2}{s^2+2}\,.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\} = \frac{2}{(s^2+1)(s^2+2)} + \frac{10s}{s^2+1} = \frac{10s}{s^2+1} + \frac{2}{s^2+1} - \frac{2}{s^2+2}\,.$$

Thus

$$y = 10\cos t + 2\sin t - \sqrt{2}\sin\sqrt{2}\,t.$$

**42.** The Laplace transform of the differential equation is

$$s^2\mathscr{L}\{y\} + 9\mathscr{L}\{y\} = \frac{1}{s-1}\,.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\} = \frac{1}{(s-1)(s^2+9)} = \frac{1}{10}\cdot\frac{1}{s-1} - \frac{1}{10}\cdot\frac{1}{s^2+9} - \frac{1}{10}\cdot\frac{s}{s^2+9}\,.$$

Thus

$$y = \frac{1}{10}e^t - \frac{1}{30}\sin 3t - \frac{1}{10}\cos 3t.$$

**43.** The Laplace transform of the differential equation is

$$2\left[s^3\mathscr{L}\{y\} - s^2(0) - sy'(0) - y''(0)\right] + 3\left[s^2\mathscr{L}\{y\} - sy(0) - y'(0)\right] - 3[s\mathscr{L}\{y\} - y(0)] - 2\mathscr{L}\{y\}$$
$$= \frac{1}{s+1}\,.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\} = \frac{2s+3}{(s+1)(s-1)(2s+1)(s+2)} = \frac{1}{2}\,\frac{1}{s+1} + \frac{5}{18}\,\frac{1}{s-1} - \frac{8}{9}\,\frac{1}{s+1/2} + \frac{1}{9}\,\frac{1}{s+2}\,.$$

Thus

$$y = \frac{1}{2}e^{-t} + \frac{5}{18}e^t - \frac{8}{9}e^{-t/2} + \frac{1}{9}e^{-2t}.$$

**44.** The Laplace transform of the differential equation is

$$s^3\mathscr{L}\{y\} - s^2(0) - sy'(0) - y''(0) + 2\left[s^2\mathscr{L}\{y\} - sy(0) - y'(0)\right] - [s\mathscr{L}\{y\} - y(0)] - 2\mathscr{L}\{y\}$$
$$= \frac{3}{s^2+9}\,.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\} = \frac{s^2+12}{(s-1)(s+1)(s+2)(s^2+9)}$$
$$= \frac{13}{60}\,\frac{1}{s-1} - \frac{13}{20}\,\frac{1}{s+1} + \frac{16}{39}\,\frac{1}{s+2} + \frac{3}{130}\,\frac{s}{s^2+9} - \frac{1}{65}\,\frac{3}{s^2+9}\,.$$

Thus

$$y = \frac{13}{60}e^t - \frac{13}{20}e^{-t} + \frac{16}{39}e^{-2t} + \frac{3}{130}\cos 3t - \frac{1}{65}\sin 3t.$$

**45.** The Laplace transform of the differential equation is

$$s\mathscr{L}\{y\} + \mathscr{L}\{y\} = \frac{s+3}{s^2+6s+13}.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\begin{aligned}\mathscr{L}\{y\} &= \frac{s+3}{(s+1)(s^2+6s+13)} = \frac{1}{4}\cdot\frac{1}{s+1} - \frac{1}{4}\cdot\frac{s+1}{s^2+6s+13}\\ &= \frac{1}{4}\cdot\frac{1}{s+1} - \frac{1}{4}\left(\frac{s+3}{(s+3)^2+4} - \frac{2}{(s+3)^2+4}\right).\end{aligned}$$

Thus

$$y = \frac{1}{4}e^{-t} - \frac{1}{4}e^{-3t}\cos 2t + \frac{1}{4}e^{-3t}\sin 2t.$$

**46.** The Laplace transform of the differential equation is

$$s^2\mathscr{L}\{y\} - s\cdot 1 - 3 - 2[s\mathscr{L}\{y\} - 1] + 5\mathscr{L}\{y\} = (s^2-2s+5)\mathscr{L}\{y\} - s - 1 = 0.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\} = \frac{s+1}{s^2-2s+5} = \frac{s-1+2}{(s-1)^2+2^2} = \frac{s-1}{(s-1)^2+2^2} + \frac{2}{(s-1)^2+2^2}.$$

Thus

$$y = e^t\cos 2t + e^t\sin 2t.$$

**47.** The Laplace transform of the differential equation in the initial-value problem is

$$\left(s^2+4\right)Y(s) = \frac{10s}{s^2+25}$$

$$Y(s) = \frac{10s}{\left(s^2+4\right)\left(s^2+25\right)}$$

with the identifications $a^2 = 4$ and $b^2 = 25$ or $a^2 - b^2 = -21$ we have from Problem 34:

$$\mathscr{L}^{-1}\left\{\frac{s}{\left(s^2+a^2\right)\left(s^2+b^2\right)}\right\} = \frac{\cos bt - \cos at}{a^2-b^2}$$

$$y(t) = 10\mathscr{L}^{-1}\left\{\frac{s}{\left(s^2+4\right)\left(s^2+25\right)}\right\} = 10\cdot\frac{\cos 5t - \cos 2t}{-21}$$

$$y(t) = \frac{10}{21}\cos 2t - \frac{10}{21}\cos 5t$$

**48.** The Laplace transform of the differential equation in the initial-value problem is

$$\left(s^2+2\right)Y(s) = \frac{4}{s^2}$$

$$Y(s) = \frac{1}{s^2\left(s^2+2\right)}$$

with the identification $a^2 = 2$ and $a^3 = 2\sqrt{2}$ we have from problem 44:

$$\mathscr{L}^{-1}\left\{\frac{1}{s^2\left(s^2+a^2\right)}\right\} = \frac{at - \sin at}{a^3}$$

$$\begin{aligned} y(t) &= 4\mathscr{L}^{-1}\left\{\frac{1}{s^2\left(s^2+2\right)}\right\} = 4\cdot\frac{\sqrt{2}\,t - \sin\sqrt{2}\,t}{2\sqrt{2}} \\ y(t) &= 2t - \sqrt{2}\,\sin\sqrt{2}\,t. \end{aligned}$$

**49.** **(a)** Differentiating $f(t) = te^{at}$ we get $f'(t) = ate^{at} + e^{at}$ so $\mathscr{L}\{ate^{at} + e^{at}\} = s\mathscr{L}\{te^{at}\}$, where we have used $f(0) = 0$. Writing the equation as

$$a\mathscr{L}\{te^{at}\} + \mathscr{L}\left\{e^{at}\right\} = s\mathscr{L}\left\{te^{at}\right\}$$

and solving for $\mathscr{L}\{te^{at}\}$ we get

$$\mathscr{L}\left\{te^{at}\right\} = \frac{1}{s-a}\mathscr{L}\left\{e^{at}\right\} = \frac{1}{(s-a)^2}\,.$$

**(b)** Starting with $f(t) = t\sin kt$ we have

$$\begin{aligned} f'(t) &= kt\cos kt + \sin kt \\ f''(t) &= -k^2t\sin kt + 2k\cos kt. \end{aligned}$$

Then

$$\mathscr{L}\left\{-k^2t\sin t + 2k\cos kt\right\} = s^2\mathscr{L}\{t\sin kt\}$$

where we have used $f(0) = 0$ and $f'(0) = 0$. Writing the above equation as

$$-k^2\mathscr{L}\{t\sin kt\} + 2k\mathscr{L}\{\cos kt\} = s^2\mathscr{L}\{t\sin kt\}$$

and solving for $\mathscr{L}\{t\sin kt\}$ gives

$$\mathscr{L}\{t\sin kt\} = \frac{2k}{s^2+k^2}\mathscr{L}\{\cos kt\} = \frac{2k}{s^2+k^2}\,\frac{s}{s^2+k^2} = \frac{2ks}{(s^2+k^2)^2}\,.$$

**50.** Let $f_1(t) = 1$ and

$$f_2(t) = \begin{cases} 1, & t \ge 0, t \ne 1 \\ 0, & t = 1 \end{cases}$$

Then $\mathscr{L}\{f_1(t)\} = \mathscr{L}\{f_2(t)\} = 1/s$, but $f_1(t) \ne f_2(t)$.

**51.** For $y'' - 4y' = 6e^{3t} - 3e^{-t}$ the transfer function is $W(s) = 1/(s^2 - 4s)$. The zero-input response is

$$y_0(t) = \mathscr{L}^{-1}\left\{\frac{s-5}{s^2-4s}\right\} = \mathscr{L}^{-1}\left\{\frac{5}{4}\cdot\frac{1}{s} - \frac{1}{4}\cdot\frac{1}{s-4}\right\} = \frac{5}{4} - \frac{1}{4}e^{4t}\,,$$

and the zero-state response is

$$\begin{aligned} y_1(t) &= \mathscr{L}^{-1}\left\{\frac{6}{(s-3)(s^2-4s)} - \frac{3}{(s+1)(s^2-4s)}\right\} \\ &= \mathscr{L}^{-1}\left\{\frac{27}{20}\cdot\frac{1}{s-4} - \frac{2}{s-3} + \frac{5}{4}\cdot\frac{1}{s} - \frac{3}{5}\cdot\frac{1}{s+1}\right\} \\ &= \frac{27}{20}e^{4t} - 2e^{3t} + \frac{5}{4} - \frac{3}{5}e^{-t}. \end{aligned}$$

**52.** From Theorem 7.2.2, if $f$ and $f'$ are continuous and of exponential order, $\mathscr{L}\{f'(t)\} = sF(s) - f(0)$. From Theorem 7.2.3, $\lim_{s\to\infty} \mathscr{L}\{f'(t)\} = 0$ so

$$\lim_{s\to\infty}[sF(s) - f(0)] = 0 \qquad \text{and} \qquad \lim_{s\to\infty} F(s) = f(0).$$

For $f(t) = \cos kt$,

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty} s\,\frac{s}{s^2+k^2} = 1 = f(0).$$

## 7.3 Operational Properties I

**1.** $\mathscr{L}\left\{te^{10t}\right\} = \dfrac{1}{(s-10)^2}$

**2.** $\mathscr{L}\left\{te^{-6t}\right\} = \dfrac{1}{(s+6)^2}$

**3.** $\mathscr{L}\left\{t^3e^{-2t}\right\} = \dfrac{3!}{(s+2)^4}$

**4.** $\mathscr{L}\left\{t^{10}e^{-7t}\right\} = \dfrac{10!}{(s+7)^{11}}$

**5.** $\mathscr{L}\left\{t\left(e^t + e^{2t}\right)^2\right\} = \mathscr{L}\left\{te^{2t} + 2te^{3t} + te^{4t}\right\} = \dfrac{1}{(s-2)^2} + \dfrac{2}{(s-3)^2} + \dfrac{1}{(s-4)^2}$

**6.** $\mathscr{L}\left\{e^{2t}(t-1)^2\right\} = \mathscr{L}\left\{t^2e^{2t} - 2te^{2t} + e^{2t}\right\} = \dfrac{2}{(s-2)^3} - \dfrac{2}{(s-2)^2} + \dfrac{1}{s-2}$

**7.** $\mathscr{L}\left\{e^t\sin 3t\right\} = \dfrac{3}{(s-1)^2+9}$

**8.** $\mathscr{L}\left\{e^{-2t}\cos 4t\right\} = \dfrac{s+2}{(s+2)^2+16}$

**9.** $\mathscr{L}\left\{(1 - e^t + 3e^{-4t})\cos 5t\right\} = \mathscr{L}\left\{\cos 5t - e^t\cos 5t + 3e^{-4t}\cos 5t\right\}$

$$= \frac{s}{s^2+25} - \frac{s-1}{(s-1)^2+25} + \frac{3(s+4)}{(s+4)^2+25}$$

**10.** $\mathscr{L}\left\{e^{3t}\left(9 - 4t + 10\sin\dfrac{t}{2}\right)\right\} = \mathscr{L}\left\{9e^{3t} - 4te^{3t} + 10e^{3t}\sin\dfrac{t}{2}\right\}$

$$= \frac{9}{s-3} - \frac{4}{(s-3)^2} + \frac{5}{(s-3)^2+1/4}$$

**11.** $\mathscr{L}^{-1}\left\{\dfrac{1}{(s+2)^3}\right\} = \mathscr{L}^{-1}\left\{\dfrac{1}{2}\,\dfrac{2}{(s+2)^3}\right\} = \dfrac{1}{2}t^2e^{-2t}$

**12.** $\mathscr{L}^{-1}\left\{\frac{1}{(s-1)^4}\right\}=\frac{1}{6}\mathscr{L}^{-1}\left\{\frac{3!}{(s-1)^4}\right\}=\frac{1}{6}t^3e^t$

**13.** $\mathscr{L}^{-1}\left\{\frac{1}{s^2-6s+10}\right\}=\mathscr{L}^{-1}\left\{\frac{1}{(s-3)^2+1^2}\right\}=e^{3t}\sin t$

**14.** $\mathscr{L}^{-1}\left\{\frac{1}{s^2+2s+5}\right\}=\mathscr{L}^{-1}\left\{\frac{1}{2}\,\frac{2}{(s+1)^2+2^2}\right\}=\frac{1}{2}e^{-t}\sin 2t$

**15.** $\mathscr{L}^{-1}\left\{\frac{s}{s^2+4s+5}\right\}=\mathscr{L}^{-1}\left\{\frac{s+2}{(s+2)^2+1^2}-2\frac{1}{(s+2)^2+1^2}\right\}=e^{-2t}\cos t-2e^{-2t}\sin t$

**16.** $\mathscr{L}^{-1}\left\{\frac{2s+5}{s^2+6s+34}\right\}=\mathscr{L}^{-1}\left\{2\frac{(s+3)}{(s+3)^2+5^2}-\frac{1}{5}\,\frac{5}{(s+3)^2+5^2}\right\}$

$$=2e^{-3t}\cos 5t-\frac{1}{5}e^{-3t}\sin 5t$$

**17.** $\mathscr{L}^{-1}\left\{\frac{s}{(s+1)^2}\right\}=\mathscr{L}^{-1}\left\{\frac{s+1-1}{(s+1)^2}\right\}=\mathscr{L}^{-1}\left\{\frac{1}{s+1}-\frac{1}{(s+1)^2}\right\}=e^{-t}-te^{-t}$

**18.** $\mathscr{L}^{-1}\left\{\frac{5s}{(s-2)^2}\right\}=\mathscr{L}^{-1}\left\{\frac{5(s-2)+10}{(s-2)^2}\right\}=\mathscr{L}^{-1}\left\{\frac{5}{s-2}+\frac{10}{(s-2)^2}\right\}=5e^{2t}+10te^{2t}$

**19.** $\mathscr{L}^{-1}\left\{\frac{2s-1}{s^2(s+1)^3}\right\}=\mathscr{L}^{-1}\left\{\frac{5}{s}-\frac{1}{s^2}-\frac{5}{s+1}-\frac{4}{(s+1)^2}-\frac{3}{2}\,\frac{2}{(s+1)^3}\right\}$

$$=5-t-5e^{-t}-4te^{-t}-\frac{3}{2}t^2e^{-t}$$

**20.** $\mathscr{L}^{-1}\left\{\frac{(s+1)^2}{(s+2)^4}\right\}=\mathscr{L}^{-1}\left\{\frac{1}{(s+2)^2}-\frac{2}{(s+2)^3}+\frac{1}{6}\,\frac{3!}{(s+2)^4}\right\}=te^{-2t}-t^2e^{-2t}+\frac{1}{6}t^3e^{-2t}$

**21.** The Laplace transform of the differential equation is

$$s\mathscr{L}\{y\}-y(0)+4\mathscr{L}\{y\}=\frac{1}{s+4}\,.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\}=\frac{1}{(s+4)^2}+\frac{2}{s+4}\,.$$

Thus

$$y=te^{-4t}+2e^{-4t}.$$

**22.** The Laplace transform of the differential equation is

$$s\mathscr{L}\{y\}-\mathscr{L}\{y\}=\frac{1}{s}+\frac{1}{(s-1)^2}\,.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\}=\frac{1}{s(s-1)}+\frac{1}{(s-1)^3}=-\frac{1}{s}+\frac{1}{s-1}+\frac{1}{(s-1)^3}\,.$$

Thus

$$y=-1+e^t+\frac{1}{2}t^2e^t.$$

**23.** The Laplace transform of the differential equation is

$$s^2\mathscr{L}\{y\} - sy(0) - y'(0) + 2\left[s\mathscr{L}\{y\} - y(0)\right] + \mathscr{L}\{y\} = 0.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\} = \frac{s+3}{(s+1)^2} = \frac{1}{s+1} + \frac{2}{(s+1)^2}\,.$$

Thus

$$y = e^{-t} + 2te^{-t}.$$

**24.** The Laplace transform of the differential equation is

$$s^2\mathscr{L}\{y\} - sy(0) - y'(0) - 4\left[s\mathscr{L}\{y\} - y(0)\right] + 4\mathscr{L}\{y\} = \frac{6}{(s-2)^4}\,.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\} = \frac{1}{20}\,\frac{5!}{(s-2)^6}\,.$$

Thus, $y = \frac{1}{20}t^5e^{2t}$.

**25.** The Laplace transform of the differential equation is

$$s^2\mathscr{L}\{y\} - sy(0) - y'(0) - 6\left[s\mathscr{L}\{y\} - y(0)\right] + 9\mathscr{L}\{y\} = \frac{1}{s^2}\,.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\} = \frac{1+s^2}{s^2(s-3)^2} = \frac{2}{27}\,\frac{1}{s} + \frac{1}{9}\,\frac{1}{s^2} - \frac{2}{27}\,\frac{1}{s-3} + \frac{10}{9}\,\frac{1}{(s-3)^2}\,.$$

Thus

$$y = \frac{2}{27} + \frac{1}{9}t - \frac{2}{27}e^{3t} + \frac{10}{9}te^{3t}.$$

**26.** The Laplace transform of the differential equation is

$$s^2\mathscr{L}\{y\} - sy(0) - y'(0) - 4\left[s\mathscr{L}\{y\} - y(0)\right] + 4\mathscr{L}\{y\} = \frac{6}{s^4}\,.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\} = \frac{s^5 - 4s^4 + 6}{s^4(s-2)^2} = \frac{3}{4}\,\frac{1}{s} + \frac{9}{8}\,\frac{1}{s^2} + \frac{3}{4}\,\frac{2}{s^3} + \frac{1}{4}\,\frac{3!}{s^4} + \frac{1}{4}\,\frac{1}{s-2} - \frac{13}{8}\,\frac{1}{(s-2)^2}\,.$$

Thus

$$y = \frac{3}{4} + \frac{9}{8}t + \frac{3}{4}t^2 + \frac{1}{4}t^3 + \frac{1}{4}e^{2t} - \frac{13}{8}te^{2t}.$$

**27.** The Laplace transform of the differential equation is

$$s^2\mathscr{L}\{y\} - sy(0) - y'(0) - 6\left[s\mathscr{L}\{y\} - y(0)\right] + 13\mathscr{L}\{y\} = 0.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\} = -\frac{3}{s^2 - 6s + 13} = -\frac{3}{2}\,\frac{2}{(s-3)^2 + 2^2}\,.$$

Thus

$$y = -\frac{3}{2}e^{3t}\sin 2t.$$

**28.** The Laplace transform of the differential equation is

$$2\left[s^2\mathscr{L}\{y\} - sy(0)\right] + 20\left[s\mathscr{L}\{y\} - y(0)\right] + 51\mathscr{L}\{y\} = 0.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\} = \frac{4s + 40}{2s^2 + 20s + 51} = \frac{2s + 20}{(s+5)^2 + 1/2} = \frac{2(s+5)}{(s+5)^2 + 1/2} + \frac{10}{(s+5)^2 + 1/2}\,.$$

Thus

$$y = 2e^{-5t}\cos\left(t/\sqrt{2}\,\right) + 10\sqrt{2}\,e^{-5t}\sin\left(t/\sqrt{2}\,\right).$$

**29.** The Laplace transform of the differential equation is

$$s^2\mathscr{L}\{y\} - sy(0) - y'(0) - \left[s\mathscr{L}\{y\} - y(0)\right] = \frac{s-1}{(s-1)^2 + 1}\,.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\} = \frac{1}{s(s^2 - 2s + 2)} = \frac{1}{2}\,\frac{1}{s} - \frac{1}{2}\,\frac{s-1}{(s-1)^2 + 1} + \frac{1}{2}\,\frac{1}{(s-1)^2 + 1}\,.$$

Thus

$$y = \frac{1}{2} - \frac{1}{2}e^t\cos t + \frac{1}{2}e^t\sin t.$$

**30.** The Laplace transform of the differential equation is

$$s^2\mathscr{L}\{y\} - sy(0) - y'(0) - 2\left[s\mathscr{L}\{y\} - y(0)\right] + 5\mathscr{L}\{y\} = \frac{1}{s} + \frac{1}{s^2}\,.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\begin{aligned}\mathscr{L}\{y\} &= \frac{4s^2 + s + 1}{s^2(s^2 - 2s + 5)} = \frac{7}{25}\,\frac{1}{s} + \frac{1}{5}\,\frac{1}{s^2} + \frac{-7s/25 + 109/25}{s^2 - 2s + 5}\\ &= \frac{7}{25}\,\frac{1}{s} + \frac{1}{5}\,\frac{1}{s^2} - \frac{7}{25}\,\frac{s-1}{(s-1)^2 + 2^2} + \frac{51}{25}\,\frac{2}{(s-1)^2 + 2^2}\,.\end{aligned}$$

Thus

$$y = \frac{7}{25} + \frac{1}{5}t - \frac{7}{25}e^t\cos 2t + \frac{51}{25}e^t\sin 2t.$$

**31.** Taking the Laplace transform of both sides of the differential equation and letting $c = y(0)$ we obtain

$$\begin{aligned}
\mathscr{L}\{y''\} + \mathscr{L}\{2y'\} + \mathscr{L}\{y\} &= 0 \\
s^2\mathscr{L}\{y\} - sy(0) - y'(0) + 2s\mathscr{L}\{y\} - 2y(0) + \mathscr{L}\{y\} &= 0 \\
s^2\mathscr{L}\{y\} - cs - 2 + 2s\mathscr{L}\{y\} - 2c + \mathscr{L}\{y\} &= 0 \\
\left(s^2 + 2s + 1\right)\mathscr{L}\{y\} &= cs + 2c + 2 \\
\mathscr{L}\{y\} &= \frac{cs}{(s+1)^2} + \frac{2c+2}{(s+1)^2} \\
&= c\frac{s+1-1}{(s+1)^2} + \frac{2c+2}{(s+1)^2} \\
&= \frac{c}{s+1} + \frac{c+2}{(s+1)^2}\,.
\end{aligned}$$

Therefore,

$$y(t) = c\mathscr{L}^{-1}\left\{\frac{1}{s+1}\right\} + (c+2)\mathscr{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} = ce^{-t} + (c+2)te^{-t}.$$

To find $c$ we let $y(1) = 2$. Then $2 = ce^{-1} + (c+2)e^{-1} = 2(c+1)e^{-1}$ and $c = e - 1$. Thus

$$y(t) = (e-1)e^{-t} + (e+1)te^{-t}.$$

**32.** Taking the Laplace transform of both sides of the differential equation and letting $c = y'(0)$ we obtain

$$\begin{aligned}
\mathscr{L}\{y''\} + \mathscr{L}\{8y'\} + \mathscr{L}\{20y\} &= 0 \\
s^2\mathscr{L}\{y\} - y'(0) + 8s\mathscr{L}\{y\} + 20\mathscr{L}\{y\} &= 0 \\
s^2\mathscr{L}\{y\} - c + 8s\mathscr{L}\{y\} + 20\mathscr{L}\{y\} &= 0 \\
(s^2 + 8s + 20)\mathscr{L}\{y\} &= c \\
\mathscr{L}\{y\} = \frac{c}{s^2 + 8s + 20} &= \frac{c}{(s+4)^2 + 4}\,.
\end{aligned}$$

Therefore,

$$y(t) = \mathscr{L}^{-1}\left\{\frac{c}{(s+4)^2 + 4}\right\} = \frac{c}{2}e^{-4t}\sin 2t = c_1e^{-4t}\sin 2t.$$

To find $c_1$ we let $y'(\pi) = 0$. Then $0 = y'(\pi) = c_1e^{-4\pi}$ and $c_1 = 0$. Thus, $y(t) = 0$. (Since the differential equation is homogeneous and both boundary conditions are 0, we can see immediately that $y(t) = 0$ is a solution. We have shown that it is the only solution.)

**33.** Recall from Section 3.8 that $mx'' = -kx - \beta x'$. Now $\beta = 7/8$, $m = W/g = 4/32 = 1/8$ slug, and $4 = 2k$ so that $k = 2$ lb/ft. Thus, the differential equation is $x'' + 7x' + 16x = 0$. The initial conditions are $x(0) = -3/2$ and $x'(0) = 0$. The Laplace transform of the differential equation is

$$s^2\mathscr{L}\{x\} + \frac{3}{2}s + 7s\mathscr{L}\{x\} + \frac{21}{2} + 16\mathscr{L}\{x\} = 0.$$

Solving for $\mathscr{L}\{x\}$ we obtain

$$\mathscr{L}\{x\} = \frac{-3s/2 - 21/2}{s^2 + 7s + 16} = -\frac{3}{2}\,\frac{s + 7/2}{(s + 7/2)^2 + (\sqrt{15}/2)^2} - \frac{7\sqrt{15}}{10}\,\frac{\sqrt{15}/2}{(s + 7/2)^2 + (\sqrt{15}/2)^2}\,.$$

Thus

$$x = -\frac{3}{2}e^{-7t/2}\cos\frac{\sqrt{15}}{2}\,t - \frac{7\sqrt{15}}{10}\,e^{-7t/2}\sin\frac{\sqrt{15}}{2}\,t.$$

**34.** The differential equation is

$$\frac{d^2q}{dt^2} + 20\frac{dq}{dt} + 200q = 150, \quad q(0) = q'(0) = 0.$$

The Laplace transform of this equation is

$$s^2\mathscr{L}\{q\} + 20s\mathscr{L}\{q\} + 200\mathscr{L}\{q\} = \frac{150}{s}\,.$$

Solving for $\mathscr{L}\{q\}$ we obtain

$$\mathscr{L}\{q\} = \frac{150}{s(s^2 + 20s + 200)} = \frac{3}{4}\,\frac{1}{s} - \frac{3}{4}\,\frac{s + 10}{(s + 10)^2 + 10^2} - \frac{3}{4}\,\frac{10}{(s + 10)^2 + 10^2}\,.$$

Thus

$$q(t) = \frac{3}{4} - \frac{3}{4}e^{-10t}\cos 10t - \frac{3}{4}e^{-10t}\sin 10t$$

and

$$i(t) = q'(t) = 15e^{-10t}\sin 10t.$$

**35.** The differential equation is

$$\frac{d^2q}{dt^2} + 2\lambda dqdt + \omega^2 q = \frac{E_0}{L}, \quad q(0) = q'(0) = 0.$$

The Laplace transform of this equation is

$$s^2\mathscr{L}\{q\} + 2\lambda s\mathscr{L}\{q\} + \omega^2\mathscr{L}\{q\} = \frac{E_0}{L}\frac{1}{s}$$

or

$$\left(s^2 + 2\lambda s + \omega^2\right)\mathscr{L}\{q\} = \frac{E_0}{L}\frac{1}{s}.$$

Solving for $\mathscr{L}\{q\}$ and using partial fractions we obtain

$$\mathscr{L}\{q\} = \frac{E_0}{L}\left(\frac{1/\omega^2}{s} - \frac{(1/\omega^2)s + 2\lambda/\omega^2}{s^2 + 2\lambda s + \omega^2}\right) = \frac{E_0}{L\omega^2}\left(\frac{1}{s} - \frac{s + 2\lambda}{s^2 + 2\lambda s + \omega^2}\right).$$

For $\lambda > \omega$ we write $s^2 + 2\lambda s + \omega^2 = (s+\lambda)^2 - \left(\lambda^2 - \omega^2\right)$, so (recalling that $\omega^2 = 1/LC$)

$$\mathscr{L}\{q\} = E_0 C\left(\frac{1}{s} - \frac{s+\lambda}{(s+\lambda)^2 - (\lambda^2 - \omega^2)} - \frac{\lambda}{(s+\lambda)^2 - (\lambda^2 - \omega^2)}\right).$$

Thus for $\lambda > \omega$,

$$q(t) = E_0 C\left[1 - e^{-\lambda t}\left(\cosh\sqrt{\lambda^2 - \omega^2}\,t - \frac{\lambda}{\sqrt{\lambda^2 - \omega^2}}\sinh\sqrt{\lambda^2 - \omega^2}\,t\right)\right].$$

For $\lambda < \omega$ we write $s^2 + 2\lambda s + \omega^2 = (s+\lambda)^2 + \left(\omega^2 - \lambda^2\right)$, so

$$\mathscr{L}\{q\} = E_0 C\left(\frac{1}{s} - \frac{s+\lambda}{(s+\lambda)^2 + (\omega^2 - \lambda^2)} - \frac{\lambda}{(s+\lambda)^2 + (\omega^2 - \lambda^2)}\right).$$

Thus for $\lambda < \omega$,

$$q(t) = E_0 C\left[1 - e^{-\lambda t}\left(\cos\sqrt{\omega^2 - \lambda^2}\,t - \frac{\lambda}{\sqrt{\omega^2 - \lambda^2}}\sin\sqrt{\omega^2 - \lambda^2}\,t\right)\right].$$

For $\lambda = \omega$, $s^2 + 2\lambda + \omega^2 = (s+\lambda)^2$ and

$$\mathscr{L}\{q\} = \frac{E_0}{L}\frac{1}{s(s+\lambda)^2} = \frac{E_0}{L}\left(\frac{1/\lambda^2}{s} - \frac{1/\lambda^2}{s+\lambda} - \frac{1/\lambda}{(s+\lambda)^2}\right) = \frac{E_0}{L\lambda^2}\left(\frac{1}{s} - \frac{1}{s+\lambda} - \frac{\lambda}{(s+\lambda)^2}\right).$$

Thus for $\lambda = \omega$,

$$q(t) = E_0 C\left(1 - e^{-\lambda t} - \lambda t e^{-\lambda t}\right).$$

**36.** The differential equation is

$$R\,\frac{dq}{dt}+\frac{1}{C}\,q=E_0e^{-kt},\quad q(0)=0.$$

The Laplace transform of this equation is

$$Rs\mathscr{L}\{q\}+\frac{1}{C}\mathscr{L}\{q\}=E_0\,\frac{1}{s+k}\,.$$

Solving for $\mathscr{L}\{q\}$ we obtain

$$\mathscr{L}\{q\}=\frac{E_0C}{(s+k)(RCs+1)}=\frac{E_0/R}{(s+k)(s+1/RC)}\,.$$

When $1/RC\neq k$ we have by partial fractions

$$\mathscr{L}\{q\}=\frac{E_0}{R}\left(\frac{1/(1/RC-k)}{s+k}-\frac{1/(1/RC-k)}{s+1/RC}\right)=\frac{E_0}{R}\,\frac{1}{1/RC-k}\left(\frac{1}{s+k}-\frac{1}{s+1/RC}\right).$$

Thus

$$q(t)=\frac{E_0C}{1-kRC}\left(e^{-kt}-e^{-t/RC}\right).$$

When $1/RC=k$ we have

$$\mathscr{L}\{q\}=\frac{E_0}{R}\,\frac{1}{(s+k)^2}\,.$$

Thus

$$q(t)=\frac{E_0}{R}\,te^{-kt}=\frac{E_0}{R}\,te^{-t/RC}.$$

**37.** $\mathscr{L}\left\{(t-1)\mathscr{U}(t-1)\right\}=\dfrac{e^{-s}}{s^2}$

**38.** $\mathscr{L}\left\{e^{2-t}\mathscr{U}(t-2)\right\}=\mathscr{L}\left\{e^{-(t-2)}\mathscr{U}(t-2)\right\}=\dfrac{e^{-2s}}{s+1}$

**39.** $\mathscr{L}\left\{t\mathscr{U}(t-2)\right\}=\mathscr{L}\{(t-2)\mathscr{U}(t-2)+2\mathscr{U}(t-2)\}=\dfrac{e^{-2s}}{s^2}+\dfrac{2e^{-2s}}{s}$

Alternatively, (16) of this section could be used:

$$\mathscr{L}\{t\mathscr{U}(t-2)\}=e^{-2s}\mathscr{L}\{t+2\}=e^{-2s}\left(\frac{1}{s^2}+\frac{2}{s}\right).$$

**40.** $\mathscr{L}\left\{(3t+1)\mathscr{U}(t-1)\right\}=3\mathscr{L}\left\{(t-1)\mathscr{U}(t-1)\right\}+4\mathscr{L}\left\{\mathscr{U}(t-1)\right\}=\dfrac{3e^{-s}}{s^2}+\dfrac{4e^{-s}}{s}$

Alternatively, (16) of this section could be used:

$$\mathscr{L}\{(3t+1)\mathscr{U}(t-1)\}=e^{-s}\mathscr{L}\{3t+4\}=e^{-s}\left(\frac{3}{s^2}+\frac{4}{s}\right).$$

**41.** $\mathscr{L}\{\cos 2t\,\mathscr{U}(t-\pi)\} = \mathscr{L}\{\cos 2(t-\pi)\,\mathscr{U}(t-\pi)\} = \dfrac{se^{-\pi s}}{s^2+4}$

Alternatively, (16) of this section could be used:

$$\mathscr{L}\{\cos 2t\,\mathscr{U}(t-\pi)\} = e^{-\pi s}\mathscr{L}\{\cos 2(t+\pi)\} = e^{-\pi s}\mathscr{L}\{\cos 2t\} = e^{-\pi s}\,\frac{s}{s^2+4}.$$

**42.** $\mathscr{L}\left\{\sin t\,\mathscr{U}\left(t-\dfrac{\pi}{2}\right)\right\} = \mathscr{L}\left\{\cos\left(t-\dfrac{\pi}{2}\right)\mathscr{U}\left(t-\dfrac{\pi}{2}\right)\right\} = \dfrac{se^{-\pi s/2}}{s^2+1}$

Alternatively, (16) of this section could be used:

$$\mathscr{L}\left\{\sin t\,\mathscr{U}\left(t-\frac{\pi}{2}\right)\right\} = e^{-\pi s/2}\mathscr{L}\left\{\sin\left(t+\frac{\pi}{2}\right)\right\} = e^{-\pi s/2}\mathscr{L}\{\cos t\} = e^{-\pi s/2}\,\frac{s}{s^2+1}.$$

**43.** $\mathscr{L}^{-1}\left\{\dfrac{e^{-2s}}{s^3}\right\} = \mathscr{L}^{-1}\left\{\dfrac{1}{2}\cdot\dfrac{2}{s^3}e^{-2s}\right\} = \dfrac{1}{2}(t-2)^2\,\mathscr{U}(t-2)$

**44.** $\mathscr{L}^{-1}\left\{\dfrac{(1+e^{-2s})^2}{s+2}\right\} = \mathscr{L}^{-1}\left\{\dfrac{1}{s+2}+\dfrac{2e^{-2s}}{s+2}+\dfrac{e^{-4s}}{s+2}\right\}$

$$= e^{-2t} + 2e^{-2(t-2)}\,\mathscr{U}(t-2) + e^{-2(t-4)}\,\mathscr{U}(t-4)$$

**45.** $\mathscr{L}^{-1}\left\{\dfrac{e^{-\pi s}}{s^2+1}\right\} = \sin(t-\pi)\,\mathscr{U}(t-\pi) = -\sin t\,\mathscr{U}(t-\pi)$

**46.** $\mathscr{L}^{-1}\left\{\dfrac{se^{-\pi s/2}}{s^2+4}\right\} = \cos 2\left(t-\dfrac{\pi}{2}\right)\mathscr{U}\left(t-\dfrac{\pi}{2}\right) = -\cos 2t\,\mathscr{U}\left(t-\dfrac{\pi}{2}\right)$

**47.** $\mathscr{L}^{-1}\left\{\dfrac{e^{-s}}{s(s+1)}\right\} = \mathscr{L}^{-1}\left\{\dfrac{e^{-s}}{s}-\dfrac{e^{-s}}{s+1}\right\} = \mathscr{U}(t-1) - e^{-(t-1)}\,\mathscr{U}(t-1)$

**48.** $\mathscr{L}^{-1}\left\{\dfrac{e^{-2s}}{s^2(s-1)}\right\} = \mathscr{L}^{-1}\left\{-\dfrac{e^{-2s}}{s}-\dfrac{e^{-2s}}{s^2}+\dfrac{e^{-2s}}{s-1}\right\}$

$$= -\mathscr{U}(t-2) - (t-2)\,\mathscr{U}(t-2) + e^{t-2}\,\mathscr{U}(t-2)$$

**49.** **(c)** **50.** **(e)** **51.** **(f)** **52.** **(b)** **53.** **(a)** **54.** **(d)**

**55.** $\mathscr{L}\{2-4\,\mathscr{U}(t-3)\} = \dfrac{2}{s} - \dfrac{4}{s}e^{-3s}$

**56.** $\mathscr{L}\{1-\mathscr{U}(t-4)+\mathscr{U}(t-5)\} = \dfrac{1}{s} - \dfrac{e^{-4s}}{s} + \dfrac{e^{-5s}}{s}$

**57.** $\mathscr{L}\{t^2\,\mathscr{U}(t-1)\} = \mathscr{L}\left\{\left[(t-1)^2+2t-1\right]\mathscr{U}(t-1)\right\}$

$$= \mathscr{L}\left\{\left[(t-1)^2+2(t-1)-1\right]\mathscr{U}(t-1)\right\} = \left(\frac{2}{s^3}+\frac{2}{s^2}+\frac{1}{s}\right)e^{-s}$$

Alternatively, by (16) of this section,

$$\mathscr{L}\{t^2\,\mathscr{U}(t-1)\} = e^{-s}\mathscr{L}\{t^2+2t+1\} = e^{-s}\left(\frac{2}{s^3}+\frac{2}{s^2}+\frac{1}{s}\right).$$

**58.** $\mathscr{L}\left\{\sin t\,\mathscr{U}\left(t-\dfrac{3\pi}{2}\right)\right\}=\mathscr{L}\left\{-\cos\left(t-\dfrac{3\pi}{2}\right)\mathscr{U}\left(t-\dfrac{3\pi}{2}\right)\right\}=-\dfrac{se^{-3\pi s/2}}{s^2+1}$

**59.** $\mathscr{L}\{t-t\,\mathscr{U}(t-2)\}=\mathscr{L}\{t-(t-2)\mathscr{U}(t-2)-2\mathscr{U}(t-2)\}=\dfrac{1}{s^2}-\dfrac{e^{-2s}}{s^2}-\dfrac{2e^{-2s}}{s}$

**60.** $\mathscr{L}\{\sin t-\sin t\,\mathscr{U}(t-2\pi)\}=\mathscr{L}\{\sin t-\sin(t-2\pi)\mathscr{U}(t-2\pi)\}=\dfrac{1}{s^2+1}-\dfrac{e^{-2\pi s}}{s^2+1}$

**61.** $\mathscr{L}\{f(t)\}=\mathscr{L}\{\mathscr{U}(t-a)-\mathscr{U}(t-b)\}=\dfrac{e^{-as}}{s}-\dfrac{e^{-bs}}{s}$

**62.** $\mathscr{L}\{f(t)\}=\mathscr{L}\{\mathscr{U}(t-1)+\mathscr{U}(t-2)+\mathscr{U}(t-3)+\cdots\}=\dfrac{e^{-s}}{s}+\dfrac{e^{-2s}}{s}+\dfrac{e^{-3s}}{s}+\cdots$

$$=\frac{1}{s}\,\frac{e^{-s}}{1-e^{-s}}$$

**63.** The Laplace transform of the differential equation is

$$s\mathscr{L}\{y\}-y(0)+\mathscr{L}\{y\}=\frac{5}{s}e^{-s}.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\}=\frac{5e^{-s}}{s(s+1)}=5e^{-s}\left[\frac{1}{s}-\frac{1}{s+1}\right].$$

Thus

$$y=5\mathscr{U}(t-1)-5e^{-(t-1)}\mathscr{U}(t-1).$$

**64.** The Laplace transform of the differential equation is

$$s\mathscr{L}\{y\}-y(0)+\mathscr{L}\{y\}=\frac{1}{s}-\frac{2}{s}e^{-s}.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\}=\frac{1}{s(s+1)}-\frac{2e^{-s}}{s(s+1)}=\frac{1}{s}-\frac{1}{s+1}-2e^{-s}\left[\frac{1}{s}-\frac{1}{s+1}\right].$$

Thus

$$y=1-e^{-t}-2\left[1-e^{-(t-1)}\right]\mathscr{U}(t-1).$$

**65.** The Laplace transform of the differential equation is

$$s\mathscr{L}\{y\}-y(0)+2\mathscr{L}\{y\}=\frac{1}{s^2}-e^{-s}\frac{s+1}{s^2}.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\}=\frac{1}{s^2(s+2)}-e^{-s}\frac{s+1}{s^2(s+2)}=-\frac{1}{4}\frac{1}{s}+\frac{1}{2}\frac{1}{s^2}+\frac{1}{4}\frac{1}{s+2}-e^{-s}\left[\frac{1}{4}\frac{1}{s}+\frac{1}{2}\frac{1}{s^2}-\frac{1}{4}\frac{1}{s+2}\right].$$

Thus

$$y=-\frac{1}{4}+\frac{1}{2}t+\frac{1}{4}e^{-2t}-\left[\frac{1}{4}+\frac{1}{2}(t-1)-\frac{1}{4}e^{-2(t-1)}\right]\mathscr{U}(t-1).$$

**66.** The Laplace transform of the differential equation is

$$s^2\mathscr{L}\{y\} - sy(0) - y'(0) + 4\mathscr{L}\{y\} = \frac{1}{s} - \frac{e^{-s}}{s}.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\} = \frac{1-s}{s(s^2+4)} - e^{-s}\frac{1}{s(s^2+4)} = \frac{1}{4}\frac{1}{s} - \frac{1}{4}\frac{s}{s^2+4} - \frac{1}{2}\frac{2}{s^2+4} - e^{-s}\left[\frac{1}{4}\frac{1}{s} - \frac{1}{4}\frac{s}{s^2+4}\right].$$

Thus

$$y = \frac{1}{4} - \frac{1}{4}\cos 2t - \frac{1}{2}\sin 2t - \left[\frac{1}{4} - \frac{1}{4}\cos 2(t-1)\right]\mathscr{U}(t-1).$$

**67.** The Laplace transform of the differential equation is

$$s^2\mathscr{L}\{y\} - sy(0) - y'(0) + 4\mathscr{L}\{y\} = e^{-2\pi s}\frac{1}{s^2+1}.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\} = \frac{s}{s^2+4} + e^{-2\pi s}\left[\frac{1}{3}\frac{1}{s^2+1} - \frac{1}{6}\frac{2}{s^2+4}\right].$$

Thus

$$y = \cos 2t + \left[\frac{1}{3}\sin(t-2\pi) - \frac{1}{6}\sin 2(t-2\pi)\right]\mathscr{U}(t-2\pi).$$

**68.** The Laplace transform of the differential equation is

$$s^2\mathscr{L}\{y\} - sy(0) - y'(0) - 5\left[s\mathscr{L}\{y\} - y(0)\right] + 6\mathscr{L}\{y\} = \frac{e^{-s}}{s}.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\begin{aligned}\mathscr{L}\{y\} &= e^{-s}\frac{1}{s(s-2)(s-3)} + \frac{1}{(s-2)(s-3)}\\ &= e^{-s}\left[\frac{1}{6}\frac{1}{s} - \frac{1}{2}\frac{1}{s-2} + \frac{1}{3}\frac{1}{s-3}\right] - \frac{1}{s-2} + \frac{1}{s-3}.\end{aligned}$$

Thus

$$y = \left[\frac{1}{6} - \frac{1}{2}e^{2(t-1)} + \frac{1}{3}e^{3(t-1)}\right]\mathscr{U}(t-1) - e^{2t} + e^{3t}.$$

**69.** The Laplace transform of the differential equation is

$$s^2\mathscr{L}\{y\} - sy(0) - y'(0) + \mathscr{L}\{y\} = \frac{e^{-\pi s}}{s} - \frac{e^{-2\pi s}}{s}.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\} = e^{-\pi s}\left[\frac{1}{s} - \frac{s}{s^2+1}\right] - e^{-2\pi s}\left[\frac{1}{s} - \frac{s}{s^2+1}\right] + \frac{1}{s^2+1}.$$

Thus

$$y = [1 - \cos(t-\pi)]\mathscr{U}(t-\pi) - [1 - \cos(t-2\pi)]\mathscr{U}(t-2\pi) + \sin t.$$

**70.** The Laplace transform of the differential equation is

$$s^2\mathscr{L}\{y\} - sy(0) - y'(0) + 4\left[s\mathscr{L}\{y\} - y(0)\right] + 3\mathscr{L}\{y\} = \frac{1}{s} - \frac{e^{-2s}}{s} - \frac{e^{-4s}}{s} + \frac{e^{-6s}}{s}.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\begin{aligned}\mathscr{L}\{y\} &= \frac{1}{3}\frac{1}{s} - \frac{1}{2}\frac{1}{s+1} + \frac{1}{6}\frac{1}{s+3} - e^{-2s}\left[\frac{1}{3}\frac{1}{s} - \frac{1}{2}\frac{1}{s+1} + \frac{1}{6}\frac{1}{s+3}\right]\\ &\quad - e^{-4s}\left[\frac{1}{3}\frac{1}{s} - \frac{1}{2}\frac{1}{s+1} + \frac{1}{6}\frac{1}{s+3}\right] + e^{-6s}\left[\frac{1}{3}\frac{1}{s} - \frac{1}{2}\frac{1}{s+1} + \frac{1}{6}\frac{1}{s+3}\right].\end{aligned}$$

Thus

$$\begin{aligned}y &= \frac{1}{3} - \frac{1}{2}e^{-t} + \frac{1}{6}e^{-3t} - \left[\frac{1}{3} - \frac{1}{2}e^{-(t-2)} + \frac{1}{6}e^{-3(t-2)}\right]\mathscr{U}(t-2)\\ &\quad - \left[\frac{1}{3} - \frac{1}{2}e^{-(t-4)} + \frac{1}{6}e^{-3(t-4)}\right]\mathscr{U}(t-4) + \left[\frac{1}{3} - \frac{1}{2}e^{-(t-6)} + \frac{1}{6}e^{-3(t-6)}\right]\mathscr{U}(t-6).\end{aligned}$$

**71.** Recall from Section 3.8 that $mx'' = -kx + f(t)$. Now $m = W/g = 32/32 = 1$ slug, and $32 = 2k$ so that $k = 16$ lb/ft. Thus, the differential equation is $x'' + 16x = f(t)$. The initial conditions are $x(0) = 0$, $x'(0) = 0$. Also, since

$$f(t) = \begin{cases} 20t, & 0 \le t < 5 \\ 0, & t \ge 5 \end{cases}$$

and $20t = 20(t-5) + 100$ we can write

$$f(t) = 20t - 20t\mathscr{U}(t-5) = 20t - 20(t-5)\mathscr{U}(t-5) - 100\mathscr{U}(t-5).$$

The Laplace transform of the differential equation is

$$s^2\mathscr{L}\{x\} + 16\mathscr{L}\{x\} = \frac{20}{s^2} - \frac{20}{s^2}e^{-5s} - \frac{100}{s}e^{-5s}.$$

Solving for $\mathscr{L}\{x\}$ we obtain

$$\begin{aligned}\mathscr{L}\{x\} &= \frac{20}{s^2(s^2+16)} - \frac{20}{s^2(s^2+16)}e^{-5s} - \frac{100}{s(s^2+16)}e^{-5s}\\ &= \left(\frac{5}{4}\cdot\frac{1}{s^2} - \frac{5}{16}\cdot\frac{4}{s^2+16}\right)\left(1 - e^{-5s}\right) - \left(\frac{25}{4}\cdot\frac{1}{s} - \frac{25}{4}\cdot\frac{s}{s^2+16}\right)e^{-5s}.\end{aligned}$$

Thus

$$\begin{aligned}x(t) &= \frac{5}{4}t - \frac{5}{16}\sin 4t - \left[\frac{5}{4}(t-5) - \frac{5}{16}\sin 4(t-5)\right]\mathscr{U}(t-5) - \left[\frac{25}{4} - \frac{25}{4}\cos 4(t-5)\right]\mathscr{U}(t-5)\\ &= \frac{5}{4}t - \frac{5}{16}\sin 4t - \frac{5}{4}t\mathscr{U}(t-5) + \frac{5}{16}\sin 4(t-5)\mathscr{U}(t-5) + \frac{25}{4}\cos 4(t-5)\mathscr{U}(t-5).\end{aligned}$$

**72.** Recall from Section 3.8 that $mx'' = -kx + f(t)$. Now $m = W/g = 32/32 = 1$ slug, and $32 = 2k$ so that $k = 16$ lb/ft. Thus, the differential equation is $x'' + 16x = f(t)$. The initial conditions are $x(0) = 0$, $x'(0) = 0$. Also, since

$$f(t) = \begin{cases} \sin t, & 0 \le t < 2\pi \\ 0, & t \ge 2\pi \end{cases}$$

and $\sin t = \sin{(t - 2\pi)}$ we can write

$$f(t) = \sin t - \sin{(t - 2\pi)}\mathscr{U}(t - 2\pi).$$

The Laplace transform of the differential equation is

$$s^2\mathscr{L}\{x\} + 16\mathscr{L}\{x\} = \frac{1}{s^2+1} - \frac{1}{s^2+1}e^{-2\pi s}.$$

Solving for $\mathscr{L}\{x\}$ we obtain

$$\begin{aligned}\mathscr{L}\{x\} &= \frac{1}{(s^2+16)\,(s^2+1)} - \frac{1}{(s^2+16)\,(s^2+1)}e^{-2\pi s} \\ &= \frac{-1/15}{s^2+16} + \frac{1/15}{s^2+1} - \left[\frac{-1/15}{s^2+16} + \frac{1/15}{s^2+1}\right]e^{-2\pi s}.\end{aligned}$$

Thus

$$\begin{aligned}x(t) &= -\frac{1}{60}\sin 4t + \frac{1}{15}\sin t + \frac{1}{60}\sin 4(t-2\pi)\mathscr{U}(t-2\pi) - \frac{1}{15}\sin{(t-2\pi)}\mathscr{U}(t-2\pi) \\ &= \begin{cases} -\dfrac{1}{60}\sin 4t + \dfrac{1}{15}\sin t, & 0 \le t < 2\pi \\ 0, & t \ge 2\pi. \end{cases}\end{aligned}$$

**73.** The differential equation is

$$2.5\,\frac{dq}{dt} + 12.5q = 5\mathscr{U}(t-3).$$

The Laplace transform of this equation is

$$s\mathscr{L}\{q\} + 5\mathscr{L}\{q\} = \frac{2}{s}\,e^{-3s}.$$

Solving for $\mathscr{L}\{q\}$ we obtain

$$\mathscr{L}\{q\} = \frac{2}{s(s+5)}\,e^{-3s} = \left(\frac{2}{5}\cdot\frac{1}{s} - \frac{2}{5}\cdot\frac{1}{s+5}\right)e^{-3s}.$$

Thus

$$q(t) = \frac{2}{5}\mathscr{U}(t-3) - \frac{2}{5}e^{-5(t-3)}\mathscr{U}(t-3).$$

**74.** The differential equation is

$$10\,\frac{dq}{dt}+10q=30e^t-30e^t\mathscr{U}(t-1.5).$$

The Laplace transform of this equation is

$$s\mathscr{L}\{q\}-q_0+\mathscr{L}\{q\}=\frac{3}{s-1}-\frac{3e^{1.5}}{s-1.5}\,e^{-1.5s}.$$

Solving for $\mathscr{L}\{q\}$ we obtain

$$\mathscr{L}\{q\}=\left(q_0-\frac{3}{2}\right)\cdot\frac{1}{s+1}+\frac{3}{2}\cdot\frac{1}{s-1}-3e^{1.5}\left(\frac{-2/5}{s+1}+\frac{2/5}{s-1.5}\right)e^{-1.5s}.$$

Thus

$$q(t)=\left(q_0-\frac{3}{2}\right)e^{-t}+\frac{3}{2}e^t+\frac{6}{5}e^{1.5}\left(e^{-(t-1.5)}-e^{1.5(t-1.5)}\right)\mathscr{U}(t-1.5).$$

**75.** **(a)** The differential equation is

$$\frac{di}{dt}+10i=\sin t+\cos\left(t-\frac{3\pi}{2}\right)\mathscr{U}\left(t-\frac{3\pi}{2}\right),\quad i(0)=0.$$

The Laplace transform of this equation is

$$s\mathscr{L}\{i\}+10\mathscr{L}\{i\}=\frac{1}{s^2+1}+\frac{se^{-3\pi s/2}}{s^2+1}\,.$$

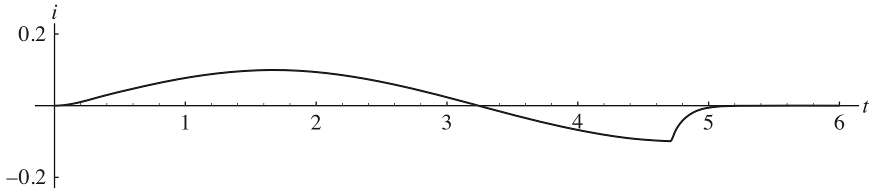
Solving for $\mathscr{L}\{i\}$ we obtain

$$\begin{aligned}\mathscr{L}\{i\}&=\frac{1}{(s^2+1)(s+10)}+\frac{s}{(s^2+1)(s+10)}\,e^{-3\pi s/2}\\&=\frac{1}{101}\left(\frac{1}{s+10}-\frac{s}{s^2+1}+\frac{10}{s^2+1}\right)+\frac{1}{101}\left(\frac{-10}{s+10}+\frac{10s}{s^2+1}+\frac{1}{s^2+1}\right)e^{-3\pi s/2}.\end{aligned}$$

Thus

$$\begin{aligned}i(t)&=\frac{1}{101}\left(e^{-10t}-\cos t+10\sin t\right)\\&\quad+\frac{1}{101}\left[-10e^{-10(t-3\pi/2)}+10\cos\left(t-\frac{3\pi}{2}\right)+\sin\left(t-\frac{3\pi}{2}\right)\right]\mathscr{U}\left(t-\frac{3\pi}{2}\right).\end{aligned}$$

**(b)**



The maximum value of $i(t)$ is approximately 0.1 at $t=1.7$, the minimum is approximately $-0.1$ at 4.7.

**76.** **(a)** The differential equation is

$$50\frac{dq}{dt}+\frac{1}{0.01}q=E_0[\mathscr{U}(t-1)-\mathscr{U}(t-3)], \quad q(0)=0$$

or

$$50\frac{dq}{dt}+100q=E_0[\mathscr{U}(t-1)-\mathscr{U}(t-3)], \quad q(0)=0.$$

The Laplace transform of this equation is

$$50s\mathscr{L}\{q\}+100\mathscr{L}\{q\}=E_0\left(\frac{1}{s}e^{-s}-\frac{1}{s}e^{-3s}\right).$$

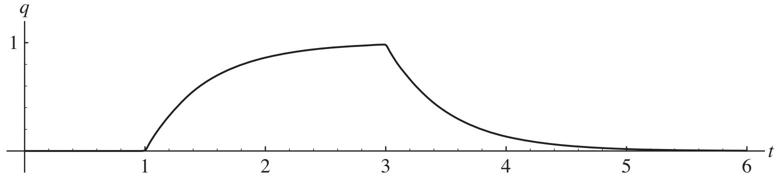
Solving for $\mathscr{L}\{q\}$ we obtain

$$\mathscr{L}\{q\}=\frac{E_0}{50}\left[\frac{e^{-s}}{s(s+2)}-\frac{e^{-3s}}{s(s+2)}\right]=\frac{E_0}{50}\left[\frac{1}{2}\left(\frac{1}{s}-\frac{1}{s+2}\right)e^{-s}-\frac{1}{2}\left(\frac{1}{s}-\frac{1}{s+2}\right)e^{-3s}\right].$$

Thus

$$q(t)=\frac{E_0}{100}\left[\left(1-e^{-2(t-1)}\right)\mathscr{U}(t-1)-\left(1-e^{-2(t-3)}\right)\mathscr{U}(t-3)\right].$$

**(b)**



The maximum value of $q(t)$ is approximately 1 at $t=3$.

**77.** The differential equation is

$$EI\frac{d^4y}{dx^4}=w_0[1-\mathscr{U}\left(x-\frac{L}{2}\right)].$$

Taking the Laplace transform of both sides and using $y(0)=y'(0)=0$ we obtain

$$s^4\mathscr{L}\{y\}-sy''(0)-y'''(0)=\frac{w_0}{EI}\frac{1}{s}\left(1-e^{-Ls/2}\right).$$

Letting $y''(0)=c_1$ and $y'''(0)=c_2$ we have

$$\mathscr{L}\{y\}=\frac{c_1}{s^3}+\frac{c_2}{s^4}+\frac{w_0}{EI}\frac{1}{s^5}\left(1-e^{-Ls/2}\right)$$

so that

$$y(x)=\frac{1}{2}c_1x^2+\frac{1}{6}c_2x^3+\frac{1}{24}\frac{w_0}{EI}\left[x^4-\left(x-\frac{L}{2}\right)^4\mathscr{U}\left(x-\frac{L}{2}\right)\right].$$

To find $c_1$ and $c_2$ we compute

$$y''(x)=c_1+c_2x+\frac{1}{2}\frac{w_0}{EI}\left[x^2-\left(x-\frac{L}{2}\right)^2\mathscr{U}\left(x-\frac{L}{2}\right)\right]$$

and

$$y'''(x)=c_2+\frac{w_0}{EI}\left[x-\left(x-\frac{L}{2}\right)\mathscr{U}\left(x-\frac{L}{2}\right)\right].$$

Then $y''(L)=y'''(L)=0$ yields the system

$$c_1 + c_2 L + \frac{1}{2}\frac{w_0}{EI}\left[L^2 - \left(\frac{L}{2}\right)^2\right] = 0 \qquad\qquad c_1 + c_2 L + \frac{3}{8}\frac{w_0 L^2}{EI} = 0$$

$$c_2 + \frac{w_0}{EI}\left(\frac{L}{2}\right) = 0. \qquad \text{or} \qquad c_2 + \frac{1}{2}\frac{w_0 L}{EI} = 0.$$

Solving for $c_1$ and $c_2$ we obtain $c_1 = w_0 L^2/8EI$ and $c_2 = -w_0 L/2EI$. Thus

$$y(x) = \frac{w_0}{EI}\left[\frac{1}{16}L^2x^2 - \frac{1}{12}Lx^3 + \frac{1}{24}x^4 - \frac{1}{24}\left(x - \frac{L}{2}\right)^4 \mathscr{U}\left(x - \frac{L}{2}\right)\right].$$

**78.** The differential equation is

$$EI\frac{d^4y}{dx^4} = w_0\left[\mathscr{U}\left(x - \frac{L}{3}\right) - \mathscr{U}\left(x - \frac{2L}{3}\right)\right].$$

Taking the Laplace transform of both sides and using $y(0) = y'(0) = 0$ we obtain

$$s^4\mathscr{L}\{y\} - sy''(0) - y'''(0) = \frac{w_0}{EI}\frac{1}{s}\left(e^{-Ls/3} - e^{-2Ls/3}\right).$$

Letting $y''(0) = c_1$ and $y'''(0) = c_2$ we have

$$\mathscr{L}\{y\} = \frac{c_1}{s^3} + \frac{c_2}{s^4} + \frac{w_0}{EI}\frac{1}{s^5}\left(e^{-Ls/3} - e^{-2Ls/3}\right)$$

so that

$$y(x) = \frac{1}{2}c_1x^2 + \frac{1}{6}c_2x^3 + \frac{1}{24}\frac{w_0}{EI}\left[\left(x - \frac{L}{3}\right)^4 \mathscr{U}\left(x - \frac{L}{3}\right) - \left(x - \frac{2L}{3}\right)^4 \mathscr{U}\left(x - \frac{2L}{3}\right)\right].$$

To find $c_1$ and $c_2$ we compute

$$y''(x) = c_1 + c_2x + \frac{1}{2}\frac{w_0}{EI}\left[\left(x - \frac{L}{3}\right)^2 \mathscr{U}\left(x - \frac{L}{3}\right) - \left(x - \frac{2L}{3}\right)^2 \mathscr{U}\left(x - \frac{2L}{3}\right)\right]$$

and

$$y'''(x) = c_2 + \frac{w_0}{EI}\left[\left(x - \frac{L}{3}\right) \mathscr{U}\left(x - \frac{L}{3}\right) - \left(x - \frac{2L}{3}\right) \mathscr{U}\left(x - \frac{2L}{3}\right)\right].$$

Then $y''(L) = y'''(L) = 0$ yields the system

$$c_1 + c_2L + \frac{1}{2}\frac{w_0}{EI}\left[\left(\frac{2L}{3}\right)^2 - \left(\frac{L}{3}\right)^2\right] = 0 \qquad\qquad c_1 + c_2L + \frac{1}{6}\frac{w_0L^2}{EI} = 0$$

$$c_2 + \frac{w_0}{EI}\left[\frac{2L}{3} - \frac{L}{3}\right] = 0. \qquad \text{or} \qquad c_2 + \frac{1}{3}\frac{w_0L}{EI} = 0.$$

Solving for $c_1$ and $c_2$ we obtain $c_1 = w_0L^2/6EI$ and $c_2 = -w_0L/3EI$. Thus

$$\begin{aligned} y(x) = \frac{w_0}{EI}\Bigg(&\frac{1}{12}L^2x^2 - \frac{1}{18}Lx^3 \\ &+\frac{1}{24}\left[\left(x-\frac{L}{3}\right)^4 \mathscr{U}\left(x-\frac{L}{3}\right) - \left(x-\frac{2L}{3}\right)^4 \mathscr{U}\left(x-\frac{2L}{3}\right)\right]\Bigg). \end{aligned}$$

**79.** The differential equation is

$$EI\,\frac{d^4y}{dx^4} = \frac{2w_0}{L}\left[\frac{L}{2} - x + \left(x-\frac{L}{2}\right)\mathscr{U}\left(x-\frac{L}{2}\right)\right].$$

Taking the Laplace transform of both sides and using $y(0) = y'(0) = 0$ we obtain

$$s^4\mathscr{L}\{y\} - sy''(0) - y'''(0) = \frac{2w_0}{EIL}\left[\frac{L}{2s} - \frac{1}{s^2} + \frac{1}{s^2}\,e^{-Ls/2}\right].$$

Letting $y''(0) = c_1$ and $y'''(0) = c_2$ we have

$$\mathscr{L}\{y\} = \frac{c_1}{s^3} + \frac{c_2}{s^4} + \frac{2w_0}{EIL}\left[\frac{L}{2s^5} - \frac{1}{s^6} + \frac{1}{s^6}\,e^{-Ls/2}\right]$$

so that

$$\begin{aligned} y(x) &= \frac{1}{2}c_1x^2 + \frac{1}{6}c_2x^3 + \frac{2w_0}{EIL}\left[\frac{L}{48}x^4 - \frac{1}{120}x^5 + \frac{1}{120}\left(x-\frac{L}{2}\right)^5\mathscr{U}\left(x-\frac{L}{2}\right)\right] \\ &= \frac{1}{2}c_1x^2 + \frac{1}{6}c_2x^3 + \frac{w_0}{60EIL}\left[\frac{5L}{2}x^4 - x^5 + \left(x-\frac{L}{2}\right)^5\mathscr{U}\left(x-\frac{L}{2}\right)\right]. \end{aligned}$$

To find $c_1$ and $c_2$ we compute

$$y''(x) = c_1 + c_2x + \frac{w_0}{60EIL}\left[30Lx^2 - 20x^3 + 20\left(x-\frac{L}{2}\right)^3\mathscr{U}\left(x-\frac{L}{2}\right)\right]$$

and

$$y'''(x) = c_2 + \frac{w_0}{60EIL}\left[60Lx - 60x^2 + 60\left(x-\frac{L}{2}\right)^2\mathscr{U}\left(x-\frac{L}{2}\right)\right].$$

Then $y''(L) = y'''(L) = 0$ yields the system

$$\begin{aligned} c_1 + c_2L + \frac{w_0}{60EIL}\left[30L^3 - 20L^3 + \frac{5}{2}L^3\right] &= 0 \\ c_2 + \frac{w_0}{60EIL}[60L^2 - 60L^2 + 15L^2] &= 0. \end{aligned} \qquad \text{or} \qquad \begin{aligned} c_1 + c_2L + \frac{5w_0L^2}{24EI} &= 0 \\ c_2 + \frac{w_0L}{4EI} &= 0. \end{aligned}$$

Solving for $c_1$ and $c_2$ we obtain $c_1 = w_0L^2/24EI$ and $c_2 = -w_0L/4EI$. Thus

$$y(x) = \frac{w_0L^2}{48EI}x^2 - \frac{w_0L}{24EI}x^3 + \frac{w_0}{60EIL}\left[\frac{5L}{2}x^4 - x^5 + \left(x - \frac{L}{2}\right)^5 \mathscr{U}\left(x - \frac{L}{2}\right)\right].$$

**80.** The differential equation is

$$EI\frac{d^4y}{dx^4} = w_0\left[1 - \mathscr{U}\left(x - \frac{L}{2}\right)\right].$$

Taking the Laplace transform of both sides and using $y(0) = y'(0) = 0$ we obtain

$$s^4\mathscr{L}\{y\} - sy''(0) - y'''(0) = \frac{w_0}{EI}\frac{1}{s}\left(1 - e^{-Ls/2}\right).$$

Letting $y''(0) = c_1$ and $y'''(0) = c_2$ we have

$$\mathscr{L}\{y\} = \frac{c_1}{s^3} + \frac{c_2}{s^4} + \frac{w_0}{EI}\frac{1}{s^5}\left(1 - e^{-Ls/2}\right)$$

so that

$$y(x) = \frac{1}{2}c_1x^2 + \frac{1}{6}c_2x^3 + \frac{1}{24}\frac{w_0}{EI}\left[x^4 - \left(x - \frac{L}{2}\right)^4 \mathscr{U}\left(x - \frac{L}{2}\right)\right].$$

To find $c_1$ and $c_2$ we compute

$$y''(x) = c_1 + c_2x + \frac{1}{2}\frac{w_0}{EI}\left[x^2 - \left(x - \frac{L}{2}\right)^2 \mathscr{U}\left(x - \frac{L}{2}\right)\right].$$

Then $y(L) = y''(L) = 0$ yields the system

$$\begin{aligned} \frac{1}{2}c_1L^2 + \frac{1}{6}c_2L^3 + \frac{1}{24}\frac{w_0}{EI}\left[L^4 - \left(\frac{L}{2}\right)^4\right] &= 0 \\ c_1 + c_2L + \frac{1}{2}\frac{w_0}{EI}\left[L^2 - \left(\frac{L}{2}\right)^2\right] &= 0. \end{aligned} \qquad \text{or} \qquad \begin{aligned} \frac{1}{2}c_1L^2 + \frac{1}{6}c_2L^3 + \frac{5w_0}{128EI}L^4 &= 0 \\ c_1 + c_2L + \frac{3w_0}{8EI}L^2 &= 0. \end{aligned}$$

Solving for $c_1$ and $c_2$ we obtain $c_1 = \frac{9}{128}w_0L^2/EI$ and $c_2 = -\frac{57}{128}w_0L/EI$. Thus

$$y(x) = \frac{w_0}{EI}\left[\frac{9}{256}L^2x^2 - \frac{19}{256}Lx^3 + \frac{1}{24}x^4 - \frac{1}{24}\left(x - \frac{L}{2}\right)^4 \mathscr{U}\left(x - \frac{L}{2}\right)\right].$$

**81.** **(a)** The temperature $T$ of the cake inside the oven is modeled by

$$\frac{dT}{dt} = k(T - T_m)$$

where $T_m$ is the ambient temperature of the oven. For $0 \le t \le 4$, we have

$$T_m = 70 + \frac{300 - 70}{4 - 0}t = 70 + 57.5t.$$

Hence for $t \ge 0$,

$$T_m = \begin{cases} 70 + 57.5t, & 0 \le t < 4 \\ 300, & t \ge 4. \end{cases}$$

In terms of the unit step function,

$$T_m = (70 + 57.5t)[1 - \mathscr{U}(t - 4)] + 300\mathscr{U}(t - 4) = 70 + 57.5t + (230 - 57.5t)\mathscr{U}(t - 4).$$

The initial-value problem is then

$$\frac{dT}{dt} = k[T - 70 - 57.5t - (230 - 57.5t)\mathscr{U}(t - 4)], \qquad T(0) = 70.$$

**(b)** Let $t(s) = \mathscr{L}\{T(t)\}$. Transforming the equation, using $230 - 57.5t = -57.5(t - 4)$ and Theorem 7.3.2, gives

$$st(s) - 70 = k\left(t(s) - \frac{70}{s} - \frac{57.5}{s^2} + \frac{57.5}{s^2}e^{-4s}\right)$$

or

$$t(s) = \frac{70}{s - k} - \frac{70k}{s(s - k)} - \frac{57.5k}{s^2(s - k)} + \frac{57.5k}{s^2(s - k)}e^{-4s}.$$

After using partial functions, the inverse transform is then

$$T(t) = 70 + 57.5\left(\frac{1}{k} + t - \frac{1}{k}e^{kt}\right) - 57.5\left(\frac{1}{k} + t - 4 - \frac{1}{k}e^{k(t-4)}\right)\mathscr{U}(t - 4).$$

Of course, the obvious question is: What is $k$? If the cake is supposed to bake for, say, 20 minutes, then $T(20) = 300$. That is,

$$300 = 70 + 57.5\left(\frac{1}{k} + 20 - \frac{1}{k}e^{20k}\right) - 57.5\left(\frac{1}{k} + 16 - \frac{1}{k}e^{16k}\right).$$

But this equation has no physically meaningful solution. This should be no surprise since the model predicts the asymptotic behavior $T(t) \to 300$ as $t$ increases. Using $T(20) = 299$ instead, we find, with the help of a CAS, that $k \approx -0.3$.

**82.** In order to apply Theorem 7.3.2 we need the function to have the form $f(t - a)\mathscr{U}(t - a)$. To accomplish this rewrite the functions given in the forms shown below.

**(a)** $2t + 1 = 2(t - 1 + 1) + 1 = 2(t - 1) + 3$ **(b)** $e^t = e^{t-5+5} = e^5e^{t-5}$

**(c)** $\cos t = -\cos(t - \pi)$ **(d)** $t^2 - 3t = (t - 2)^2 + (t - 2) - 2$

**83.** **(a)** From Theorem 7.3.1 we have $\mathscr{L}\{te^{kti}\} = 1/(s-ki)^2$. Then, using Euler's formula,

$$\begin{aligned}\mathscr{L}\{te^{kti}\} &= \mathscr{L}\{t\cos kt + it\sin kt\} = \mathscr{L}\{t\cos kt\} + i\mathscr{L}\{t\sin kt\}\\ &= \frac{1}{(s-ki)^2} = \frac{(s+ki)^2}{(s^2+k^2)^2} = \frac{s^2-k^2}{(s^2+k^2)^2} + i\,\frac{2ks}{(s^2+k^2)^2}\,.\end{aligned}$$

Equating real and imaginary parts we have

$$\mathscr{L}\{t\cos kt\} = \frac{s^2-k^2}{(s^2+k^2)^2} \quad\text{and}\quad \mathscr{L}\{t\sin kt\} = \frac{2ks}{(s^2+k^2)^2}\,.$$

**(b)** The Laplace transform of the differential equation is

$$s^2\mathscr{L}\{x\} + \omega^2\mathscr{L}\{x\} = \frac{s}{s^2+\omega^2}\,.$$

Solving for $\mathscr{L}\{x\}$ we obtain $\mathscr{L}\{x\} = s/(s^2+\omega^2)^2$. Thus $x = (1/2\omega)t\sin\omega t$.

## 7.4 Operational Properties II

**1.** $\mathscr{L}\{te^{-10t}\} = -\dfrac{d}{ds}\left(\dfrac{1}{s+10}\right) = \dfrac{1}{(s+10)^2}$

**2.** $\mathscr{L}\{t^3e^t\} = (-1)^3\dfrac{d^3}{ds^3}\left(\dfrac{1}{s-1}\right) = \dfrac{6}{(s-1)^4}$

**3.** $\mathscr{L}\{t\cos 2t\} = -\dfrac{d}{ds}\left(\dfrac{s}{s^2+4}\right) = \dfrac{s^2-4}{(s^2+4)^2}$

**4.** $\mathscr{L}\{t\sinh 3t\} = -\dfrac{d}{ds}\left(\dfrac{3}{s^2-9}\right) = \dfrac{6s}{(s^2-9)^2}$

**5.** $\mathscr{L}\{t^2\sinh t\} = \dfrac{d^2}{ds^2}\left(\dfrac{1}{s^2-1}\right) = \dfrac{6s^2+2}{(s^2-1)^3}$

**6.** $\mathscr{L}\{t^2\cos t\} = \dfrac{d^2}{ds^2}\left(\dfrac{s}{s^2+1}\right) = \dfrac{d}{ds}\left(\dfrac{1-s^2}{(s^2+1)^2}\right) = \dfrac{2s\left(s^2-3\right)}{(s^2+1)^3}$

**7.** $\mathscr{L}\left\{te^{2t}\sin 6t\right\} = -\dfrac{d}{ds}\left(\dfrac{6}{(s-2)^2+36}\right) = \dfrac{12(s-2)}{[(s-2)^2+36]^2}$

**8.** $\mathscr{L}\left\{te^{-3t}\cos 3t\right\} = -\dfrac{d}{ds}\left(\dfrac{s+3}{(s+3)^2+9}\right) = \dfrac{(s+3)^2-9}{[(s+3)^2+9]^2}$

**9.** The Laplace transform of the differential equation is

$$s\mathscr{L}\{y\} + \mathscr{L}\{y\} = \frac{2s}{(s^2+1)^2}\,.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\} = \frac{2s}{(s+1)(s^2+1)^2} = -\frac{1}{2}\,\frac{1}{s+1} - \frac{1}{2}\,\frac{1}{s^2+1} + \frac{1}{2}\,\frac{s}{s^2+1} + \frac{1}{(s^2+1)^2} + \frac{s}{(s^2+1)^2}\,.$$

Thus

$$\begin{aligned} y(t) &= -\frac{1}{2}e^{-t} - \frac{1}{2}\sin t + \frac{1}{2}\cos t + \frac{1}{2}(\sin t - t\cos t) + \frac{1}{2}t\sin t \\ &= -\frac{1}{2}e^{-t} + \frac{1}{2}\cos t - \frac{1}{2}t\cos t + \frac{1}{2}t\sin t. \end{aligned}$$

**10.** The Laplace transform of the differential equation is

$$s\mathscr{L}\{y\} - \mathscr{L}\{y\} = \frac{2(s-1)}{((s-1)^2+1)^2}.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\} = \frac{2}{((s-1)^2+1)^2}.$$

Thus

$$y = e^t\sin t - te^t\cos t.$$

**11.** The Laplace transform of the differential equation is

$$s^2\mathscr{L}\{y\} - sy(0) - y'(0) + 9\mathscr{L}\{y\} = \frac{s}{s^2+9}.$$

Letting $y(0) = 2$ and $y'(0) = 5$ and solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\} = \frac{2s^3+5s^2+19s-45}{(s^2+9)^2} = \frac{2s}{s^2+9} + \frac{5}{s^2+9} + \frac{s}{(s^2+9)^2}.$$

Thus

$$y = 2\cos 3t + \frac{5}{3}\sin 3t + \frac{1}{6}t\sin 3t.$$

**12.** The Laplace transform of the differential equation is

$$s^2\mathscr{L}\{y\} - sy(0) - y'(0) + \mathscr{L}\{y\} = \frac{1}{s^2+1}.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\} = \frac{s^3-s^2+s}{(s^2+1)^2} = \frac{s}{s^2+1} - \frac{1}{s^2+1} + \frac{1}{(s^2+1)^2}.$$

Thus

$$y = \cos t - \sin t + \left(\frac{1}{2}\sin t - \frac{1}{2}t\cos t\right) = \cos t - \frac{1}{2}\sin t - \frac{1}{2}t\cos t.$$

**13.** The Laplace transform of the differential equation is

$$s^2\mathscr{L}\{y\} - sy(0) - y'(0) + 16\mathscr{L}\{y\} = \mathscr{L}\{\cos 4t - \cos 4t\,\mathscr{U}(t-\pi)\}$$

or by (16) of Section 7.3 in the text,

$$\begin{aligned}(s^2+16)\mathscr{L}\{y\} &= 1+\frac{s}{s^2+16}-e^{-\pi s}\mathscr{L}\left\{\cos 4(t+\pi)\right\}\\ &= 1+\frac{s}{s^2+16}-e^{-\pi s}\mathscr{L}\{\cos 4t\} = 1+\frac{s}{s^2+16}-\frac{s}{s^2+16}e^{-\pi s}\,.\end{aligned}$$

Thus

$$\mathscr{L}\{y\} = \frac{1}{s^2+16}+\frac{s}{(s^2+16)^2}-\frac{s}{(s^2+16)^2}e^{-\pi s}$$

and

$$y = \frac{1}{4}\sin 4t+\frac{1}{8}t\sin 4t-\frac{1}{8}(t-\pi)\sin 4(t-\pi)\mathscr{U}(t-\pi).$$

**14.** The Laplace transform of the differential equation is

$$s^2\mathscr{L}\{y\}-sy(0)-y'(0)+\mathscr{L}\{y\} = \mathscr{L}\left\{1-\mathscr{U}\left(t-\frac{\pi}{2}\right)+\sin t\,\mathscr{U}\left(t-\frac{\pi}{2}\right)\right\}$$

or

$$\begin{aligned}(s^2+1)\mathscr{L}\{y\} &= s+\frac{1}{s}-\frac{1}{s}\,e^{-\pi s/2}+e^{-\pi s/2}\mathscr{L}\left\{\sin\left(t+\frac{\pi}{2}\right)\right\}\\ &= s+\frac{1}{s}-\frac{1}{s}\,e^{-\pi s/2}+e^{-\pi s/2}\mathscr{L}\{\cos t\}\\ &= s+\frac{1}{s}-\frac{1}{s}\,e^{-\pi s/2}+\frac{s}{s^2+1}\,e^{-\pi s/2}.\end{aligned}$$

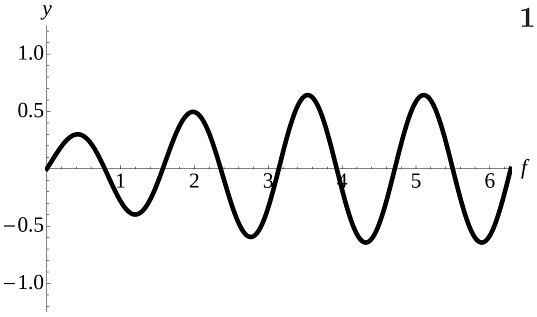
Thus

$$\begin{aligned}\mathscr{L}\{y\} &= \frac{s}{s^2+1}+\frac{1}{s(s^2+1)}-\frac{1}{s(s^2+1)}e^{-\pi s/2}+\frac{s}{(s^2+1)^2}\,e^{-\pi s/2}\\ &= \frac{s}{s^2+1}+\frac{1}{s}-\frac{s}{s^2+1}-\left(\frac{1}{s}-\frac{s}{s^2+1}\right)e^{-\pi s/2}+\frac{s}{(s^2+1)^2}\,e^{-\pi s/2}\\ &= \frac{1}{s}-\left(\frac{1}{s}-\frac{s}{s^2+1}\right)e^{-\pi s/2}+\frac{s}{(s^2+1)^2}\,e^{-\pi s/2}\end{aligned}$$

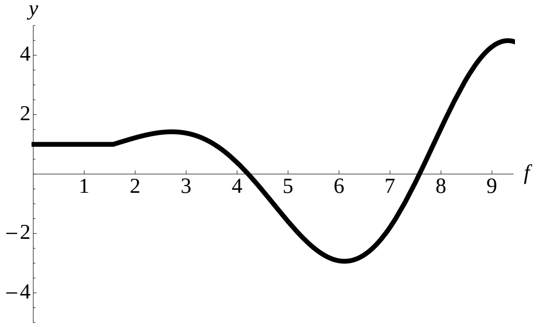
and

$$\begin{aligned}y &= 1-\left[1-\cos\left(t-\frac{\pi}{2}\right)\right]\mathscr{U}\left(t-\frac{\pi}{2}\right)+\frac{1}{2}\left(t-\frac{\pi}{2}\right)\sin\left(t-\frac{\pi}{2}\right)\mathscr{U}\left(t-\frac{\pi}{2}\right)\\ &= 1-(1-\sin t)\mathscr{U}\left(t-\frac{\pi}{2}\right)-\frac{1}{2}\left(t-\frac{\pi}{2}\right)\cos t\,\mathscr{U}\left(t-\frac{\pi}{2}\right).\end{aligned}$$

**15.**



**16.**



**17.** From (7) of Section 7.2 in the text along with Theorem 7.4.1,

$$\mathscr{L}\{ty''\} = -\frac{d}{ds}\mathscr{L}\{y''\} = -\frac{d}{ds}[s^2Y(s) - sy(0) - y'(0)] = -s^2\frac{dY}{ds} - 2sY + y(0),$$

so that the transform of the given second-order differential equation is the linear first-order differential equation in $Y(s)$:

$$s^2Y' + 3sY = -\frac{4}{s^3} \qquad \text{or} \qquad Y' + \frac{3}{s}Y = -\frac{4}{s^5}.$$

The solution of the latter equation is $Y(s) = 4/s^4 + c/s^3$, so

$$y(t) = \mathscr{L}^{-1}\{Y(s)\} = \frac{2}{3}t^3 + \frac{c}{2}t^2.$$

**18.** From Theorem 7.4.1 in the text

$$\mathscr{L}\{ty'\} = -\frac{d}{ds}\mathscr{L}\{y'\} = -\frac{d}{ds}[sY(s) - y(0)] = -s\frac{dY}{ds} - Y$$

so that the transform of the given second-order differential equation is the linear first-order differential equation in $Y(s)$:

$$Y' + \left(\frac{3}{s} - 2s\right)Y = -\frac{10}{s}.$$

Using the integrating factor $s^3e^{-s^2}$, the last equation yields

$$Y(s) = \frac{5}{s^3} + \frac{c}{s^3}e^{s^2}.$$

But if $Y(s)$ is the Laplace transform of a piecewise-continuous function of exponential order, we must have, in view of Theorem 7.2.3, $\lim_{s\to\infty} Y(s) = 0$. In order to obtain this condition we require $c = 0$. Hence

$$y(t) = \mathscr{L}^{-1}\left\{\frac{5}{s^3}\right\} = \frac{5}{2}t^2.$$

**19.** Identify $f(\tau) = 4\tau$ and $g(t-\tau) = 3(t-\tau)^2$. Therefore,

$$f * g = \int_0^t (4\tau)\,3(t-\tau)^2\,d\tau = 12\int_0^t \left(t^2\tau - 2t\tau^2 + \tau^3\right)d\tau = 12\left(t^2\cdot\frac{1}{2}t^2 - 2t\cdot\frac{1}{3}t^3 + \frac{1}{4}t^4\right) = t^4$$

Then

$$\mathscr{L}\{f*g\} = \mathscr{L}\left\{t^4\right\} = \frac{4!}{s^5} = \frac{24}{s^5}$$

**20.** Identify $f(\tau) = \tau$ and $g(t-\tau) = e^{-t+\tau}$. Therefore,

$$f * g = \int_0^t \tau e^{-t+\tau}\,d\tau = e^{-t}\int_0^t \tau e^{\tau}\,d\tau = e^{-t}\left(te^t - e^t + 1\right) = t - 1 + e^{-t}$$

Then

$$\mathscr{L}\{f*g\} = \mathscr{L}\left\{t - 1 + e^{-t}\right\} = \frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1} = \frac{1}{s^2(s+1)}$$

**21.** Identify $f(\tau) = e^{-\tau}$ and $g(t-\tau) = e^{t-\tau}$. Therefore,

$$f * g = \int_0^t e^{-\tau} e^{t-\tau}\, d\tau = e^t \int_0^t e^{-2\tau}\, d\tau = e^t \cdot \left( -\frac{1}{2} e^{-2t} + \frac{1}{2} \right) = -\frac{1}{2} e^{-t} + \frac{1}{2} e^t$$

Then

$$\mathscr{L}\{f * g\} = \mathscr{L}\left\{ -\frac{1}{2} e^{-t} + \frac{1}{2} e^t \right\} = -\frac{1}{2}\,\frac{1}{s+1} + \frac{1}{2}\,\frac{1}{s-1} = \frac{1}{s^2 - 1}$$

**22.** Identify $f(\tau) = \cos 2\tau$ and $g(t-\tau) = e^{t-\tau}$. Therefore,

$$\begin{aligned} f * g &= \int_0^t (\cos 2\tau)\, e^{t-\tau}\, d\tau = e^t \int_0^t e^{-\tau} \cos 2\tau\, d\tau = e^t \cdot \left( -\frac{1}{5} e^{-t} \cos 2t + \frac{1}{5} + \frac{2}{5} e^{-t} \sin 2t \right) \\ &= -\frac{1}{5} \cos 2t + \frac{2}{5} \sin 2t + \frac{1}{5} e^t \end{aligned}$$

Then

$$\begin{aligned} \mathscr{L}\{f * g\} &= \mathscr{L}\left\{ -\frac{1}{5} \cos 2t + \frac{2}{5} \sin 2t + \frac{1}{5} e^t \right\} \\ &= -\frac{1}{5}\,\frac{s}{s^2+4} + \frac{2}{5}\,\frac{2}{s^2+4} + \frac{1}{5}\,\frac{1}{s-1} = \frac{s}{(s-1)\left(s^2+4\right)} \end{aligned}$$

**23.** $\mathscr{L}\left\{1 * t^3\right\} = \dfrac{1}{s}\,\dfrac{3!s^4}{=}\,\dfrac{6}{s^5}$

**24.** $\mathscr{L}\left\{t^2 * te^t\right\} = \dfrac{2}{s^3(s-1)^2}$

**25.** $\mathscr{L}\left\{e^{-t} * e^t \cos t\right\} = \dfrac{s-1}{(s+1)\left[(s-1)^2+1\right]}$

**26.** $\mathscr{L}\left\{e^{2t} * \sin t\right\} = \dfrac{1}{(s-2)(s^2+1)}$

**27.** $\mathscr{L}\left\{ \displaystyle\int_0^t e^{\tau}\, d\tau \right\} = \dfrac{1}{s} \mathscr{L}\{e^t\} = \dfrac{1}{s(s-1)}$

**28.** $\mathscr{L}\left\{ \displaystyle\int_0^t \cos\tau\, d\tau \right\} = \dfrac{1}{s} \mathscr{L}\{\cos t\} = \dfrac{s}{s(s^2+1)} = \dfrac{1}{s^2+1}$

**29.** $\mathscr{L}\left\{ \displaystyle\int_0^t e^{-\tau} \cos\tau\, d\tau \right\} = \dfrac{1}{s} \mathscr{L}\left\{ e^{-t} \cos t \right\} = \dfrac{1}{s}\,\dfrac{s+1}{(s+1)^2+1} = \dfrac{s+1}{s\left(s^2+2s+2\right)}$

**30.** $\mathscr{L}\left\{ \displaystyle\int_0^t \tau \sin\tau\, d\tau \right\} = \dfrac{1}{s} \mathscr{L}\{t \sin t\} = \dfrac{1}{s} \left( -\dfrac{d}{ds}\,\dfrac{1}{s^2+1} \right) = -\dfrac{1}{s}\,\dfrac{-2s}{\left(s^2+1\right)^2} = \dfrac{2}{\left(s^2+1\right)^2}$

**31.** $\mathscr{L}\left\{ \displaystyle\int_0^t \tau e^{t-\tau}\, d\tau \right\} = \mathscr{L}\{t\} \mathscr{L}\{e^t\} = \dfrac{1}{s^2(s-1)}$

**32.** $\mathscr{L}\left\{ \displaystyle\int_0^t \sin\tau \cos(t-\tau)\, d\tau \right\} = \mathscr{L}\{\sin t\} \mathscr{L}\{\cos t\} = \dfrac{s}{\left(s^2+1\right)^2}$

**33.** $\mathscr{L}\left\{ t \displaystyle\int_0^t \sin\tau\, d\tau \right\} = -\dfrac{d}{ds} \mathscr{L}\left\{ \displaystyle\int_0^t \sin\tau\, d\tau \right\} = -\dfrac{d}{ds} \left( \dfrac{1}{s}\,\dfrac{1}{s^2+1} \right) = \dfrac{3s^2+1}{s^2\left(s^2+1\right)^2}$

**34.** $\mathscr{L}\left\{t\int_0^t \tau e^{-\tau}\,d\tau\right\} = -\frac{d}{ds}\mathscr{L}\left\{\int_0^t \tau e^{-\tau}\,d\tau\right\} = -\frac{d}{ds}\left(\frac{1}{s}\,\frac{1}{(s+1)^2}\right) = \frac{3s+1}{s^2(s+1)^3}$

**35.** $\mathscr{L}^{-1}\left\{\frac{1}{s(s-1)}\right\} = \mathscr{L}^{-1}\left\{\frac{1/(s-1)}{s}\right\} = \int_0^t e^\tau\,d\tau = e^t - 1$

**36.** $\mathscr{L}^{-1}\left\{\frac{1}{s^2(s-1)}\right\} = \mathscr{L}^{-1}\left\{\frac{1/s(s-1)}{s}\right\} = \int_0^t (e^\tau - 1)\,d\tau = e^t - t - 1$

**37.** $\mathscr{L}^{-1}\left\{\frac{1}{s^3(s-1)}\right\} = \mathscr{L}^{-1}\left\{\frac{1/s^2(s-1)}{s}\right\} = \int_0^t (e^\tau - \tau - 1)\,d\tau = e^t - \frac{1}{2}t^2 - t - 1$

**38.** Using $\mathscr{L}^{-1}\left\{\frac{1}{(s-a)^2}\right\} = te^{at}$, (8) in the text gives

$$\mathscr{L}^{-1}\left\{\frac{1}{s(s-a)^2}\right\} = \int_0^t \tau e^{a\tau}\,d\tau = \frac{1}{a^2}(ate^{at} - e^{at} + 1).$$

**39.** **(a)** The result in (4) in the text is $\mathscr{L}^{-1}\{F(s)G(s)\} = f * g$, so identify

$$F(s) = \frac{2k^3}{(s^2+k^2)^2} \qquad \text{and} \qquad G(s) = \frac{4s}{s^2+k^2}\,.$$

Then

$$f(t) = \sin kt - kt\cos kt \qquad \text{and} \qquad g(t) = 4\cos kt$$

so

$$\mathscr{L}^{-1}\left\{\frac{8k^3s}{(s^2+k^2)^3}\right\} = \mathscr{L}^{-1}\{F(s)G(s)\} = f * g = 4\int_0^t f(\tau)g(t-\tau)\,dt$$

$$= 4\int_0^t (\sin k\tau - k\tau\cos k\tau)\cos k(t-\tau)\,d\tau.$$

Using a CAS to evaluate the integral we get

$$\mathscr{L}^{-1}\left\{\frac{8k^3s}{(s^2+k^2)^3}\right\} = t\sin kt - kt^2\cos kt.$$

**(b)** Observe from part (a) that

$$\mathscr{L}\left\{t(\sin kt - kt\cos kt)\right\} = \frac{8k^3s}{(s^2+k^2)^3}\,,$$

and from Theorem 7.4.1 that $\mathscr{L}\left\{tf(t)\right\} = -F'(s)$. We saw in (5) in the text that

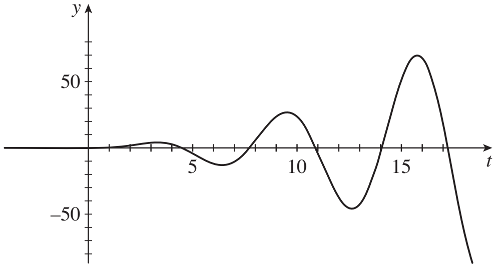
$$\mathscr{L}\{\sin kt - kt\cos kt\} = 2k^3/(s^2+k^2)^2,$$

so

$$\mathscr{L}\left\{t(\sin kt - kt\cos kt)\right\} = -\frac{d}{ds}\,\frac{2k^3}{(s^2+k^2)^2} = \frac{8k^3s}{(s^2+k^2)^3}\,.$$

**40.** The Laplace transform of the differential equation is

$$s^2\mathscr{L}\{y\} + \mathscr{L}\{y\} = \frac{1}{(s^2+1)} + \frac{2s}{(s^2+1)^2}\,.$$



Thus

$$\mathscr{L}\{y\} = \frac{1}{(s^2+1)^2} + \frac{2s}{(s^2+1)^3}$$

and, using Problem 39 with $k = 1$,

$$y = \frac{1}{2}(\sin t - t\cos t) + \frac{1}{4}(t\sin t - t^2\cos t).$$

**41.** The Laplace transform of the given equation is

$$\mathscr{L}\{f\} + \mathscr{L}\{t\}\mathscr{L}\{f\} = \mathscr{L}\{t\}.$$

Solving for $\mathscr{L}\{f\}$ we obtain $\mathscr{L}\{f\} = \dfrac{1}{s^2+1}$. Thus, $f(t) = \sin t$.

**42.** The Laplace transform of the given equation is

$$\mathscr{L}\{f\} = \mathscr{L}\{2t\} - 4\mathscr{L}\{\sin t\}\mathscr{L}\{f\}.$$

Solving for $\mathscr{L}\{f\}$ we obtain

$$\mathscr{L}\{f\} = \frac{2s^2+2}{s^2(s^2+5)} = \frac{2}{5}\frac{1}{s^2} + \frac{8}{5\sqrt{5}}\frac{\sqrt{5}}{s^2+5}\,.$$

Thus

$$f(t) = \frac{2}{5}t + \frac{8}{5\sqrt{5}}\sin\sqrt{5}\,t.$$

**43.** The Laplace transform of the given equation is

$$\mathscr{L}\{f\} = \mathscr{L}\left\{te^t\right\} + \mathscr{L}\{t\}\mathscr{L}\{f\}.$$

Solving for $\mathscr{L}\{f\}$ we obtain

$$\mathscr{L}\{f\} = \frac{s^2}{(s-1)^3(s+1)} = \frac{1}{8}\frac{1}{s-1} + \frac{3}{4}\frac{1}{(s-1)^2} + \frac{1}{4}\frac{2}{(s-1)^3} - \frac{1}{8}\frac{1}{s+1}\,.$$

Thus

$$f(t) = \frac{1}{8}e^t + \frac{3}{4}te^t + \frac{1}{4}t^2e^t - \frac{1}{8}e^{-t}$$

**44.** The Laplace transform of the given equation is

$$\mathscr{L}\{f\} + 2\mathscr{L}\{\cos t\}\mathscr{L}\{f\} = 4\mathscr{L}\left\{e^{-t}\right\} + \mathscr{L}\{\sin t\}.$$

Solving for $\mathscr{L}\{f\}$ we obtain

$$\mathscr{L}\{f\} = \frac{4s^2+s+5}{(s+1)^3} = \frac{4}{s+1} - \frac{7}{(s+1)^2} + 4\frac{2}{(s+1)^3}.$$

Thus

$$f(t) = 4e^{-t} - 7te^{-t} + 4t^2e^{-t}.$$

**45.** The Laplace transform of the given equation is

$$\mathscr{L}\{f\} + \mathscr{L}\{1\}\mathscr{L}\{f\} = \mathscr{L}\{1\}.$$

Solving for $\mathscr{L}\{f\}$ we obtain $\mathscr{L}\{f\} = \dfrac{1}{s+1}$. Thus, $f(t) = e^{-t}$.

**46.** The Laplace transform of the given equation is

$$\mathscr{L}\{f\} = \mathscr{L}\{\cos t\} + \mathscr{L}\left\{e^{-t}\right\}\mathscr{L}\{f\}.$$

Solving for $\mathscr{L}\{f\}$ we obtain

$$\mathscr{L}\{f\} = \frac{s}{s^2+1} + \frac{1}{s^2+1}.$$

Thus

$$f(t) = \cos t + \sin t.$$

**47.** The Laplace transform of the given equation is

$$\begin{aligned}\mathscr{L}\{f\} &= \mathscr{L}\{1\} + \mathscr{L}\{t\} - \mathscr{L}\left\{\frac{8}{3}\int_0^t (t-\tau)^3 f(\tau)\,d\tau\right\}\\ &= \frac{1}{s} + \frac{1}{s^2} + \frac{8}{3}\mathscr{L}\{t^3\}\mathscr{L}\{f\} = \frac{1}{s} + \frac{1}{s^2} + \frac{16}{s^4}\mathscr{L}\{f\}.\end{aligned}$$

Solving for $\mathscr{L}\{f\}$ we obtain

$$\mathscr{L}\{f\} = \frac{s^2(s+1)}{s^4-16} = \frac{1}{8}\frac{1}{s+2} + \frac{3}{8}\frac{1}{s-2} + \frac{1}{4}\frac{2}{s^2+4} + \frac{1}{2}\frac{s}{s^2+4}.$$

Thus

$$f(t) = \frac{1}{8}e^{-2t} + \frac{3}{8}e^{2t} + \frac{1}{4}\sin 2t + \frac{1}{2}\cos 2t.$$

**48.** The Laplace transform of the given equation is

$$\mathscr{L}\{t\} - 2\mathscr{L}\{f\} = \mathscr{L}\left\{e^t - e^{-t}\right\}\mathscr{L}\{f\}.$$

Solving for $\mathscr{L}\{f\}$ we obtain

$$\mathscr{L}\{f\} = \frac{s^2-1}{2s^4} = \frac{1}{2}\frac{1}{s^2} - \frac{1}{12}\frac{3!}{s^4}.$$

Thus

$$f(t) = \frac{1}{2}t - \frac{1}{12}t^3.$$

**49.** The Laplace transform of the given equation is

$$s\mathscr{L}\{y\} - y(0) = \mathscr{L}\{1\} - \mathscr{L}\{\sin t\} - \mathscr{L}\{1\}\mathscr{L}\{y\}.$$

Solving for $\mathscr{L}\{y\}$ we obtain

$$\mathscr{L}\{y\} = \frac{s^2 - s + 1}{(s^2+1)^2} = \frac{1}{s^2+1} - \frac{1}{2}\,\frac{2s}{(s^2+1)^2}\,.$$

Thus

$$y = \sin t - \frac{1}{2}\,t\sin t.$$

**50.** The Laplace transform of the given equation is

$$s\mathscr{L}\{y\} - y(0) + 6\mathscr{L}\{y\} + 9\mathscr{L}\{1\}\mathscr{L}\{y\} = \mathscr{L}\{1\}.$$

Solving for $\mathscr{L}\{y\}$ we obtain $\mathscr{L}\{y\} = \dfrac{1}{(s+3)^2}$. Thus, $y = te^{-3t}$.

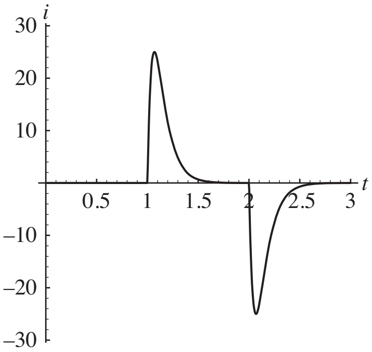
**51.** The differential equation is

$$0.1\,\frac{di}{dt} + 3i + \frac{1}{0.05}\int_0^t i(\tau)\,d\tau = 100\,[\mathscr{U}(t-1) - \mathscr{U}(t-2)]$$

or

$$\frac{di}{dt} + 30i + 200\int_0^t i(\tau)\,d\tau = 1000\,[\mathscr{U}(t-1) - \mathscr{U}(t-2)]\,,$$

where $i(0) = 0$. The Laplace transform of the differential equation is



$$s\mathscr{L}\{i\} - y(0) + 30\mathscr{L}\{i\} + \frac{200}{s}\mathscr{L}\{i\} = \frac{1000}{s}(e^{-s} - e^{-2s}).$$

Solving for $\mathscr{L}\{i\}$ we obtain

$$\mathscr{L}\{i\} = \frac{1000e^{-s} - 1000e^{-2s}}{s^2 + 30s + 200} = \left(\frac{100}{s+10} - \frac{100}{s+20}\right)(e^{-s} - e^{-2s}).$$

Thus

$$i(t) = 100\left(e^{-10(t-1)} - e^{-20(t-1)}\right)\mathscr{U}(t-1) - 100\left(e^{-10(t-2)} - e^{-20(t-2)}\right)\mathscr{U}(t-2).$$

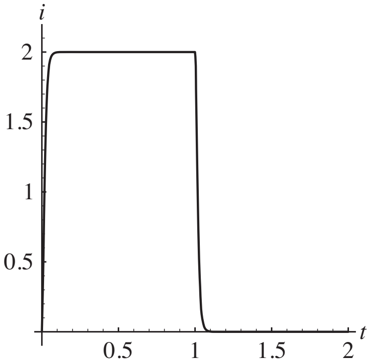
**52.** The differential equation is

$$0.005\,\frac{di}{dt} + i + \frac{1}{0.02}\int_0^t i(\tau)\,d\tau = 100\left[t - (t-1)\mathscr{U}(t-1)\right]$$

or

$$\frac{di}{dt} + 200i + 10{,}000\int_0^t i(\tau)\,d\tau = 20{,}000\left[t - (t-1)\mathscr{U}(t-1)\right],$$

where $i(0) = 0$. The Laplace transform of the differential equation is



$$s\mathscr{L}\{i\} + 200\mathscr{L}\{i\} + \frac{10{,}000}{s}\mathscr{L}\{i\} = 20{,}000\left(\frac{1}{s^2} - \frac{1}{s^2}e^{-s}\right).$$

Solving for $\mathscr{L}\{i\}$ we obtain

$$\mathscr{L}\{i\} = \frac{20{,}000}{s(s+100)^2}(1-e^{-s}) = \left[\frac{2}{s} - \frac{2}{s+100} - \frac{200}{(s+100)^2}\right](1-e^{-s}).$$

Thus

$$i(t) = 2 - 2e^{-100t} - 200te^{-100t} - 2\mathscr{U}(t-1) + 2e^{-100(t-1)}\mathscr{U}(t-1) + 200(t-1)e^{-100(t-1)}\mathscr{U}(t-1).$$

**53.** $\mathscr{L}\{f(t)\} = \dfrac{1}{1-e^{-2as}}\left[\displaystyle\int_0^a e^{-st}\,dt - \int_a^{2a} e^{-st}\,dt\right] = \dfrac{(1-e^{-as})^2}{s(1-e^{-2as})} = \dfrac{1-e^{-as}}{s(1+e^{-as})}$

**54.** $\mathscr{L}\{f(t)\} = \dfrac{1}{1-e^{-2as}}\displaystyle\int_0^a e^{-st}\,dt = \dfrac{1}{s(1+e^{-as})}$

**55.** Using integration by parts,

$$\mathscr{L}\{f(t)\} = \frac{1}{1-e^{-bs}}\int_0^b \frac{a}{b}\,te^{-st}\,dt = \frac{a}{s}\left(\frac{1}{bs} - \frac{1}{e^{bs}-1}\right).$$

**56.** $\mathscr{L}\{f(t)\} = \dfrac{1}{1-e^{-2s}}\left[\displaystyle\int_0^1 te^{-st}\,dt + \int_1^2 (2-t)e^{-st}\,dt\right] = \dfrac{1-e^{-s}}{s^2(1+e^{-s})}$

**57.** $\mathscr{L}\{f(t)\} = \dfrac{1}{1-e^{-\pi s}}\displaystyle\int_0^\pi e^{-st}\sin t\,dt = \dfrac{1}{s^2+1}\cdot\dfrac{e^{\pi s/2}+e^{-\pi s/2}}{e^{\pi s/2}-e^{-\pi s/2}} = \dfrac{1}{s^2+1}\coth\dfrac{\pi s}{2}$

**58.** $\mathscr{L}\{f(t)\} = \dfrac{1}{1-e^{-2\pi s}}\displaystyle\int_0^\pi e^{-st}\sin t\,dt = \dfrac{1}{s^2+1}\cdot\dfrac{1}{1-e^{-\pi s}}$

**59.** The differential equation is $L\,di/dt + Ri = E(t)$, where $i(0) = 0$. The Laplace transform of the equation is

$$Ls\mathscr{L}\{i\} + R\mathscr{L}\{i\} = \mathscr{L}\{E(t)\}.$$

From Problem 53 we have $\mathscr{L}\{E(t)\} = (1-e^{-s})/s(1+e^{-s})$. Thus

$$(Ls+R)\mathscr{L}\{i\} = \frac{1-e^{-s}}{s(1+e^{-s})}$$

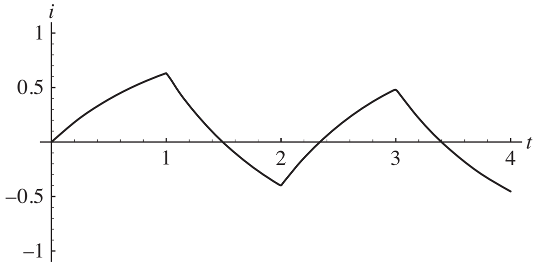
and

$$\begin{aligned}\mathscr{L}\{i\} &= \frac{1}{L}\,\frac{1-e^{-s}}{s(s+R/L)(1+e^{-s})} = \frac{1}{L}\,\frac{1-e^{-s}}{s(s+R/L)}\,\frac{1}{1+e^{-s}}\\ &= \frac{1}{R}\left(\frac{1}{s}-\frac{1}{s+R/L}\right)(1-e^{-s})(1-e^{-s}+e^{-2s}-e^{-3s}+e^{-4s}-\cdots)\\ &= \frac{1}{R}\left(\frac{1}{s}-\frac{1}{s+R/L}\right)(1-2e^{-s}+2e^{-2s}-2e^{-3s}+2e^{-4s}-\cdots).\end{aligned}$$

Therefore,

$$\begin{aligned}i(t) &= \frac{1}{R}\left(1-e^{-Rt/L}\right)-\frac{2}{R}\left(1-e^{-R(t-1)/L}\right)\mathscr{U}(t-1)\\ &\quad +\frac{2}{R}\left(1-e^{-R(t-2)/L}\right)\mathscr{U}(t-2)-\frac{2}{R}\left(1-e^{-R(t-3)/L}\right)\mathscr{U}(t-3)+\cdots\\ &= \frac{1}{R}\left(1-e^{-Rt/L}\right)+\frac{2}{R}\sum_{n=1}^{\infty}\left(1-e^{-R(t-n)/L}\right)\mathscr{U}(t-n).\end{aligned}$$

The graph of $i(t)$ with $L=1$ and $R=1$ is shown below.



**60.** The differential equation is $L\,di/dt+Ri=E(t)$, where $i(0)=0$. The Laplace transform of the equation is

$$Ls\mathscr{L}\{i\}+R\mathscr{L}\{i\}=\mathscr{L}\{E(t)\}.$$

From Problem 55 we have

$$\mathscr{L}\{E(t)\}=\frac{1}{s}\left(\frac{1}{s}-\frac{1}{e^{s}-1}\right)=\frac{1}{s^2}-\frac{1}{s}\,\frac{1}{e^{s}-1}\,.$$

Thus

$$(Ls+R)\mathscr{L}\{i\}=\frac{1}{s^2}-\frac{1}{s}\,\frac{1}{e^{s}-1}$$

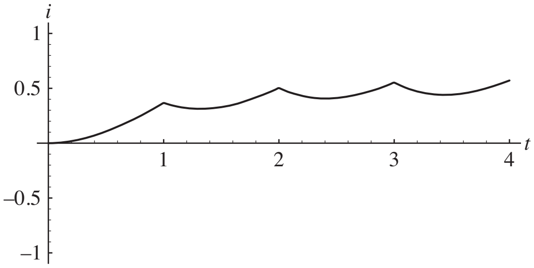
and

$$\begin{aligned}\mathscr{L}\{i\} &= \frac{1}{L}\,\frac{1}{s^2(s+R/L)}-\frac{1}{L}\,\frac{1}{s(s+R/L)}\,\frac{1}{e^{s}-1}\\ &= \frac{1}{R}\left(\frac{1}{s^2}-\frac{L}{R}\,\frac{1}{s}+\frac{L}{R}\,\frac{1}{s+R/L}\right)-\frac{1}{R}\left(\frac{1}{s}-\frac{1}{s+R/L}\right)\left(e^{-s}+e^{-2s}+e^{-3s}+\cdots\right).\end{aligned}$$

Therefore

$$\begin{aligned} i(t) &= \frac{1}{R}\left(t - \frac{L}{R} + \frac{L}{R}e^{-Rt/L}\right) - \frac{1}{R}\left(1 - e^{-R(t-1)/L}\right)\mathscr{U}(t-1) \\ &\quad - \frac{1}{R}\left(1 - e^{-R(t-2)/L}\right)\mathscr{U}(t-2) - \frac{1}{R}\left(1 - e^{-R(t-3)/L}\right)\mathscr{U}(t-3) - \cdots \\ &= \frac{1}{R}\left(t - \frac{L}{R} + \frac{L}{R}e^{-Rt/L}\right) - \frac{1}{R}\sum_{n=1}^{\infty}\left(1 - e^{-R(t-n)/L}\right)\mathscr{U}(t-n). \end{aligned}$$

The graph of $i(t)$ with $L = 1$ and $R = 1$ is shown below.



**61.** The differential equation is $x'' + 2x' + 10x = 20f(t)$, where $f(t)$ is the meander function in Problem 53 with $a = \pi$. Using the initial conditions $x(0) = x'(0) = 0$ and taking the Laplace transform we obtain

$$\begin{aligned} (s^2 + 2s + 10)\mathscr{L}\{x(t)\} &= \frac{20}{s}(1 - e^{-\pi s})\frac{1}{1 + e^{-\pi s}} \\ &= \frac{20}{s}(1 - e^{-\pi s})(1 - e^{-\pi s} + e^{-2\pi s} - e^{-3\pi s} + \cdots) \\ &= \frac{20}{s}(1 - 2e^{-\pi s} + 2e^{-2\pi s} - 2e^{-3\pi s} + \cdots) \\ &= \frac{20}{s} + \frac{40}{s}\sum_{n=1}^{\infty}(-1)^n e^{-n\pi s}. \end{aligned}$$

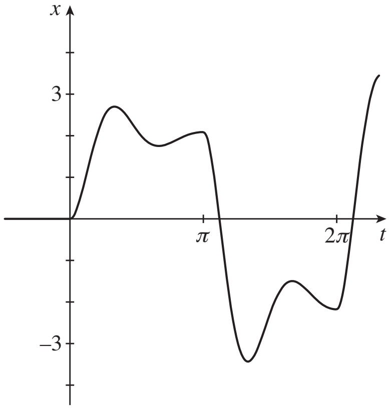
Then

$$\begin{aligned} \mathscr{L}\{x(t)\} &= \frac{20}{s(s^2 + 2s + 10)} + \frac{40}{s(s^2 + 2s + 10)}\sum_{n=1}^{\infty}(-1)^n e^{-n\pi s} \\ &= \frac{2}{s} - \frac{2s + 4}{s^2 + 2s + 10} + \sum_{n=1}^{\infty}(-1)^n\left[\frac{4}{s} - \frac{4s + 8}{s^2 + 2s + 10}\right]e^{-n\pi s} \\ &= \frac{2}{s} - \frac{2(s + 1) + 2}{(s + 1)^2 + 9} + 4\sum_{n=1}^{\infty}(-1)^n\left[\frac{1}{s} - \frac{(s + 1) + 1}{(s + 1)^2 + 9}\right]e^{-n\pi s} \end{aligned}$$

and

$$x(t) = 2\left(1 - e^{-t}\cos 3t - \frac{1}{3}e^{-t}\sin 3t\right)$$
$$+ 4\sum_{n=1}^{\infty}(-1)^n\left[1 - e^{-(t-n\pi)}\cos 3(t-n\pi) - \frac{1}{3}e^{-(t-n\pi)}\sin 3(t-n\pi)\right]\mathscr{U}(t-n\pi).$$

The graph of $x(t)$ is shown below.



**62.** The differential equation is $x'' + 2x' + x = 5f(t)$, where $f(t)$ is the square wave function with $a = \pi$. Using the initial conditions $x(0) = x'(0) = 0$ and taking the Laplace transform, we obtain

$$(s^2 + 2s + 1)\mathscr{L}\{x(t)\} = \frac{5}{s}\,\frac{1}{1 + e^{-\pi s}} = \frac{5}{s}\left(1 - e^{-\pi s} + e^{-2\pi s} - e^{-3\pi s} + e^{-4\pi s} - \cdots\right)$$
$$= \frac{5}{s}\sum_{n=0}^{\infty}(-1)^n e^{-n\pi s}.$$

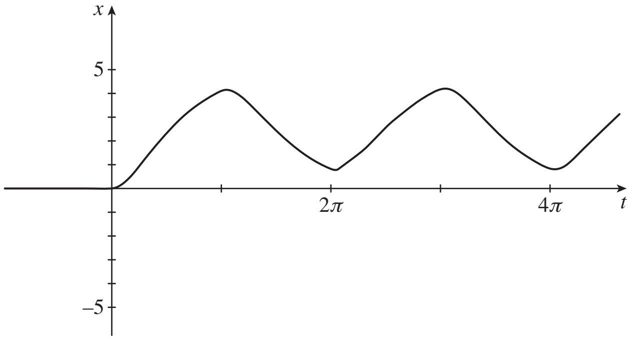
Then

$$\mathscr{L}\{x(t)\} = \frac{5}{s(s+1)^2}\sum_{n=0}^{\infty}(-1)^n e^{-n\pi s} = 5\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{s} - \frac{1}{s+1} - \frac{1}{(s+1)^2}\right)e^{-n\pi s}$$

and

$$x(t) = 5\sum_{n=0}^{\infty}(-1)^n(1 - e^{-(t-n\pi)} - (t-n\pi)e^{-(t-n\pi)})\mathscr{U}(t-n\pi).$$

The graph of $x(t)$ is shown below.



**63.** $f(t) = -\frac{1}{t}\mathscr{L}^{-1}\left\{\frac{d}{ds}\left[\ln(s-3) - \ln(s+1)\right]\right\} = -\frac{1}{t}\mathscr{L}^{-1}\left\{\frac{1}{s-3} - \frac{1}{s+1}\right\}$

$= -\frac{1}{t}\left(e^{3t} - e^{-t}\right)$

**64.** The transform of Bessel's equation is

$$-\frac{d}{ds}[s^2Y(s) - sy(0) - y'(0)] + sY(s) - y(0) - \frac{d}{ds}Y(s) = 0$$

or, after simplifying and using the initial condition, $(s^2+1)Y' + sY = 0$. This equation is both separable and linear. Solving gives $Y(s) = c/\sqrt{s^2+1}$. Now $Y(s) = \mathscr{L}\{J_0(t)\}$, where $J_0$ has a derivative that is continuous and of exponential order, implies by Problem 46 of Exercises 7.2 that

$$1 = J_0(0) = \lim_{s\to\infty} sY(s) = c\lim_{s\to\infty}\frac{s}{\sqrt{s^2+k^2}} = c$$

so $c = 1$ and

$$Y(s) = \frac{1}{\sqrt{s^2+1}} \qquad \text{or} \qquad \mathscr{L}\{J_0(t)\} = \frac{1}{\sqrt{s^2+1}}.$$

**65. (a)** Using Theorem 7.4.1, the Laplace transform of the differential equation is

$$-\frac{d}{ds}\left[s^2Y - sy(0) - y'(0)\right] + sY - y(0) + \frac{d}{ds}\left[sY - y(0)\right] + nY$$

$$= -\frac{d}{ds}\left[s^2Y\right] + sY + \frac{d}{ds}\left[sY\right] + nY$$

$$= -s^2\left(\frac{dY}{ds}\right) - 2sY + sY + s\left(\frac{dY}{ds}\right) + Y + nY$$

$$= (s - s^2)\left(\frac{dY}{ds}\right) + (1 + n - s)Y = 0.$$

Separating variables, we find

$$\frac{dY}{Y} = \frac{1+n-s}{s^2-s}\,ds = \left(\frac{n}{s-1} - \frac{1+n}{s}\right)ds$$

$$\ln Y = n\ln(s-1) - (1+n)\ln s + c$$

$$Y = c_1\frac{(s-1)^n}{s^{1+n}}.$$

Since the differential equation is homogeneous, any constant multiple of a solution will still be a solution, so for convenience we take $c_1 = 1$. The following polynomials are solutions of Laguerre's differential equation:

$$n = 0: \quad L_0(t) = \mathscr{L}^{-1}\left\{\frac{1}{s}\right\} = 1$$

$$n = 1: \quad L_1(t) = \mathscr{L}^{-1}\left\{\frac{s-1}{s^2}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{s} - \frac{1}{s^2}\right\} = 1 - t$$

$$n = 2: \quad L_2(t) = \mathscr{L}^{-1}\left\{\frac{(s-1)^2}{s^3}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{s} - \frac{2}{s^2} + \frac{1}{s^3}\right\} = 1 - 2t + \frac{1}{2}t^2$$

$$n = 3: \quad L_3(t) = \mathscr{L}^{-1}\left\{\frac{(s-1)^3}{s^4}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{s} - \frac{3}{s^2} + \frac{3}{s^3} - \frac{1}{s^4}\right\} = 1 - 3t + \frac{3}{2}t^2 - \frac{1}{6}t^3$$

$$\begin{aligned} n = 4: \quad L_4(t) &= \mathscr{L}^{-1}\left\{\frac{(s-1)^4}{s^5}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{s} - \frac{4}{s^2} + \frac{6}{s^3} - \frac{4}{s^4} + \frac{1}{s^5}\right\} \\ &= 1 - 4t + 3t^2 - \frac{2}{3}t^3 + \frac{1}{24}t^4. \end{aligned}$$

**(b)** Letting $f(t) = t^n e^{-t}$ we note that $f^{(k)}(0) = 0$ for $k = 0, 1, 2, \ldots, n-1$ and $f^{(n)}(0) = n!$. Now, by the first translation theorem,

$$\begin{aligned} \mathscr{L}\left\{\frac{e^t}{n!}\frac{d^n}{dt^n}\,t^n e^{-t}\right\} &= \frac{1}{n!}\mathscr{L}\{e^t f^{(n)}(t)\} = \frac{1}{n!}\mathscr{L}\{f^{(n)}(t)\}\bigg|_{s\to s-1} \\ &= \frac{1}{n!}\left[s^n\mathscr{L}\{t^n e^{-t}\} - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)\right]_{s\to s-1} \\ &= \frac{1}{n!}\left[s^n\mathscr{L}\{t^n e^{-t}\}\right]_{s\to s-1} \\ &= \frac{1}{n!}\left[s^n\frac{n!}{(s+1)^{n+1}}\right]_{s\to s-1} = \frac{(s-1)^n}{s^{n+1}} = Y, \end{aligned}$$

where $Y = \mathscr{L}\{L_n(t)\}$. Thus

$$L_n(t) = \frac{e^t}{n!}\frac{d^n}{dt^n}(t^n e^{-t}), \quad n = 0, 1, 2, \ldots .$$

**66.** Let $\mathscr{L}\left\{e^{-t^2}\right\} = F(s)$ then

$$\begin{aligned} \mathscr{L}\{y''\} + \mathscr{L}\{y\} &= \mathscr{L}\left\{e^{-t^2}\right\} \\ s^2 Y(s) + Y(s) &= F(s) \\ Y(s) &= \frac{1}{s^2+1}\cdot F(s) \end{aligned}$$

Now take the inverse transform and use the inverse form of the convolution theorem to get

$$y(t) = \mathscr{L}^{-1}\left\{\frac{1}{s^2+1}\cdot F(s)\right\} = \sin t * e^{-t^2} = \int_0^t \sin\tau\, e^{-(t-\tau)^2}\,d\tau$$

or

$$y(t) = \mathscr{L}^{-1}\left\{F(s)\cdot\frac{1}{s^2+1}\right\} = e^{-t^2} * \sin t = \int_0^t e^{-\tau^2}\sin(t-\tau)\,d\tau$$

**67.** Take the transform of both sides of the equation to get

$$\mathscr{L}\{f(t)\} = \mathscr{L}\{e^t\} + \mathscr{L}\left\{e^t\cdot\int_0^t f(\tau)e^{-\tau}\,d\tau\right\}$$

$$F(s) = \frac{1}{s-1} + \left[\mathscr{L}\{f(t)e^{-t}\}\,\mathscr{L}\{1\}\right]_{s\to s-1}$$

$$F(s) = \frac{1}{s-1} + \left[\mathscr{L}\{f(t)e^{-t}\}\right]_{s\to s-1}\cdot\left[\mathscr{L}\{1\}\right]_{s\to s-1}$$

$$F(s) = frac1s - 1 + [F(s+1)]_{s\to s-1}\cdot\left[\frac{1}{s}\right]_{s\to s-1}$$

$$F(s) = \frac{1}{s-1} + F(s)\cdot\frac{1}{s-1}$$

Solve the last equation for $F(s)$ to get $F(s) = 1/(s-2)$ therefore $f(t) = e^{2t}$.

**68.** **(a)** $E(t) = \sum_{n=0}^{\infty}(-1)^k\mathscr{U}(t-k) = \mathscr{U}(t) - \mathscr{U}(t-1) + \mathscr{U}(t-2) + \mathscr{U}(t-3) + \cdots$

$$= \begin{cases} 1, & 0 \le t < 1 \\ 0, & 1 \le t < 2 \\ 1, & 2 \le t < 3 \\ \vdots & \end{cases}$$

**(b)** $\mathscr{L}\{E(t)\} = \frac{1}{s} - \frac{1}{s}e^{-s} + \frac{1}{s}e^{-2s} - \frac{1}{s}e^{-3s} = \frac{1}{s}\Big(\overbrace{1 - e^{-s} + e^{-2s} - e^{-3s} + \cdots}^{\text{geometric series with } r=-e^{-s}}\Big)$

$$= \frac{1}{s}\left[\frac{1}{1-(-e^{-s})}\right] = \frac{1}{s\,(1+e^{-s})}.$$

**69.** We know that

$$\mathscr{L}\{t\sin 4t\| = -\frac{d}{ds}\left(\frac{4}{s^2+16}\right) = \frac{8s}{(s^2+16)^2},$$

and so for $s > 0$,

$$F(s) = \int_0^{\infty} te^{-st}\sin 4t\,dt = \frac{8s}{(s^2+16)^2}.$$

Therefore

$$F(2) = \int_0^{\infty} te^{2t}\sin 4t\,dt = \frac{8\cdot 2}{(2^2+16)^2} = \frac{16}{400} = \frac{1}{25}.$$

**70.** **(a)** $\mathscr{L}\left\{\frac{\sin at}{t}\right\} = \int_s^\infty \frac{a}{u^2+a^2}\,du = a\cdot\frac{1}{a}\arctan\frac{u}{a}\Big|_s^\infty = \frac{\pi}{2} - \arctan\frac{s}{a} = \arctan\frac{a}{s}$

**(b)**
$$\mathscr{L}\left\{\frac{2\,(1-\cos kt)}{t}\right\} = \int_s^\infty \left[\frac{2}{u} - \frac{2u}{u^2+k^2}\right] du = \left[2\ln u - \ln\left(u^2+k^2\right)\right]_s^\infty$$
$$= \left[2\ln u - \ln\left(u^2+k^2\right)\right]_s^\infty = \left[\ln\frac{u^2}{u^2+k^2}\right]_s^\infty$$
$$= \ln 1 - \ln\frac{s^2}{s^2+k^2} = \ln\frac{s^2+k^2}{s^2}$$

**71.** **(a)** Using the definition of the Laplace transform and integration by parts,
$$\mathscr{L}\{\ln t\} = \int_0^\infty e^{-st}\ln t\,dt = e^{-st}\,(t\ln t - t)\Big|_0^\infty + s\int_0^\infty e^{-st}\,(t\ln t - t)\;dt$$
$$= s\mathscr{L}\{t\ln t\} - s\mathscr{L}\{t\} = s\mathscr{L}\{t\ln t\} - \frac{1}{s}.$$

**(b)** Letting $Y(s) = \mathscr{L}\{\ln t\}$ we have
$$Y = s\left(-\frac{d}{ds}Y\right) - \frac{1}{s}$$
by Theorem 7.4.1. That is,
$$s\frac{dY}{ds} + Y = -\frac{1}{s}$$
$$\frac{d}{ds}[sY] = -\frac{1}{s}$$
$$sY = -\ln s + c$$
$$Y = \frac{c}{s} - \frac{1}{s}\ln s,\, s > 0.$$

**(c)** Now $Y(1) = \mathscr{L}\{\ln t\}\Big|_{s=1} = \int_0^\infty e^{-t}\ln t\,dt = -\gamma$. Thus
$$-\gamma = Y(1) = c - \ln 1 = c$$
and
$$Y = \mathscr{L}\{\ln t\} = -\frac{\gamma}{s} - \frac{1}{s}\ln s.$$

**72.** The output for the first three lines of the program are
$$9y[t] + 6y'[t] + y''[t] == t\,\sin[t]$$
$$1 - 2s + 9Y + s^2Y + 6(-2+sY) == \frac{2s}{(1+s^2)^2}$$
$$Y \to -\left(\frac{-11 - 4s - 22s^2 - 4s^3 - 11s^4 - 2s^5}{(1+s^2)^2(9+6s+s^2)}\right)$$

The fourth line is the same as the third line with $Y \to$ removed. The final line of output shows a solution involving complex coefficients of $e^{it}$ and $e^{-it}$. To get the solution in more standard form write the last line as two lines:

```
euler={E^(It)->Cos[t] + I Sin[t], E^(-It)->Cos[t] - I Sin[t]}
InverseLaplaceTransform[Y, s, t]/.euler//Expand
```

We see that the solution is

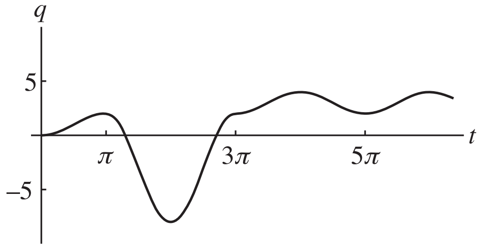
$$y(t) = \left(\frac{487}{250} + \frac{247}{50}t\right)e^{-3t} + \frac{1}{250}\left(13\cos t - 15t\cos t - 9\sin t + 20t\sin t\right).$$

**73.** The solution is

$$y(t) = \frac{1}{6}e^{t} - \frac{1}{6}e^{-t/2}\cos\frac{\sqrt{15}\,t}{2} - \frac{1}{2\sqrt{15}}e^{-t/2}\sin\frac{\sqrt{15}\,t}{2}.$$

**74.** The solution is

$$q(t) = 1 - \cos t + (6 + 6\cos t)\mathscr{U}(t - 3\pi) - (4 + 4\cos t)\mathscr{U}(t - \pi).$$



## 7.5 The Dirac Delta Function

**1.** The Laplace transform of the differential equation yields

$$\mathscr{L}\{y\} = \frac{1}{s-3}e^{-2s}$$

so that

$$y = e^{3(t-2)}\mathscr{U}(t-2).$$

**2.** The Laplace transform of the differential equation yields

$$\mathscr{L}\{y\} = \frac{2}{s+1} + \frac{e^{-s}}{s+1}$$

so that

$$y = 2e^{-t} + e^{-(t-1)}\mathscr{U}(t-1).$$

**3.** The Laplace transform of the differential equation yields

$$\mathscr{L}\{y\} = \frac{1}{s^2+1}\left(1+e^{-2\pi s}\right)$$

so that

$$y = \sin t + \sin t\,\mathscr{U}(t-2\pi).$$

**4.** The Laplace transform of the differential equation yields

$$\mathscr{L}\{y\} = \frac{1}{4}\,\frac{4}{s^2+16}e^{-2\pi s}$$

so that

$$y = \frac{1}{4}\sin 4(t-2\pi)\mathscr{U}(t-2\pi) = \frac{1}{4}\sin 4t\,\mathscr{U}(t-2\pi).$$

**5.** The Laplace transform of the differential equation yields

$$\mathscr{L}\{y\} = \frac{1}{s^2+1}\left(e^{-\pi s/2}+e^{-3\pi s/2}\right)$$

so that

$$\begin{aligned} y &= \sin\left(t-\frac{\pi}{2}\right)\mathscr{U}\left(t-\frac{\pi}{2}\right) + \sin\left(t-\frac{3\pi}{2}\right)\mathscr{U}\left(t-\frac{3\pi}{2}\right) \\ &= -\cos t\,\mathscr{U}\left(t-\frac{\pi}{2}\right) + \cos t\,\mathscr{U}\left(t-\frac{3\pi}{2}\right). \end{aligned}$$

**6.** The Laplace transform of the differential equation yields

$$\mathscr{L}\{y\} = \frac{s}{s^2+1} + \frac{1}{s^2+1}(e^{-2\pi s}+e^{-4\pi s})$$

so that

$$y = \cos t + \sin t[\mathscr{U}(t-2\pi)+\mathscr{U}(t-4\pi)].$$

**7.** The Laplace transform of the differential equation yields

$$\mathscr{L}\{y\} = \frac{1}{s^2+2s}(1+e^{-s}) = \left[\frac{1}{2}\,\frac{1}{s} - \frac{1}{2}\,\frac{1}{s+2}\right](1+e^{-s})$$

so that

$$y = \frac{1}{2} - \frac{1}{2}e^{-2t} + \left[\frac{1}{2} - \frac{1}{2}e^{-2(t-1)}\right]\mathscr{U}(t-1).$$

**8.** The Laplace transform of the differential equation yields

$$\mathscr{L}\{y\} = \frac{s+1}{s^2(s-2)} + \frac{1}{s(s-2)}\,e^{-2s} = \frac{3}{4}\,\frac{1}{s-2} - \frac{3}{4}\,\frac{1}{s} - \frac{1}{2}\,\frac{1}{s^2} + \left[\frac{1}{2}\,\frac{1}{s-2} - \frac{1}{2}\,\frac{1}{s}\right]e^{-2s}$$

so that

$$y = \frac{3}{4}e^{2t} - \frac{3}{4} - \frac{1}{2}t + \left[\frac{1}{2}e^{2(t-2)} - \frac{1}{2}\right]\mathscr{U}(t-2).$$

**9.** The Laplace transform of the differential equation yields

$$\mathscr{L}\{y\} = \frac{1}{(s+2)^2+1}e^{-2\pi s}$$

so that

$$y = e^{-2(t-2\pi)}\sin t\,\mathscr{U}(t-2\pi).$$

**10.** The Laplace transform of the differential equation yields

$$\mathscr{L}\{y\} = \frac{1}{(s+1)^2}e^{-s}$$

so that

$$y = (t-1)e^{-(t-1)}\mathscr{U}(t-1).$$

**11.** The Laplace transform of the differential equation yields

$$\begin{aligned}\mathscr{L}\{y\} &= \frac{4+s}{s^2+4s+13} + \frac{e^{-\pi s}+e^{-3\pi s}}{s^2+4s+13}\\ &= \frac{2}{3}\frac{3}{(s+2)^2+3^2} + \frac{s+2}{(s+2)^2+3^2} + \frac{1}{3}\frac{3}{(s+2)^2+3^2}\left(e^{-\pi s}+e^{-3\pi s}\right)\end{aligned}$$

so that

$$\begin{aligned}y = {}&\frac{2}{3}e^{-2t}\sin 3t + e^{-2t}\cos 3t + \frac{1}{3}e^{-2(t-\pi)}\sin 3(t-\pi)\mathscr{U}(t-\pi)\\ &+ \frac{1}{3}e^{-2(t-3\pi)}\sin 3(t-3\pi)\mathscr{U}(t-3\pi).\end{aligned}$$

**12.** The Laplace transform of the differential equation yields

$$\begin{aligned}\mathscr{L}\{y\} &= \frac{1}{(s-1)^2(s-6)} + \frac{e^{-2s}+e^{-4s}}{(s-1)(s-6)}\\ &= -\frac{1}{25}\frac{1}{s-1} - \frac{1}{5}\frac{1}{(s-1)^2} + \frac{1}{25}\frac{1}{s-6} + \left[-\frac{1}{5}\frac{1}{s-1} + \frac{1}{5}\frac{1}{s-6}\right]\left(e^{-2s}+e^{-4s}\right)\end{aligned}$$

so that

$$y = -\frac{1}{25}e^t - \frac{1}{5}te^t + \frac{1}{25}e^{6t} + \left[-\frac{1}{5}e^{t-2} + \frac{1}{5}e^{6(t-2)}\right]\mathscr{U}(t-2) + \left[-\frac{1}{5}e^{t-4} + \frac{1}{5}e^{6(t-4)}\right]\mathscr{U}(t-4).$$

**13.** The Laplace transform of the differential equation yields

$$\mathscr{L}\{y''\} + \mathscr{L}\{y\} = \sum_{k=1}^{\infty}\mathscr{L}\left\{\delta\left(t-k\pi\right)\right\}$$

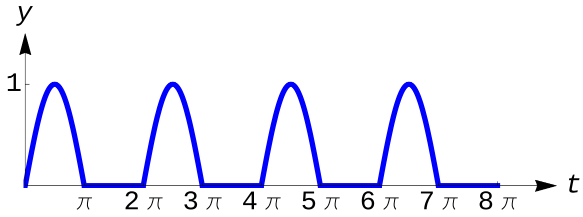
$$\left(s^2+1\right)Y(s) = 1 + \sum k = 1^{\infty}e^{-k\pi s}$$

$$Y(s) = \frac{1}{s^2+1} + \sum_{k=1}^{\infty}\frac{e^{-k\pi s}}{s^2+1}$$

so that

$$\begin{aligned}
y(t) &= \sin t + \sum_{k=1}^{\infty} \sin(t-k\pi)\mathscr{U}(t-k\pi) \\
&= \sin t + \sin t \sum_{k=1}^{\infty} (-1)^k \mathscr{U}(t-k\pi) \qquad \longleftarrow \qquad \sin(t-k\pi) = (-1)^k \sin t \\
&= \sin t - \sin t\mathscr{U}(t-\pi) + \sin t\mathscr{U}(t-2\pi) - \sin t\mathscr{U}(t-3\pi) + \sin t\mathscr{U}(t-4\pi) - \cdots
\end{aligned}$$

$$= \begin{cases} \sin t, & 0 \le t < \pi \\ 0, & \pi \le t < 2\pi \\ \sin t, & 2\pi \le t < 3\pi \\ 0, & 3\pi \le t < 4\pi \\ \sin t, & 4\pi \le t < 5\pi \\ 0, & 5\pi \le t < 6\pi \\ \sin t, & 6\pi \le t < 7\pi \\ 0, & 7\pi \le t < 8\pi \\ \vdots \end{cases}$$



The graph of $y(t)$ given on the right is the half-wave rectification of $\sin t$. Also, see Figure 7.4.11 in Exercises 7.4.

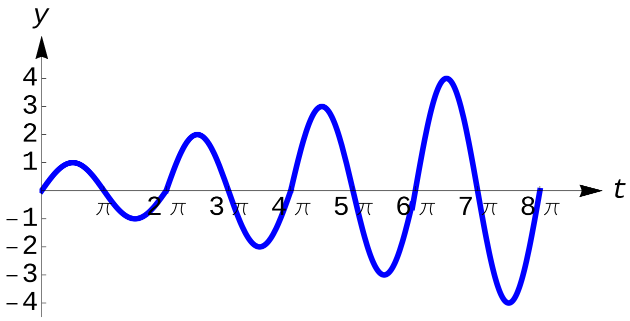
**14.** The Laplace transform of the differential equation yields

$$\begin{aligned}
\mathscr{L}\{y''\} + \mathscr{L}\{y\} &= \sum_{k=1}^{\infty} \mathscr{L}\{\delta(t-2k\pi)\} \\
\left(s^2+1\right) Y(s) &= 1 + \sum_{k=1}^{\infty} e^{-2k\pi s} \\
Y(s) &= \frac{1}{s^2+1} + \sum_{k=1}^{\infty} \frac{e^{-2k\pi s}}{s^2+1}
\end{aligned}$$

so that

$$\begin{aligned}
y(t) &= \sin t + \sum_{k=1}^{\infty} \sin(t-2k\pi)\mathscr{U}(t-2k\pi) \\
&= \sin t + \sin t \sum_{k=1}^{\infty} (-1)^k \mathscr{U}(t-2k\pi) \qquad \longleftarrow \qquad \sin(t-2k\pi) = \sin t \\
&= \sin t + \sin t\mathscr{U}(t-2\pi) + \sin t\mathscr{U}(t-4\pi) + \sin t\mathscr{U}(t-6\pi) + \sin t\mathscr{U}(t-8\pi) + \cdots
\end{aligned}$$

$$= \begin{cases} \sin t, & 0 \le t < \pi \\ 2\sin t, & 2\pi \le t < 4\pi \\ 3\sin t, & 4\pi \le t < 6\pi \\ 4\sin t, & 6\pi \le t < 8\pi \\ \vdots & \end{cases}$$



The graph of $y(t)$ is given on the right.

**15.** Take the transform of the equation $y^{(4)} = \frac{w_0}{EI}\delta\left(x - \frac{L}{2}\right)$ and solve for $Y(s)$ to get

$$s^2Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0) = \frac{w_0}{EI}e^{(-L/2)s}$$

$$s^4Y(s) - sy''(0) - y'''(0) = \frac{w_0}{EI}e^{(-L/2)s}$$

$$Y(s) = \frac{1}{s^3}y''(0) + \frac{1}{s^4}y'''(0) + \frac{w_0}{EI}\frac{1}{s^4}e^{(-L/2)s}$$

The inverse transform is then $y(x) = \frac{1}{2}y''(0)x^2 + \frac{1}{6}y'''(0)x^3 + \frac{w_0}{6EI}\left(x - \frac{L}{2}\right)^3 \mathscr{U}\left(x - \frac{L}{2}\right)$. Using the conditions $y''(L) = y'''(L) = 0$ to find $y''(0)$ and $y'''(0)$ we get

$$y(x) = \frac{1}{2}\left(\frac{w_0L}{2EI}\right)x^2 - \frac{1}{6}\left(\frac{w_0}{EI}\right)x^3 + \frac{w_0}{6EI}\left(x - \frac{L}{2}\right)^3 \mathscr{U}\left(x - \frac{L}{2}\right)$$

We could also write the solution as

$$y(x) = \begin{cases} \left(\frac{w_0L}{4EI}\right)x^2 - \left(\frac{w_0}{6EI}\right)x^3, & 0 \le x < \frac{L}{2} \\ \left(\frac{w_0L}{4EI}\right)x^2 - \left(\frac{w_0}{6EI}\right)x^3 + \frac{w_0}{6EI}\left(x - \frac{L}{2}\right)^3, & \frac{L}{2} \le x \le L \end{cases}$$

**16.** From Problem 15 we have $y(x) = \frac{1}{2}y''(0)x^2 + \frac{1}{6}y'''(0)x^3 + \frac{w_0}{6EI}\left(x - \frac{L}{2}\right)^3 \mathscr{U}\left(x - \frac{L}{2}\right)$. Using the conditions $y(L) = y'(L) = 0$ to find $y''(0)$ and $y'''(0)$ we get the system of equations

$$\begin{cases} (24L^2)y''(0) + (8L^3)y'''(0) = -\frac{w_0L^3}{EI} \\ (8L)y''(0) + (4L^2)y'''(0) = -\frac{w_0L^2}{EI} \end{cases}$$

Solving the system leads to the solution

$$y(x) = \frac{1}{2}\left(\frac{w_0L}{8EI}\right)x^2 + \frac{1}{6}\left(-\frac{w_0}{2EI}\right)x^3 + \frac{w_0}{6EI}\left(x - \frac{L}{2}\right)^3 \mathscr{U}\left(x - \frac{L}{2}\right)$$

We could also write the solution as

$$y(x) = \begin{cases} \left(\dfrac{w_0L}{16EI}\right)x^2 - \left(\dfrac{w_0}{12EI}\right)x^3, & 0 \le x < \dfrac{L}{2} \\[2ex] \left(\dfrac{w_0L}{16EI}\right)x^2 - \left(\dfrac{w_0}{12EI}\right)x^3 + \dfrac{w_0}{6EI}\left(x - \dfrac{L}{2}\right)^3, & \dfrac{L}{2} \le x \le L \end{cases}$$

**17.** You should disagree. Although formal manipulations of the Laplace transform lead to $y(t) = \frac{1}{3}e^{-t}\sin 3t$ in both cases, this function does not satisfy the initial condition $y'(0) = 0$ of the second initial-value problem.

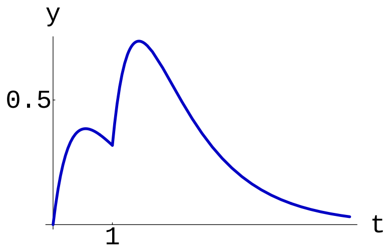
**18.** From (7) in (*i*) of the *Remarks* it follows with the identifications $f(t) = e^te^{-st} = e^{(1-s)t}$ and $t_0 = 1$ that

$$\mathscr{L}\left\{e^t\delta(t-1)\right\} = \int_0^\infty e^{(1-s)t}\delta(t-1)\,dt = f(1) = e^{1-s} = ee^{-s}.$$

Using the initial conditions $y(0) = 0$ and $y'(0) = 2$ we then have

$$\begin{aligned} \mathscr{L}\{y''\} + 4\mathscr{L}\{y'\} + 3\mathscr{L}\{y\} &= \mathscr{L}\left\{e^t\delta(t-1)\right\} \\ (s+1)(s+3)Y(s) &= 2 + ee^{-s} \\ Y(s) &= \frac{2}{(s+1)(s+3)} + \frac{ee^{-s}}{(s+1)(s+3)} \\ &= \frac{1}{s+1} - \frac{1}{s+3} + \frac{e}{2}\left[\frac{1}{s+1} - \frac{1}{s+3}\right]e^{-s} \\ y(t) &= e^{-t} - e^{-3t} + \frac{e}{2}\left(e^{-(t-1)} - e^{-3(t-1)}\right)\mathscr{U}(t-1). \end{aligned}$$

The graph of $y(t)$ is given on the right.



## 7.6 Systems of Linear Differential Equations

**1.** Taking the Laplace transform of the system gives

$$\begin{aligned} s\mathscr{L}\{x\} &= -\mathscr{L}\{x\} + \mathscr{L}\{y\} \\ s\mathscr{L}\{y\} - 1 &= 2\mathscr{L}\{x\} \end{aligned}$$

so that

$$\mathscr{L}\{x\} = \frac{1}{(s-1)(s+2)} = \frac{1}{3}\frac{1}{s-1} - \frac{1}{3}\frac{1}{s+2}$$

and

$$\mathscr{L}\{y\} = \frac{1}{s} + \frac{2}{s(s-1)(s+2)} = \frac{2}{3}\frac{1}{s-1} + \frac{1}{3}\frac{1}{s+2}.$$

Then

$$x = \frac{1}{3}e^{t} - \frac{1}{3}e^{-2t} \qquad \text{and} \qquad y = \frac{2}{3}e^{t} + \frac{1}{3}e^{-2t}.$$

**2.** Taking the Laplace transform of the system gives

$$\begin{aligned} s\mathscr{L}\{x\} - 1 &= 2\mathscr{L}\{y\} + \frac{1}{s-1} \\ s\mathscr{L}\{y\} - 1 &= 8\mathscr{L}\{x\} - \frac{1}{s^2} \end{aligned}$$

so that

$$\mathscr{L}\{y\} = \frac{s^3+7s^2-s+1}{s(s-1)(s^2-16)} = \frac{1}{16}\frac{1}{s} - \frac{8}{15}\frac{1}{s-1} + \frac{173}{96}\frac{1}{s-4} - \frac{53}{160}\frac{1}{s+4}$$

and

$$y = \frac{1}{16} - \frac{8}{15}e^{t} + \frac{173}{96}e^{4t} - \frac{53}{160}e^{-4t}.$$

Then

$$x = \frac{1}{8}y' + \frac{1}{8}t = \frac{1}{8}t - \frac{1}{15}e^{t} + \frac{173}{192}e^{4t} + \frac{53}{320}e^{-4t}.$$

**3.** Taking the Laplace transform of the system gives

$$\begin{aligned} s\mathscr{L}\{x\} + 1 &= \mathscr{L}\{x\} - 2\mathscr{L}\{y\} \\ s\mathscr{L}\{y\} - 2 &= 5\mathscr{L}\{x\} - \mathscr{L}\{y\} \end{aligned}$$

so that

$$\mathscr{L}\{x\} = \frac{-s-5}{s^2+9} = -\frac{s}{s^2+9} - \frac{5}{3}\frac{3}{s^2+9}$$

and

$$x = -\cos 3t - \frac{5}{3}\sin 3t.$$

Then

$$y = \frac{1}{2}x - \frac{1}{2}x' = 2\cos 3t - \frac{7}{3}\sin 3t.$$

**4.** Taking the Laplace transform of the system gives

$$\begin{aligned} (s+3)\mathscr{L}\{x\} + s\mathscr{L}\{y\} &= \frac{1}{s} \\ (s-1)\mathscr{L}\{x\} + (s-1)\mathscr{L}\{y\} &= \frac{1}{s-1} \end{aligned}$$

so that

$$\mathscr{L}\{y\} = \frac{5s-1}{3s(s-1)^2} = -\frac{1}{3}\frac{1}{s} + \frac{1}{3}\frac{1}{s-1} + \frac{4}{3}\frac{1}{(s-1)^2}$$

and

$$\mathscr{L}\{x\} = \frac{1-2s}{3s(s-1)^2} = \frac{1}{3}\frac{1}{s} - \frac{1}{3}\frac{1}{s-1} - \frac{1}{3}\frac{1}{(s-1)^2}\,.$$

Then

$$x = \frac{1}{3} - \frac{1}{3}e^t - \frac{1}{3}te^t \qquad \text{and} \qquad y = -\frac{1}{3} + \frac{1}{3}e^t + \frac{4}{3}te^t.$$

**5.** Taking the Laplace transform of the system gives

$$(2s-2)\mathscr{L}\{x\} + s\mathscr{L}\{y\} = \frac{1}{s}$$

$$(s-3)\mathscr{L}\{x\} + (s-3)\mathscr{L}\{y\} = \frac{2}{s}$$

so that

$$\mathscr{L}\{x\} = \frac{-s-3}{s(s-2)(s-3)} = -\frac{1}{2}\frac{1}{s} + \frac{5}{2}\frac{1}{s-2} - \frac{2}{s-3}$$

and

$$\mathscr{L}\{y\} = \frac{3s-1}{s(s-2)(s-3)} = -\frac{1}{6}\frac{1}{s} - \frac{5}{2}\frac{1}{s-2} + \frac{8}{3}\frac{1}{s-3}\,.$$

Then

$$x = -\frac{1}{2} + \frac{5}{2}e^{2t} - 2e^{3t} \qquad \text{and} \qquad y = -\frac{1}{6} - \frac{5}{2}e^{2t} + \frac{8}{3}e^{3t}.$$

**6.** Taking the Laplace transform of the system gives

$$(s+1)\mathscr{L}\{x\} - (s-1)\mathscr{L}\{y\} = -1$$

$$s\mathscr{L}\{x\} + (s+2)\mathscr{L}\{y\} = 1$$

so that

$$\mathscr{L}\{y\} = \frac{s+1/2}{s^2+s+1} = \frac{s+1/2}{(s+1/2)^2 + (\sqrt{3}/2)^2}$$

and

$$\mathscr{L}\{x\} = \frac{-3/2}{s^2+s+1} = -\sqrt{3}\,\frac{\sqrt{3}/2}{(s+1/2)^2 + (\sqrt{3}/2)^2}\,.$$

Then

$$y = e^{-t/2}\cos\frac{\sqrt{3}}{2}\,t \qquad \text{and} \qquad x = -\sqrt{3}\,e^{-t/2}\sin\frac{\sqrt{3}}{2}\,t.$$

**7.** Taking the Laplace transform of the system gives

$$(s^2+1)\mathscr{L}\{x\} - \mathscr{L}\{y\} = -2$$

$$-\mathscr{L}\{x\} + (s^2+1)\mathscr{L}\{y\} = 1$$

so that

$$\mathscr{L}\{x\} = \frac{-2s^2-1}{s^4+2s^2} = -\frac{1}{2}\frac{1}{s^2} - \frac{3}{2}\frac{1}{s^2+2}$$

and

$$x = -\frac{1}{2}t - \frac{3}{2\sqrt{2}}\sin\sqrt{2}\,t.$$

Then

$$y = x'' + x = -\frac{1}{2}t + \frac{3}{2\sqrt{2}}\sin\sqrt{2}\,t.$$

**8.** Taking the Laplace transform of the system gives

$$\begin{aligned}(s+1)\mathscr{L}\{x\} + \mathscr{L}\{y\} &= 1\\ 4\mathscr{L}\{x\} - (s+1)\mathscr{L}\{y\} &= 1\end{aligned}$$

so that

$$\mathscr{L}\{x\} = \frac{s+2}{s^2+2s+5} = \frac{s+1}{(s+1)^2+2^2} + \frac{1}{2}\frac{2}{(s+1)^2+2^2}$$

and

$$\mathscr{L}\{y\} = \frac{-s+3}{s^2+2s+5} = -\frac{s+1}{(s+1)^2+2^2} + 2\frac{2}{(s+1)^2+2^2}\,.$$

Then

$$x = e^{-t}\cos 2t + \frac{1}{2}e^{-t}\sin 2t \qquad \text{and} \qquad y = -e^{-t}\cos 2t + 2e^{-t}\sin 2t.$$

**9.** Adding the equations and then subtracting them gives

$$\begin{aligned}\frac{d^2x}{dt^2} &= \frac{1}{2}t^2 + 2t\\ \frac{d^2y}{dt^2} &= \frac{1}{2}t^2 - 2t.\end{aligned}$$

Taking the Laplace transform of the system gives

$$\mathscr{L}\{x\} = 8\frac{1}{s} + \frac{1}{24}\frac{4!}{s^5} + \frac{1}{3}\frac{3!}{s^4}$$

and

$$\mathscr{L}\{y\} = \frac{1}{24}\frac{4!}{s^5} - \frac{1}{3}\frac{3!}{s^4}$$

so that

$$x = 8 + \frac{1}{24}t^4 + \frac{1}{3}t^3 \qquad \text{and} \qquad y = \frac{1}{24}t^4 - \frac{1}{3}t^3.$$

**10.** Taking the Laplace transform of the system gives

$$\begin{aligned}(s-4)\mathscr{L}\{x\} + s^3\mathscr{L}\{y\} &= \frac{6}{s^2+1}\\ (s+2)\mathscr{L}\{x\} - 2s^3\mathscr{L}\{y\} &= 0\end{aligned}$$

so that

$$\mathscr{L}\{x\} = \frac{4}{(s-2)(s^2+1)} = \frac{4}{5}\frac{1}{s-2} - \frac{4}{5}\frac{s}{s^2+1} - \frac{8}{5}\frac{1}{s^2+1}$$

and

$$\mathscr{L}\{y\} = \frac{2s+4}{s^3(s-2)(s^2+1)} = \frac{1}{s} - \frac{2}{s^2} - 2\frac{2}{s^3} + \frac{1}{5}\,\frac{1}{s-2} - \frac{6}{5}\,\frac{s}{s^2+1} + \frac{8}{5}\,\frac{1}{s^2+1}\,.$$

Then

$$x = \frac{4}{5}e^{2t} - \frac{4}{5}\cos t - \frac{8}{5}\sin t$$

and

$$y = 1 - 2t - 2t^2 + \frac{1}{5}e^{2t} - \frac{6}{5}\cos t + \frac{8}{5}\sin t.$$

**11.** Taking the Laplace transform of the system gives

$$\begin{aligned} s^2\mathscr{L}\{x\} + 3(s+1)\mathscr{L}\{y\} &= 2 \\ s^2\mathscr{L}\{x\} + 3\mathscr{L}\{y\} &= \frac{1}{(s+1)^2} \end{aligned}$$

so that

$$\mathscr{L}\{x\} = -\frac{2s+1}{s^3(s+1)} = \frac{1}{s} + \frac{1}{s^2} + \frac{1}{2}\,\frac{2}{s^3} - \frac{1}{s+1}\,.$$

Then

$$x = 1 + t + \frac{1}{2}t^2 - e^{-t}$$

and

$$y = \frac{1}{3}te^{-t} - \frac{1}{3}x'' = \frac{1}{3}te^{-t} + \frac{1}{3}e^{-t} - \frac{1}{3}\,.$$

**12.** Taking the Laplace transform of the system gives

$$\begin{aligned} (s-4)\mathscr{L}\{x\} + 2\mathscr{L}\{y\} &= \frac{2e^{-s}}{s} \\ -3\mathscr{L}\{x\} + (s+1)\mathscr{L}\{y\} &= \frac{1}{2} + \frac{e^{-s}}{s} \end{aligned}$$

so that

$$\begin{aligned} \mathscr{L}\{x\} &= \frac{-1}{(s-1)(s-2)} + e^{-s}\,\frac{2}{(s-1)(s-2)} \\ &= \frac{1}{s-1} - \frac{1}{s-2} + e^{-s}\left[-2\,\frac{1}{s-1} + 2\,\frac{1}{s-2}\right] \end{aligned}$$

and

$$\begin{aligned} \mathscr{L}\{y\} &= \frac{s/2-2}{(s-1)(s-2)} + e^{-s}\,\frac{2/s+1}{(s-1)(s-2)} \\ &= \frac{3}{2}\,\frac{1}{s-1} - \frac{1}{2}\,\frac{1}{s-2} + e^{-s}\left[\frac{1}{s} - 3\,\frac{1}{s-1} + 2\,\frac{1}{s-2}\right]. \end{aligned}$$

Then

$$x = e^t - e^{2t} + \left[-2e^{t-1} + 2e^{2(t-1)}\right]\mathscr{U}(t-1)$$

and

$$y = \frac{3}{2}\,e^t - \frac{1}{2}\,e^{2t} + \left[1 - 3e^{t-1} + 2e^{2(t-1)}\right]\mathscr{U}(t-1).$$

**13.** The system is

$$\begin{aligned} x_1'' &= -3x_1 + 2(x_2 - x_1) \\ x_2'' &= -2(x_2 - x_1) \end{aligned}$$

$$x_1(0) = 0 \qquad x_1'(0) = 1 \qquad x_2(0) = 1 \qquad x_2'(0) = 0.$$

Taking the Laplace transform of the system gives

$$\begin{aligned} (s^2+5)\mathscr{L}\{x_1\} - 2\mathscr{L}\{x_2\} &= 1 \\ -2\mathscr{L}\{x_1\} + (s^2+2)\mathscr{L}\{x_2\} &= s \end{aligned}$$

so that

$$\mathscr{L}\{x_1\} = \frac{s^2+2s+2}{s^4+7s^2+6} = \frac{2}{5}\,\frac{s}{s^2+1} + \frac{1}{5}\,\frac{1}{s^2+1} - \frac{2}{5}\,\frac{s}{s^2+6} + \frac{4}{5\sqrt{6}}\,\frac{\sqrt{6}}{s^2+6}$$

and

$$\mathscr{L}\{x_2\} = \frac{s^3+5s+2}{(s^2+1)(s^2+6)} = \frac{4}{5}\,\frac{s}{s^2+1} + \frac{2}{5}\,\frac{1}{s^2+1} + \frac{1}{5}\,\frac{s}{s^2+6} - \frac{2}{5\sqrt{6}}\,\frac{\sqrt{6}}{s^2+6}\,.$$

Then

$$x_1 = \frac{2}{5}\cos t + \frac{1}{5}\sin t - \frac{2}{5}\cos\sqrt{6}\,t + \frac{4}{5\sqrt{6}}\sin\sqrt{6}\,t$$

and

$$x_2 = \frac{4}{5}\cos t + \frac{2}{5}\sin t + \frac{1}{5}\cos\sqrt{6}\,t - \frac{2}{5\sqrt{6}}\sin\sqrt{6}\,t.$$

**14.** In this system $x_1$ and $x_2$ represent displacements of masses $m_1$ and $m_2$ from their equilibrium positions. Since the net forces acting on $m_1$ and $m_2$ are

$$-k_1x_1 + k_2(x_2 - x_1) \qquad \text{and} \qquad -k_2(x_2 - x_1) - k_3x_2,$$

respectively, Newton's second law of motion gives

$$\begin{aligned} m_1x_1'' &= -k_1x_1 + k_2(x_2 - x_1) \\ m_2x_2'' &= -k_2(x_2 - x_1) - k_3x_2. \end{aligned}$$

Using $k_1 = k_2 = k_3 = 1$, $m_1 = m_2 = 1$, $x_1(0) = 0$, $x_1(0) = -1$, $x_2(0) = 0$, and $x_2'(0) = 1$, and taking the Laplace transform of the system, we obtain

$$\begin{aligned} (2+s^2)\mathscr{L}\{x_1\} - \mathscr{L}\{x_2\} &= -1 \\ \mathscr{L}\{x_1\} - (2+s^2)\mathscr{L}\{x_2\} &= -1 \end{aligned}$$

so that

$$\mathscr{L}\{x_1\} = -\frac{1}{s^2+3} \qquad \text{and} \qquad \mathscr{L}\{x_2\} = \frac{1}{s^2+3}\,.$$

Then

$$x_1 = -\frac{1}{\sqrt{3}}\sin\sqrt{3}\,t \qquad \text{and} \qquad x_2 = \frac{1}{\sqrt{3}}\sin\sqrt{3}\,t.$$

**15. (a)** By Kirchhoff's first law we have $i_1 = i_2 + i_3$. By Kirchhoff's second law, on each loop we have $E(t) = Ri_1 + L_1 i_2'$ and $E(t) = Ri_1 + L_2 i_3'$ or $L_1 i_2' + Ri_2 + Ri_3 = E(t)$ and $L_2 i_3' + Ri_2 + Ri_3 = E(t)$.

**(b)** Taking the Laplace transform of the system

$$\begin{aligned} 0.01 i_2' + 5i_2 + 5i_3 &= 100 \\ 0.0125 i_3' + 5i_2 + 5i_3 &= 100 \end{aligned}$$

gives

$$\begin{aligned} (s+500)\mathscr{L}\{i_2\} + 500\mathscr{L}\{i_3\} &= \frac{10{,}000}{s} \\ 400\mathscr{L}\{i_2\} + (s+400)\mathscr{L}\{i_3\} &= \frac{8{,}000}{s} \end{aligned}$$

so that

$$\mathscr{L}\{i_3\} = \frac{8{,}000}{s^2 + 900s} = \frac{80}{9}\frac{1}{s} - \frac{80}{9}\frac{1}{s+900}.$$

Then

$$i_3 = \frac{80}{9} - \frac{80}{9}e^{-900t} \qquad \text{and} \qquad i_2 = 20 - 0.0025 i_3' - i_3 = \frac{100}{9} - \frac{100}{9}e^{-900t}.$$

**(c)** $i_1 = i_2 + i_3 = 20 - 20e^{-900t}$

**16. (a)** Taking the Laplace transform of the system

$$\begin{aligned} i_2' + i_3' + 10i_2 &= 120 - 120\mathscr{U}(t-2) \\ -10i_2' + 5i_3' + 5i_3 &= 0 \end{aligned}$$

gives

$$\begin{aligned} (s+10)\mathscr{L}\{i_2\} + s\mathscr{L}\{i_3\} &= \frac{120}{s}\left(1 - e^{-2s}\right) \\ -10s\mathscr{L}\{i_2\} + 5(s+1)\mathscr{L}\{i_3\} &= 0 \end{aligned}$$

so that

$$\mathscr{L}\{i_2\} = \frac{120(s+1)}{(3s^2 + 11s + 10)s}\left(1 - e^{-2s}\right) = \left[\frac{48}{s+5/3} - \frac{60}{s+2} + \frac{12}{s}\right]\left(1 - e^{-2s}\right)$$

and

$$\mathscr{L}\{i_3\} = \frac{240}{3s^2 + 11s + 10}\left(1 - e^{-2s}\right) = \left[\frac{240}{s+5/3} - \frac{240}{s+2}\right]\left(1 - e^{-2s}\right).$$

Then

$$i_2 = 12 + 48e^{-5t/3} - 60e^{-2t} - \left[12 + 48e^{-5(t-2)/3} - 60e^{-2(t-2)}\right]\mathscr{U}(t-2)$$

and

$$i_3 = 240e^{-5t/3} - 240e^{-2t} - \left[240e^{-5(t-2)/3} - 240e^{-2(t-2)}\right]\mathscr{U}(t-2).$$

**(b)** $i_1 = i_2 + i_3 = 12 + 288e^{-5t/3} - 300e^{-2t} - \left[12 + 288e^{-5(t-2)/3} - 300e^{-2(t-2)}\right]\mathscr{U}(t-2)$

**17.** Taking the Laplace transform of the system

$$\begin{aligned} i_2' + 11i_2 + 6i_3 &= 50\sin t \\ i_3' + 6i_2 + 6i_3 &= 50\sin t \end{aligned}$$

gives

$$\begin{aligned} (s+11)\mathscr{L}\{i_2\} + 6\mathscr{L}\{i_3\} &= \frac{50}{s^2+1} \\ 6\mathscr{L}\{i_2\} + (s+6)\mathscr{L}\{i_3\} &= \frac{50}{s^2+1} \end{aligned}$$

so that

$$\mathscr{L}\{i_2\} = \frac{50s}{(s+2)(s+15)(s^2+1)} = -\frac{20}{13}\frac{1}{s+2} + \frac{375}{1469}\frac{1}{s+15} + \frac{145}{113}\frac{s}{s^2+1} + \frac{85}{113}\frac{1}{s^2+1}.$$

Then

$$i_2 = -\frac{20}{13}e^{-2t} + \frac{375}{1469}e^{-15t} + \frac{145}{113}\cos t + \frac{85}{113}\sin t$$

and

$$i_3 = \frac{25}{3}\sin t - \frac{1}{6}i_2' - \frac{11}{6}i_2 = \frac{30}{13}e^{-2t} + \frac{250}{1469}e^{-15t} - \frac{280}{113}\cos t + \frac{810}{113}\sin t.$$

**18.** Taking the Laplace transform of the system

$$\begin{aligned} 0.5i_1' + 50i_2 &= 60 \\ 0.005i_2' + i_2 - i_1 &= 0 \end{aligned}$$

gives

$$\begin{aligned} s\mathscr{L}\{i_1\} + 100\mathscr{L}\{i_2\} &= \frac{120}{s} \\ -200\mathscr{L}\{i_1\} + (s+200)\mathscr{L}\{i_2\} &= 0 \end{aligned}$$

so that

$$\mathscr{L}\{i_2\} = \frac{24{,}000}{s(s^2+200s+20{,}000)} = \frac{6}{5}\frac{1}{s} - \frac{6}{5}\frac{s+100}{(s+100)^2+100^2} - \frac{6}{5}\frac{100}{(s+100)^2+100^2}.$$

Then

$$i_2 = \frac{6}{5} - \frac{6}{5}e^{-100t}\cos 100t - \frac{6}{5}e^{-100t}\sin 100t$$

and

$$i_1 = 0.005i_2' + i_2 = \frac{6}{5} - \frac{6}{5}e^{-100t}\cos 100t.$$

**19.** Taking the Laplace transform of the system

$$\begin{aligned} 2i_1' + 50i_2 &= 60 \\ 0.005i_2' + i_2 - i_1 &= 0 \end{aligned}$$

gives

$$\begin{aligned} 2s\mathscr{L}\{i_1\} + 50\mathscr{L}\{i_2\} &= \frac{60}{s} \\ -200\mathscr{L}\{i_1\} + (s+200)\mathscr{L}\{i_2\} &= 0 \end{aligned}$$

so that

$$\begin{aligned} \mathscr{L}\{i_2\} &= \frac{6{,}000}{s(s^2+200s+5{,}000)} \\ &= \frac{6}{5}\,\frac{1}{s} - \frac{6}{5}\,\frac{s+100}{(s+100)^2-(50\sqrt{2}\,)^2} - \frac{6\sqrt{2}}{5}\,\frac{50\sqrt{2}}{(s+100)^2-(50\sqrt{2}\,)^2}\,. \end{aligned}$$

Then

$$i_2 = \frac{6}{5} - \frac{6}{5}e^{-100t}\cosh 50\sqrt{2}\,t - \frac{6\sqrt{2}}{5}e^{-100t}\sinh 50\sqrt{2}\,t$$

and

$$i_1 = 0.005i_2' + i_2 = \frac{6}{5} - \frac{6}{5}e^{-100t}\cosh 50\sqrt{2}\,t - \frac{9\sqrt{2}}{10}e^{-100t}\sinh 50\sqrt{2}\,t.$$

**20. (a)** Using Kirchhoff's first law we write $i_1 = i_2 + i_3$. Since $i_2 = dq/dt$ we have $i_1 - i_3 = dq/dt$. Using Kirchhoff's second law and summing the voltage drops across the shorter loop gives

$$E(t) = iR_1 + \frac{1}{C}q, \tag{1}$$

so that

$$i_1 = \frac{1}{R_1}E(t) - \frac{1}{R_1C}q.$$

Then

$$\frac{dq}{dt} = i_1 - i_3 = \frac{1}{R_1}E(t) - \frac{1}{R_1C}q - i_3$$

and

$$R_1\frac{dq}{dt} + \frac{1}{C}q + R_1i_3 = E(t).$$

Summing the voltage drops across the longer loop gives

$$E(t) = i_1R_1 + L\frac{di_3}{dt} + R_2i_3.$$

Combining this with (1) we obtain

$$i_1R_1 + L\frac{di_3}{dt} + R_2i_3 = i_1R_1 + \frac{1}{C}q$$

or

$$L\frac{di_3}{dt} + R_2i_3 - \frac{1}{C}q = 0.$$

**(b)** Using $L = R_1 = R_2 = C = 1$, $E(t) = 50e^{-t}\mathscr{U}(t-1) = 50e^{-1}e^{-(t-1)}\mathscr{U}(t-1)$, $q(0) = i_3(0) = 0$, and taking the Laplace transform of the system we obtain

$$\begin{aligned}(s+1)\mathscr{L}\{q\} + \mathscr{L}\{i_3\} &= \frac{50e^{-1}}{s+1}e^{-s}\\ (s+1)\mathscr{L}\{i_3\} - \mathscr{L}\{q\} &= 0,\end{aligned}$$

so that

$$\mathscr{L}\{q\} = \frac{50e^{-1}e^{-s}}{(s+1)^2+1}$$

and

$$q(t) = 50e^{-1}e^{-(t-1)}\sin(t-1)\mathscr{U}(t-1) = 50e^{-t}\sin(t-1)\mathscr{U}(t-1).$$

**21. (a)** Taking the Laplace transform of the system

$$\begin{aligned}4\theta_1'' + \theta_2'' + 8\theta_1 &= 0\\ \theta_1'' + \theta_2'' + 2\theta_2 &= 0\end{aligned}$$

gives

$$\begin{aligned}4\left(s^2+2\right)\mathscr{L}\{\theta_1\} + s^2\mathscr{L}\{\theta_2\} &= 3s\\ s^2\mathscr{L}\{\theta_1\} + \left(s^2+2\right)\mathscr{L}\{\theta_2\} &= 0\end{aligned}$$

so that

$$\left(3s^2+4\right)\left(s^2+4\right)\mathscr{L}\{\theta_2\} = -3s^3$$

or

$$\mathscr{L}\{\theta_2\} = \frac{1}{2}\,\frac{s}{s^2+4/3} - \frac{3}{2}\,\frac{s}{s^2+4}\,.$$

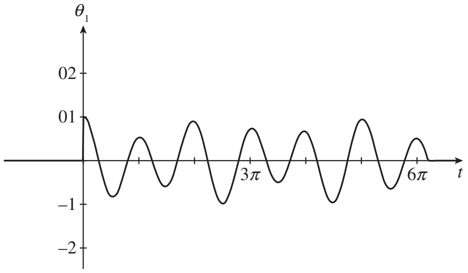
Then

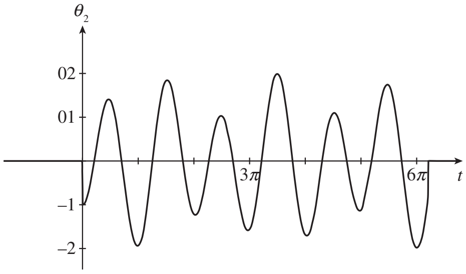
$$\theta_2 = \frac{1}{2}\cos\frac{2}{\sqrt{3}}t - \frac{3}{2}\cos 2t \qquad \text{and} \qquad \theta_1'' = -\theta_2'' - 2\theta_2$$

so that

$$\theta_1 = \frac{1}{4}\cos\frac{2}{\sqrt{3}}t + \frac{3}{4}\cos 2t.$$

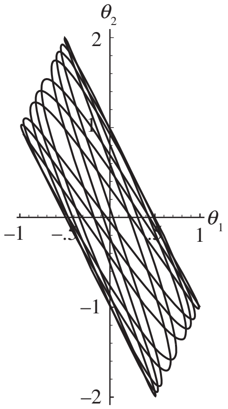
**(b)**





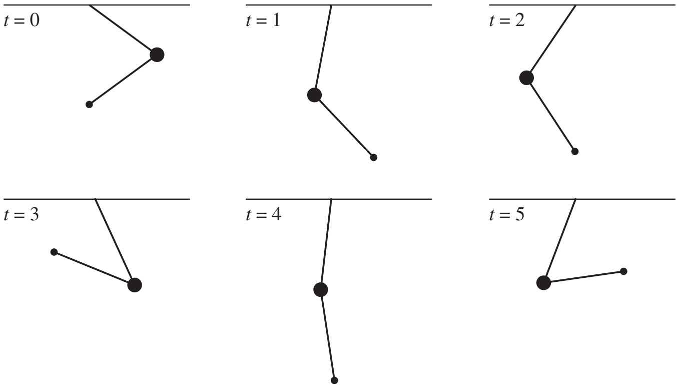
Mass $m_2$ has extreme displacements of greater magnitude. Mass $m_1$ first passes through its equilibrium position at about $t = 0.87$, and mass $m_2$ first passes through its equilibrium position at about $t = 0.66$. The motion of the pendulums is not periodic since $\cos(2t/\sqrt{3})$ has period $\sqrt{3}\,\pi$, $\cos 2t$ has period $\pi$, and the ratio of these periods is $\sqrt{3}$, which is not a rational number.

**(c)**

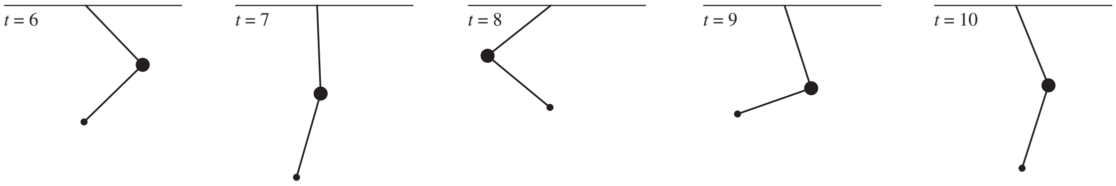


The Lissajous curve is plotted for $0 \le t \le 30$.

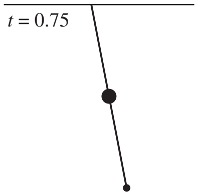
**(d)**



| $t$ | $\theta_1$ | $\theta_2$ |
|---|---|---|
| 1 | −0.2111 | 0.8263 |
| 2 | −0.6585 | 0.6438 |
| 3 | 0.4830 | −1.9145 |
| 4 | −0.1325 | 0.1715 |
| 5 | −0.4111 | 1.6951 |
| 6 | 0.8327 | −0.8662 |
| 7 | 0.0458 | −0.3186 |
| 8 | −0.9639 | 0.9452 |
| 9 | 0.3534 | −1.2741 |
| 10 | 0.4370 | −0.3502 |



**(e)** Using a CAS to solve $\theta_1(t) = \theta_2(t)$ we see that $\theta_1 = \theta_2$ (so that the double pendulum is straight out) when $t$ is about 0.75 seconds.



**(f)** To make a movie of the pendulum it is necessary to locate the mass in the plane as a function of time. Suppose that the upper arm is attached to the origin and that the equilibrium position lies along the negative $y$-axis. Then mass $m_1$ is at $(x,(t),y_1(t))$ and mass $m_2$ is at $(x_2(t),y_2(t))$, where

$$x_1(t)=16\sin\theta_1(t) \qquad \text{and} \qquad y_1(t)=-16\cos\theta_1(t)$$

and

$$x_2(t)=x_1(t)+16\sin\theta_2(t) \qquad \text{and} \qquad y_2(t)=y_1(t)-16\cos\theta_2(t).$$

A reasonable movie can be constructed by letting $t$ range from 0 to 10 in increments of 0.1 seconds.

## Chapter 7 in Review

**1.** $\mathscr{L}\{f(t)\}=\displaystyle\int_0^1 te^{-st}\,dt+\int_1^\infty (2-t)e^{-st}\,dt=\frac{1}{s^2}-\frac{2}{s^2}e^{-s}$

**2.** $\mathscr{L}\{f(t)\}=\displaystyle\int_2^4 e^{-st}\,dt=\frac{1}{s}\left(e^{-2s}-e^{-4s}\right)$

**3.** False; consider $f(t)=t^{-1/2}$.

**4.** False, since $f(t)=(e^t)^{10}=e^{10t}$.

**5.** True, since $\lim_{s\to\infty}F(s)=1\neq 0$. (See Theorem 7.2.3 in the text.)

**6.** False; consider $f(t)=1$ and $g(t)=1$.

**7.** $\mathscr{L}\left\{e^{-7t}\right\}=\dfrac{1}{s+7}$

**8.** $\mathscr{L}\left\{te^{-7t}\right\}=\dfrac{1}{(s+7)^2}$

**9.** $\mathscr{L}\{\sin 2t\}=\dfrac{2}{s^2+4}$

**10.** $\mathscr{L}\left\{e^{-3t}\sin 2t\right\}=\dfrac{2}{(s+3)^2+4}$

**11.** $\mathscr{L}\{t\sin 2t\}=-\dfrac{d}{ds}\left[\dfrac{2}{s^2+4}\right]=\dfrac{4s}{(s^2+4)^2}$

**12.** $\mathscr{L}\{\sin 2t\,\mathscr{U}(t-\pi)\}=\mathscr{L}\{\sin 2(t-\pi)\mathscr{U}(t-\pi)\}=\dfrac{2}{s^2+4}e^{-\pi s}$

**13.** $\mathscr{L}^{-1}\left\{\dfrac{20}{s^6}\right\}=\mathscr{L}^{-1}\left\{\dfrac{1}{6}\,\dfrac{5!}{s^6}\right\}=\dfrac{1}{6}t^5$

**14.** $\mathscr{L}^{-1}\left\{\dfrac{1}{3s-1}\right\}=\mathscr{L}^{-1}\left\{\dfrac{1}{3}\,\dfrac{1}{s-1/3}\right\}=\dfrac{1}{3}e^{t/3}$

**15.** $\mathscr{L}^{-1}\left\{\frac{1}{(s-5)^3}\right\} = \frac{1}{2}\mathscr{L}^{-1}\left\{\frac{2}{(s-5)^3}\right\} = \frac{1}{2}t^2e^{5t}$

**16.** $\mathscr{L}^{-1}\left\{\frac{1}{s^2-5}\right\} = \mathscr{L}^{-1}\left\{-\frac{1}{2\sqrt{5}}\,\frac{1}{s+\sqrt{5}} + \frac{1}{2\sqrt{5}}\,\frac{1}{s-\sqrt{5}}\right\} = -\frac{1}{2\sqrt{5}}e^{-\sqrt{5}\,t} + \frac{1}{2\sqrt{5}}e^{\sqrt{5}\,t}$

**17.** $\mathscr{L}^{-1}\left\{\frac{s}{s^2-10s+29}\right\} = \mathscr{L}^{-1}\left\{\frac{s-5}{(s-5)^2+2^2} + \frac{5}{2}\,\frac{2}{(s-5)^2+2^2}\right\} = e^{5t}\cos 2t + \frac{5}{2}e^{5t}\sin 2t$

**18.** $\mathscr{L}^{-1}\left\{\frac{1}{s^2}e^{-5s}\right\} = (t-5)\mathscr{U}(t-5)$

**19.**
$$\begin{aligned}\mathscr{L}^{-1}\left\{\frac{s+\pi}{s^2+\pi^2}\,e^{-s}\right\} &= \mathscr{L}^{-1}\left\{\frac{s}{s^2+\pi^2}\,e^{-s} + \frac{\pi}{s^2+\pi^2}\,e^{-s}\right\}\\ &= \cos\pi(t-1)\mathscr{U}(t-1) + \sin\pi(t-1)\mathscr{U}(t-1)\end{aligned}$$

**20.** $\mathscr{L}^{-1}\left\{\frac{1}{L^2s^2+n^2\pi^2}\right\} = \frac{1}{L^2}\,\frac{L}{n\pi}\mathscr{L}^{-1}\left\{\frac{n\pi/L}{s^2+(n^2\pi^2)/L^2}\right\} = \frac{1}{Ln\pi}\sin\frac{n\pi}{L}\,t$

**21.** $\mathscr{L}\left\{e^{-5t}\right\}$ exists for $s > -5$.

**22.** $\mathscr{L}\left\{te^{8t}f(t)\right\} = -\frac{d}{ds}F(s-8)$.

**23.** $\mathscr{L}\{e^{at}f(t-k)\mathscr{U}(t-k)\} = e^{-ks}\mathscr{L}\{e^{a(t+k)}f(t)\} = e^{-ks}e^{ak}\mathscr{L}\{e^{at}f(t)\} = e^{-k(s-a)}F(s-a)$

**24.** $\mathscr{L}\left\{\int_0^t e^{a\tau}f(\tau)\,d\tau\right\} = \frac{1}{s}\mathscr{L}\{e^{at}f(t)\} = \frac{F(s-a)}{s}$, whereas

$$\mathscr{L}\left\{e^{at}\int_0^t f(\tau)\,d\tau\right\} = \mathscr{L}\left\{\int_0^t f(\tau)\,d\tau\right\}\bigg|_{s\to s-a} = \frac{F(s)}{s}\bigg|_{s\to s-a} = \frac{F(s-a)}{s-a}.$$

**25.** $f(t)\mathscr{U}(t-t_0)$

**26.** $f(t) - f(t)\mathscr{U}(t-t_0)$

**27.** $f(t-t_0)\mathscr{U}(t-t_0)$

**28.** $f(t) - f(t)\mathscr{U}(t-t_0) + f(t)\mathscr{U}(t-t_1)$

**29.** $f(t) = t - [(t-1)+1]\mathscr{U}(t-1) + \mathscr{U}(t-1) - \mathscr{U}(t-4) = t - (t-1)\mathscr{U}(t-1) - \mathscr{U}(t-4)$

$$\mathscr{L}\{f(t)\} = \frac{1}{s^2} - \frac{1}{s^2}e^{-s} - \frac{1}{s}e^{-4s}$$

$$\mathscr{L}\left\{e^tf(t)\right\} = \frac{1}{(s-1)^2} - \frac{1}{(s-1)^2}e^{-(s-1)} - \frac{1}{s-1}e^{-4(s-1)}$$

**30.** $f(t) = \sin t\,\mathscr{U}(t-\pi) - \sin t\,\mathscr{U}(t-3\pi) = -\sin(t-\pi)\mathscr{U}(t-\pi) + \sin(t-3\pi)\mathscr{U}(t-3\pi)$

$$\mathscr{L}\{f(t)\} = -\frac{1}{s^2+1}e^{-\pi s} + \frac{1}{s^2+1}e^{-3\pi s}$$

$$\mathscr{L}\left\{e^tf(t)\right\} = -\frac{1}{(s-1)^2+1}e^{-\pi(s-1)} + \frac{1}{(s-1)^2+1}e^{-3\pi(s-1)}$$

**31.** $f(t) = 2 - 2\mathscr{U}(t-2) + [(t-2)+2]\mathscr{U}(t-2) = 2 + (t-2)\mathscr{U}(t-2)$

$$\mathscr{L}\{f(t)\} = \frac{2}{s} + \frac{1}{s^2}e^{-2s}$$

$$\mathscr{L}\left\{e^t f(t)\right\} = \frac{2}{s-1} + \frac{1}{(s-1)^2}e^{-2(s-1)}$$

**32.** $f(t) = t - t\mathscr{U}(t-1) + (2-t)\mathscr{U}(t-1) - (2-t)\mathscr{U}(t-2) = t - 2(t-1)\mathscr{U}(t-1) + (t-2)\mathscr{U}(t-2)$

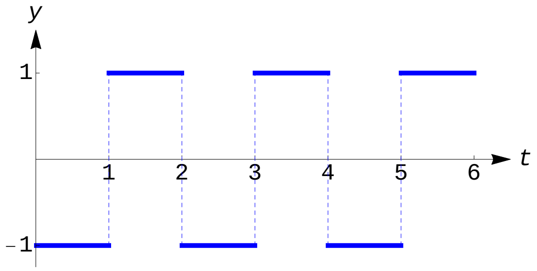
$$\mathscr{L}\{f(t)\} = \frac{1}{s^2} - \frac{2}{s^2}e^{-s} + \frac{1}{s^2}e^{-2s}$$

$$\mathscr{L}\left\{e^t f(t)\right\} = \frac{1}{(s-1)^2} - \frac{2}{(s-1)^2}e^{-(s-1)} + \frac{1}{(s-1)^2}e^{-2(s-1)}$$

**33.** The graph of

$$f(t) = -1 + 2\sum_{k=1}^{\infty}(-1)^{k+1}\,\mathscr{U}(t-k) = -1 + 2\mathscr{U}(t-1) - 2\mathscr{U}(t-2) + 2\mathscr{U}(t-3) - \cdots$$

is



One way of proceeding to find the Laplace transform is to take the transform term-by-term of the series:

$$\mathscr{L}\{f(t)\} = -\frac{1}{s} + \frac{2}{s}e^{-s} - \frac{2}{s}e^{-2s} + \frac{2}{s}e^{-3s} - \cdots \quad \longleftarrow \quad \text{geometric series}$$

For $s > 0$,

$$\mathscr{L}\{f(t)\} = -\frac{1}{s} + \frac{2}{s}\left[e^{-s} - e^{-2s} + e^{-3s} - \cdots\right] = -\frac{1}{s} + \frac{2}{s}\cdot\frac{e^{-s}}{1+e^{-s}}$$

$$= \frac{e^{-s}-1}{s\left(1+e^{-s}\right)}$$

Alternatively, since $f$ is a periodic functions it can also be defined by

$$f(t) = \begin{cases} -1, & 0 \le t < 1 \\ 1, & 1 \le t < 2, \end{cases} \quad \text{where } f(t+2) = f(t).$$

By Theorem 7.4.3 with $p = 2$ we get

$$\begin{aligned}\mathscr{L}\{f(t)\} &= \frac{1}{1-e^{-2s}}\left(\int_0^1 (-1)\,e^{-st}\,dt + \int_1^2 (1)\,e^{-st}\,dt\right)\\ &= \frac{1}{1-e^{-2s}}\left(\frac{1}{s}\,e^{-s} - \frac{1}{s} - \frac{1}{s}\,e^{-2s} + \frac{1}{s}\,e^{-s}\right) = \frac{-1}{s\,(1-e^{-2s})}\left(1 - 2e^{-s} + e^{-2s}\right)\\ &= \frac{-\left(1-e^{-s}\right)^2}{s\,(1-e^{-2s})}\end{aligned}$$

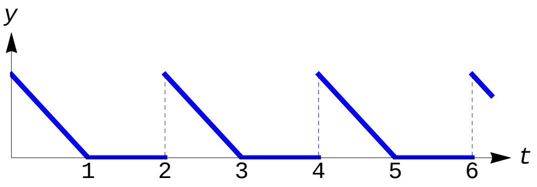
Using $1 - e^{-2s} = (1 + e^{-s})\,(1 - e^{-s})$ and algebra the last expression is the same as

$$\mathscr{L}\{f(t)\} = \frac{e^{-s}-1}{s\,(1+e^{-s})}\,.$$

**34.** The graph of

$$\begin{aligned}f(t) &= \sum_{k=0}^{\infty} (2k+1-t)\,[\mathscr{U}(t-2k) - \mathscr{U}(t-2k-1)]\\ &= (1-t)\,[\mathscr{U}(t) - \mathscr{U}(t-1)] + (3-t)\,[\mathscr{U}(t-2) - \mathscr{U}(t-3)] + \cdots\end{aligned}$$

is



Since $f$ is a periodic function it can also be defined by

$$f(t) = \begin{cases} 1-t, & 0 \le t < 1\\ 0, & 1 \le t < 2\end{cases} \quad \text{where } f(t+2) = f(t).$$

By Theorem 7.4.3 with $p = 2$ we get

$$\begin{aligned}\mathscr{L}\{f(t)\} &= \frac{1}{1-e^{-2s}}\left(\int_0^1 (1-t)\,e^{-st}\,dt + \int_1^2 0\cdot e^{-st}\,dt\right) = \frac{1}{1-e^{-2s}}\left(\frac{1}{s} + \frac{1}{s^2}\,e^{-s} - \frac{1}{s^2}\right)\\ &= \frac{s-1+e^{-s}}{s^2\,(1-e^{-2s})}\,.\end{aligned}$$

**35.** Taking the Laplace transform of the differential equation we obtain

$$\mathscr{L}\{y\} = \frac{5}{(s-1)^2} + \frac{1}{2}\,\frac{2}{(s-1)^3}$$

so that

$$y = 5te^t + \frac{1}{2}t^2e^t.$$

**36.** Taking the Laplace transform of the differential equation we obtain

$$\begin{aligned}\mathscr{L}\{y\} &= \frac{1}{(s-1)^2(s^2-8s+20)}\\ &= \frac{6}{169}\,\frac{1}{s-1} + \frac{1}{13}\,\frac{1}{(s-1)^2} - \frac{6}{169}\,\frac{s-4}{(s-4)^2+2^2} + \frac{5}{338}\,\frac{2}{(s-4)^2+2^2}\end{aligned}$$

so that

$$y = \frac{6}{169}e^t + \frac{1}{13}te^t - \frac{6}{169}e^{4t}\cos 2t + \frac{5}{338}e^{4t}\sin 2t.$$

**37.** Taking the Laplace transform of the given differential equation we obtain

$$\begin{aligned}\mathscr{L}\{y\} &= \frac{s^3+6s^2+1}{s^2(s+1)(s+5)} - \frac{1}{s^2(s+1)(s+5)}\,e^{-2s} - \frac{2}{s(s+1)(s+5)}\,e^{-2s}\\ &= -\frac{6}{25}\cdot\frac{1}{s} + \frac{1}{5}\cdot\frac{1}{s^2} + \frac{3}{2}\cdot\frac{1}{s+1} - \frac{13}{50}\cdot\frac{1}{s+5}\\ &\qquad - \left(-\frac{6}{25}\cdot\frac{1}{s} + \frac{1}{5}\cdot\frac{1}{s^2} + \frac{1}{4}\cdot\frac{1}{s+1} - \frac{1}{100}\cdot\frac{1}{s+5}\right)e^{-2s}\\ &\qquad - \left(\frac{2}{5}\cdot\frac{1}{s} - \frac{1}{2}\cdot\frac{1}{s+1} + \frac{1}{10}\cdot\frac{1}{s+5}\right)e^{-2s}\end{aligned}$$

so that

$$\begin{aligned}y = -\frac{6}{25} + \frac{1}{5}t + \frac{3}{2}e^{-t} - \frac{13}{50}e^{-5t} - \frac{4}{25}\mathscr{U}(t-2) - \frac{1}{5}(t-2)\mathscr{U}(t-2)\\ + \frac{1}{4}e^{-(t-2)}\mathscr{U}(t-2) - \frac{9}{100}\,e^{-5(t-2)}\mathscr{U}(t-2).\end{aligned}$$

**38.** Taking the Laplace transform of the differential equation we obtain

$$\begin{aligned}\mathscr{L}\{y\} &= \frac{s^3+2}{s^3(s-5)} - \frac{2+2s+s^2}{s^3(s-5)}e^{-s}\\ &= -\frac{2}{125}\,\frac{1}{s} - \frac{2}{25}\,\frac{1}{s^2} - \frac{1}{5}\,\frac{2}{s^3} + \frac{127}{125}\,\frac{1}{s-5} - \left[-\frac{37}{125}\,\frac{1}{s} - \frac{12}{25}\,\frac{1}{s^2} - \frac{1}{5}\,\frac{2}{s^3} + \frac{37}{125}\,\frac{1}{s-5}\right]e^{-s}\end{aligned}$$

so that

$$y = -\frac{2}{125} - \frac{2}{25}t - \frac{1}{5}t^2 + \frac{127}{125}e^{5t} - \left[-\frac{37}{125} - \frac{12}{25}(t-1) - \frac{1}{5}(t-1)^2 + \frac{37}{125}e^{5(t-1)}\right]\mathscr{U}(t-1).$$

**39.** The function in Figure 7.R.10 is

$$f(t) = \begin{cases} 0, & 0 \le t < 1\\ t-1, & 1 \le t < 2\\ 3-t, & 2 \le t < 3\\ 0, & t \ge 3\end{cases}$$

or

$$f(t) = (t-1)\mathscr{U}(t-1) - 2(t-2)\mathscr{U}(t-2) + (t-3)\mathscr{U}(t-3).$$

The transform of the differential equation is

$$sY(s) - 1 + 2Y(s) = \frac{e^{-s}}{s^2} - \frac{2e^{-2s}}{s^2} + \frac{e^{-3s}}{s^2}$$

so

$$Y(s) = \frac{1}{s+2} + \frac{1}{s^2(s+2)}\, e^{-s} - \frac{2}{s^2(s+2)}\, e^{-2s} + \frac{1}{s^2(s+2)}\, e^{-3s},$$

and

$$\begin{aligned} y(t) = e^{-2t} &+ \left[-\frac{1}{4} + \frac{1}{2}(t-1) + \frac{1}{4}\, e^{-2(t-1)}\right]\mathscr{U}(t-1) \\ &- 2\left[-\frac{1}{4} + \frac{1}{2}(t-2) + \frac{1}{4}\, e^{-2(t-2)}\right]\mathscr{U}(t-2) \\ &+ \left[-\frac{1}{4} + \frac{1}{2}(t-3) + \frac{1}{4}\, e^{-2(t-3)}\right]\mathscr{U}(t-3). \end{aligned}$$

**40.** The tranform of the differential equation is

$$s^2Y(s) - 3 + 5sY(s) + 4Y(s) = 12\sum_{k=0}^{\infty}(-1)^k\frac{e^{-ks}}{s}$$

so

$$\left(s^2 + 5s + 4\right)Y(s) = 3 + 12\sum_{k=0}^{\infty}(-1)^k\frac{e^{-ks}}{s}$$

and

$$Y(s) = \frac{3}{s^2+5s+4} + \sum_{k=0}^{\infty}(-1)^k\frac{12}{s\left(s^2+5s+4\right)}\, e^{-ks}.$$

Thus

$$Y(s) = \left[\frac{1}{s+1} - \frac{1}{s+4}\right] + \sum_{k=0}^{\infty}(-1)^k\left[\frac{3}{s} - \frac{4}{s+1} + \frac{1}{s-4}\right]e^{-ks}$$

and

$$y(t) = e^{-t} - e^{-4t} + \sum_{k=0}^{\infty}(-1)^k\left(3 - 4e^{-(t-k)} + e^{-4(t-k)}\right)\mathscr{U}(t-k).$$

**41.** Taking the Laplace transform of the integral equation we obtain

$$\mathscr{L}\{y\} = \frac{1}{s} + \frac{1}{s^2} + \frac{1}{2}\,\frac{2}{s^3}$$

so that

$$y(t) = 1 + t + \frac{1}{2}t^2.$$

**42.** Taking the Laplace transform of the integral equation we obtain

$$(\mathscr{L}\{f\})^2 = 6 \cdot \frac{6}{s^4} \quad \text{or} \quad \mathscr{L}\{f\} = \pm 6 \cdot \frac{1}{s^2}$$

so that $f(t) = \pm 6t$.

**43.** Taking the Laplace transform of the system gives

$$\begin{aligned} s\mathscr{L}\{x\} + \mathscr{L}\{y\} &= \frac{1}{s^2} + 1 \\ 4\mathscr{L}\{x\} + s\mathscr{L}\{y\} &= 2 \end{aligned}$$

so that

$$\mathscr{L}\{x\} = \frac{s^2 - 2s + 1}{s(s-2)(s+2)} = -\frac{1}{4}\frac{1}{s} + \frac{1}{8}\frac{1}{s-2} + \frac{9}{8}\frac{1}{s+2}\,.$$

Then

$$x = -\frac{1}{4} + \frac{1}{8}e^{2t} + \frac{9}{8}e^{-2t} \quad \text{and} \quad y = -x' + t = \frac{9}{4}e^{-2t} - \frac{1}{4}e^{2t} + t.$$

**44.** Taking the Laplace transform of the system gives

$$\begin{aligned} s^2\mathscr{L}\{x\} + s^2\mathscr{L}\{y\} &= \frac{1}{s-2} \\ 2s\mathscr{L}\{x\} + s^2\mathscr{L}\{y\} &= -\frac{1}{s-2} \end{aligned}$$

so that

$$\mathscr{L}\{x\} = \frac{2}{s(s-2)^2} = \frac{1}{2}\frac{1}{s} - \frac{1}{2}\frac{1}{s-2} + \frac{1}{(s-2)^2}$$

and

$$\mathscr{L}\{y\} = \frac{-s-2}{s^2(s-2)^2} = -\frac{3}{4}\frac{1}{s} - \frac{1}{2}\frac{1}{s^2} + \frac{3}{4}\frac{1}{s-2} - \frac{1}{(s-2)^2}\,.$$

Then

$$x = \frac{1}{2} - \frac{1}{2}e^{2t} + te^{2t} \quad \text{and} \quad y = -\frac{3}{4} - \frac{1}{2}t + \frac{3}{4}e^{2t} - te^{2t}.$$

**45.** The integral equation is

$$10i + 2\int_0^t i(\tau)\,d\tau = 2t^2 + 2t.$$

Taking the Laplace transform we obtain

$$\mathscr{L}\{i\} = \left(\frac{4}{s^3} + \frac{2}{s^2}\right)\frac{s}{10s+2} = \frac{s+2}{s^2(5s+2)} = -\frac{9}{s} + \frac{2}{s^2} + \frac{45}{5s+1} = -\frac{9}{s} + \frac{2}{s^2} + \frac{9}{s+1/5}\,.$$

Thus

$$i(t) = -9 + 2t + 9e^{-t/5}.$$

**46.** The differential equation is

$$\frac{1}{2}\frac{d^2q}{dt^2} + 10\frac{dq}{dt} + 100q = 10 - 10\mathscr{U}(t-5).$$

Taking the Laplace transform we obtain

$$\begin{aligned}\mathscr{L}\{q\} &= \frac{20}{s(s^2+20s+200)}\left(1-e^{-5s}\right)\\ &= \left[\frac{1}{10}\frac{1}{s} - \frac{1}{10}\frac{s+10}{(s+10)^2+10^2} - \frac{1}{10}\frac{10}{(s+10)^2+10^2}\right]\left(1-e^{-5s}\right)\end{aligned}$$

so that

$$\begin{aligned}q(t) &= \frac{1}{10} - \frac{1}{10}e^{-10t}\cos 10t - \frac{1}{10}e^{-10t}\sin 10t\\ &\quad - \left[\frac{1}{10} - \frac{1}{10}e^{-10(t-5)}\cos 10(t-5) - \frac{1}{10}e^{-10(t-5)}\sin 10(t-5)\right]\mathscr{U}(t-5).\end{aligned}$$

**47.** Taking the Laplace transform of the given differential equation we obtain

$$\mathscr{L}\{y\} = \frac{2w_0}{EIL}\left(\frac{L}{48}\cdot\frac{4!}{s^5} - \frac{1}{120}\cdot\frac{5!}{s^6} + \frac{1}{120}\cdot\frac{5!}{s^6}e^{-sL/2}\right) + \frac{c_1}{2}\cdot\frac{2!}{s^3} + \frac{c_2}{6}\cdot\frac{3!}{s^4}$$

so that

$$y = \frac{2w_0}{EIL}\left[\frac{L}{48}x^4 - \frac{1}{120}x^5 + \frac{1}{120}\left(x-\frac{L}{2}\right)^5\mathscr{U}\left(x-\frac{L}{2}\right) + \frac{c_1}{2}x^2 + \frac{c_2}{6}x^3\right]$$

where $y''(0)=c_1$ and $y'''(0)=c_2$. Using $y''(L)=0$ and $y'''(L)=0$ we find

$$c_1 = w_0L^2/24EI, \qquad c_2 = -w_0L/4EI.$$

Hence

$$y = \frac{w_0}{12EIL}\left[-\frac{1}{5}x^5 + \frac{L}{2}x^4 - \frac{L^2}{2}x^3 + \frac{L^3}{4}x^2 + \frac{1}{5}\left(x-\frac{L}{2}\right)^5\mathscr{U}\left(x-\frac{L}{2}\right)\right].$$

**48. (a)** In this case the boundary conditions are $y(0)=0$, $y''(0)=0$, $y(\pi)=0$, and $y''(\pi)=0$.

$$\mathscr{L}\left\{y^{(4)}\right\} + 4\mathscr{L}\{y\} = \mathscr{L}\left\{\frac{w_0}{EI}\right\}$$

$$s^4Y(s) - s^3\overbrace{y(0)}^{0} - s^2\overbrace{y'(0)}^{c_1} - s\overbrace{y''(0)}^{0} - \overbrace{y'''(0)}^{c_2} + 4Y(s) = \frac{w_0}{sEI}$$

$$\left(s^4+4\right)Y(s) = c_1s^2 + c_2 + \frac{w_0}{sEI}$$

Thus

$$\begin{aligned}
Y(s) &= c_1 \frac{s^2}{s^4+4} + c_2 \frac{1}{s^4+4} + \frac{w_0/EI}{s\left(s^4+4\right)} \\
&= c_1 \frac{s^2}{s^4+4} + c_2 \frac{1}{s^4+4} + \frac{w_0/EI}{s\left(s^2-2s+2\right)\left(s^2+2s+2\right)} \\
&= c_1 \frac{s^2}{s^4+4} + c_2 \frac{1}{s^4+4} + \frac{w_0}{8EI}\left[\frac{2}{s} - \frac{s-1}{s^2-2s+2} - \frac{s+1}{s^2+2s+2}\right] \\
&= \frac{c_1}{2}\frac{2s^2}{s^4+4} + \frac{c_2}{4}\frac{4}{s^4+4} + \frac{w_0}{8EI}\left[\frac{2}{s} - \frac{s-1}{(s-1)^2+1} - \frac{s+1}{(s+1)^2+1}\right]
\end{aligned}$$

Using the table in Appendix III the inverse transform is

$$\begin{aligned}
y(x) = \frac{c_1}{2}&\left(\sin x \cosh x + \cos x \sinh x\right) + \frac{c_2}{4}\left(\sin x \cosh x - \cos x \sinh x\right) \\
&+ \frac{w_0}{8EI}\left[2 - e^{x}\cos x - e^{-x}\cos x\right] \\
= \frac{c_1}{2}&\left(\sin x \cosh x + \cos x \sinh x\right) + \frac{c_2}{4}\left(\sin x \cosh x - \cos x \sinh x\right) + \frac{w_0}{4EI}\left[1 - \cos x \cosh x\right]
\end{aligned}$$

The remaining conditions $y(\pi) = 0$ and $y''(0) = 0$ then give

$$c_1 = \frac{w_0}{4EI}\left(1+\cosh\pi\right)\operatorname{csch}\pi, \quad \text{and} \quad c_2 = -\frac{w_0}{2EI}\left(1+\cosh\pi\right)\operatorname{csch}\pi$$

Therefore,

$$\begin{aligned}
y(x) = &\frac{w_0}{8EI}\left(1+\cosh\pi\right)\operatorname{csch}\pi\left(\sin x \cosh x + \cos x \sinh x\right) \\
&- \frac{w_0}{8EI}\left(1+\cosh\pi\right)\operatorname{csch}\pi\left(\sin x \cosh x - \cos x \sinh x\right) + \frac{w_0}{4EI}\left[1 - \cos x \cosh x\right]
\end{aligned}$$

**(b)** In this case the boundary conditions are $y(0) = y'(0) = 0$ and $y(\pi) = y'(\pi) = 0$. If we let $c_1 = y''(0)$ and $c_2 = y'''(0)$ then

$$s^4\mathscr{L}\{y\} - sy(0) - s^2y'(0) - s^3y''(0) - y'''(0) + 4\mathscr{L}\{y\} = \frac{w_0}{EI}\mathscr{L}\left\{\delta\left(x-\frac{\pi}{2}\right)\right\}$$

and

$$\mathscr{L}\{y\} = \frac{c_1}{2}\cdot\frac{2s}{s^4+4} + \frac{c_2}{4}\cdot\frac{4}{s^4+4} + \frac{w_0}{4EI}\cdot\frac{4}{s^4+4}e^{-s\pi/2}.$$

From the table of transforms we get

$$\begin{aligned}
y = &\frac{c_1}{2}\sin x \sinh x + \frac{c_2}{4}(\sin x \cosh x - \cos x \sinh x) \\
&+ \frac{w_0}{4EI}\left[\sin\left(x-\frac{\pi}{2}\right)\cosh\left(x-\frac{\pi}{2}\right) - \cos\left(x-\frac{\pi}{2}\right)\sinh\left(x-\frac{\pi}{2}\right)\right]\mathscr{U}\left(x-\frac{\pi}{2}\right)
\end{aligned}$$

Using $y(\pi) = 0$ and $y'(\pi) = 0$ we find

$$c_1 = \frac{w_0}{EI}\frac{\sinh\frac{\pi}{2}}{\sinh\pi}, \qquad c_2 = -\frac{w_0}{EI}\frac{\cosh\frac{\pi}{2}}{\sinh\pi}.$$

Hence

$$y = \frac{w_0}{2EI}\frac{\sinh\frac{\pi}{2}}{\sinh\pi}\sin x\sinh x - \frac{w_0}{4EI}\frac{\cosh\frac{\pi}{2}}{\sinh\pi}(\sin x\cosh x - \cos x\sinh x)$$
$$+\frac{w_0}{4EI}\left[\sin\left(x-\frac{\pi}{2}\right)\cosh\left(x-\frac{\pi}{2}\right) - \cos\left(x-\frac{\pi}{2}\right)\sinh\left(x-\frac{\pi}{2}\right)\right]\mathscr{U}\left(x-\frac{\pi}{2}\right).$$

**49. (a)** With $\omega^2 = g/l$ and $K = k/m$ the system of differential equations is

$$\theta_1'' + \omega^2\theta_1 = -K(\theta_1 - \theta_2)$$
$$\theta_2'' + \omega^2\theta_2 = K(\theta_1 - \theta_2).$$

Denoting the Laplace transform of $\theta(t)$ by $\Theta(s)$ we have that the Laplace transform of the system is

$$(s^2+\omega^2)\Theta_1(s) = -K\Theta_1(s) + K\Theta_2(s) + s\theta_0$$
$$(s^2+\omega^2)\Theta_2(s) = K\Theta_1(s) - K\Theta_2(s) + s\psi_0.$$

If we add the two equations, we get

$$\Theta_1(s) + \Theta_2(s) = (\theta_0 + \psi_0)\frac{s}{s^2+\omega^2}$$

which implies

$$\theta_1(t) + \theta_2(t) = (\theta_0 + \psi_0)\cos\omega t.$$

This enables us to solve for first, say, $\theta_1(t)$ and then find $\theta_2(t)$ from

$$\theta_2(t) = -\theta_1(t) + (\theta_0 + \psi_0)\cos\omega t.$$

Now solving

$$(s^2+\omega^2+K)\Theta_1(s) - K\Theta_2(s) = s\theta_0$$
$$-k\Theta_1(s) + (s^2+\omega^2+K)\Theta_2(s) = s\psi_0$$

gives

$$[(s^2+\omega^2+K)^2 - K^2]\Theta_1(s) = s(s^2+\omega^2+K)\theta_0 + Ks\psi_0.$$

Factoring the difference of two squares and using partial fractions we get

$$\Theta_1(s) = \frac{s(s^2+\omega^2+K)\theta_0 + Ks\psi_0}{(s^2+\omega^2)(s^2+\omega^2+2K)} = \frac{\theta_0+\psi_0}{2}\frac{s}{s^2+\omega^2} + \frac{\theta_0-\psi_0}{2}\frac{s}{s^2+\omega^2+2K},$$

so

$$\theta_1(t) = \frac{\theta_0+\psi_0}{2}\cos\omega t + \frac{\theta_0-\psi_0}{2}\cos\sqrt{\omega^2+2K}\,t.$$

Then from $\theta_2(t) = -\theta_1(t) + (\theta_0+\psi_0)\cos\omega t$ we get

$$\theta_2(t) = \frac{\theta_0+\psi_0}{2}\cos\omega t - \frac{\theta_0-\psi_0}{2}\cos\sqrt{\omega^2+2K}\,t.$$

**(b)** With the initial conditions $\theta_1(0)=\theta_0$, $\theta_1'(0)=0$, $\theta_2(0)=\theta_0$, $\theta_2'(0)=0$ we have

$$\theta_1(t)=\theta_0\cos\omega t, \qquad \theta_2(t)=\theta_0\cos\omega t.$$

Physically this means that both pendulums swing in the same direction as if they were free since the spring exerts no influence on the motion ($\theta_1(t)$ and $\theta_2(t)$ are free of $K$). With the initial conditions $\theta_1(0)=\theta_0$, $\theta_1'(0)=0$, $\theta_2(0)=-\theta_0$, $\theta_2'(0)=0$ we have

$$\theta_1(t)=\theta_0\cos\sqrt{\omega^2+2K}\,t, \qquad \theta_2(t)=-\theta_0\cos\sqrt{\omega^2+2K}\,t.$$

Physically this means that both pendulums swing in the opposite directions, stretching and compressing the spring. The amplitude of both displacements is $|\theta_0|$. Moreover, $\theta_1(t)=\theta_0$ and $\theta_2(t)=-\theta_0$ at precisely the same times. At these times the spring is stretched to its maximum.

**50.** **(a)** We will find the first two times for which $x'(t)=0$ and then ontain the rest of the times using periodicity of $x'(t)$. The solution of

$$x''+\omega^2x=F, \quad x(0)=x_0, \quad x'(0)=0$$

is

$$x(t)=\left(x_0-\frac{F}{\omega^2}\right)\cos\omega t+\frac{F}{\omega^2} \qquad \text{so} \qquad x'(t)=\left(x_0-\frac{F}{\omega^2}\right)(-\omega\sin\omega t)\,.$$

The latter equation is 0 when $t=\pi/\omega$. The next initial-value problem is then

$$x''+\omega^2x=-F, \quad x\left(\frac{\pi}{\omega}\right)=\frac{2F}{\omega^2}-x_0, \quad x'\left(\frac{\pi}{\omega}\right)=0,$$

where $\pi/\omega=T/2$. From the solution of this problem,

$$x(t)=\left(x_0-\frac{3F}{\omega^2}\right)\cos\omega t-\frac{F}{\omega^2},$$

we see from

$$x'(t)=\left(x_0-\frac{3F}{\omega^2}\right)(-\omega\sin\omega t)=0$$

that $t=2\pi/\omega=T$. Since $x'(t)$ has period $T=2\pi/\omega$ we can see that $x'(t)=0$ at the times

$$\frac{\pi}{\omega}+\frac{2\pi}{\omega}k \qquad \text{and} \qquad \frac{2\pi}{\omega}+\frac{2pi}{\omega}k, \qquad \text{for } k=0,1,2,\ldots$$

which are

$$0,\frac{\pi}{\omega},\frac{2\pi}{\omega},\frac{3\pi}{\omega},\ldots \quad \text{or} \quad 0,\frac{T}{2},T,\frac{3T}{2},\ldots$$

**(b)** There is no motin unless the intial displacement is such that the force of the spring is greater than the force due to friction. That is,

$$k|x_0|>f_k \quad \text{or} \quad \frac{k}{m}|x_0|>\frac{f_k}{m} \quad \text{or} \quad \omega^2|x_0|>F.$$

**(c)** From part (b), an intial displacement $x(0) = x_0$ for which $|x_0| > F/\omega^2$ will result in motion. On the other hand, an initial displacement $x_0$ for which $|x_0| \le F/\omega^2$ will be insufficient to overcome the force of friction and the system will be "dead".

**(d)** The system can be described by the initial-value problem

$$x'' + \omega^2 x = g(t), \quad x(0) = x_0, \quad x'(0) = 0,$$

where $g$ is a version of the meander function shown in Figure 7.4.6 in the text. In this case the amplitude of the function is $F$ instead of 1, and the length of each line segment is $T/2$ rather than $a$. Then

$$\mathscr{L}\left\{x''\right\} + \omega^2\alpha\left\{x\right\} = \alpha\left\{g\right\},$$
$$s^2X(s) - sx_0 + \omega^2X(s) = G(s),$$

and

$$X(s) = \frac{sx_0}{s^2 + w^2} + \frac{1}{s^2+\omega^2}G(s).$$

Now, using Problem 49 in Section 7.4 with $F$ instead of 1 and $a = T/2$, we have

$$\mathscr{L}\left\{g(t)\right\} = G(s) = \frac{F}{s}\,\frac{1-e^{-sT/2}}{1+e^{-sT/2}} = \frac{F}{s}\left[1 - 2e^{-sT/2} + 2e^{-sT} - 2e^{-3sT/2} + \cdots\right].$$

Then

$$\begin{aligned} X(s) &= x_0\,\frac{s}{s^2+\omega^2} + \frac{F}{s}\,\frac{1}{s^2+\omega^2}\left[1 - 2e^{-sT/2} + 2e^{-sT} - 2e^{-3sT/2} + \cdots\right] \\ &= x_0\,\frac{s}{s^2+\omega^2} + \frac{F}{\omega^2}\left[\frac{1}{s} - \frac{s}{s^2+\omega^2}\right]\left[1 - 2e^{-sT/2} + 2e^{-sT} - 2e^{-3sT/2} + \cdots\right] \end{aligned}$$

and

$$\begin{aligned} x(t) = x_0\cos\omega t &+ \frac{F}{\omega}(1-\cos\omega t) - \frac{2F}{\omega^2}\left[\mathscr{U}\left(t-\frac{T}{2}\right) - \cos\omega\left(t-\frac{T}{2}\right)\mathscr{U}\left(t-\frac{T}{2}\right)\right] \\ &+ \frac{2F}{\omega^2}\left[\mathscr{U}(t-T) - \cos\omega(t-T)\mathscr{U}(t-T)\right] \\ &- \frac{2F}{\omega^2}\left[\mathscr{U}\left(t-\frac{3T}{2}\right) - \cos\omega\left(t-\frac{2T}{2}\right)\mathscr{U}\left(t-\frac{3T}{2}\right)\right] + \cdots \end{aligned}$$

or

$$x(t) = \begin{cases} x_0\cos\omega t + \dfrac{F}{\omega^2}(1-\cos\omega t), & 0 \le t < T/2 \\[2ex] x_0\cos\omega t + \dfrac{F}{\omega^2}(1-\cos\omega t) - \dfrac{2F}{\omega^2}\left[1-\cos\omega\left(t-\dfrac{T}{2}\right)\right], & T/2 \le t < T \\[2ex] x_0\cos\omega t + \dfrac{F}{\omega^2}(1-\cos\omega t) - \dfrac{2F}{\omega^2}\left[1-\cos\omega\left(t-\dfrac{T}{2}\right)\right] & \\ \qquad\qquad + \dfrac{2F}{\omega^2}\left[1-\cos\omega(t-T)\right], & T \le t < 3T/2 \\ \qquad\vdots & \end{cases}$$

**(e)** The solutions from Problem 28 in Chapter 5 in Review are

$$x(t) = \begin{cases} 4.5\cos t + 1, & 0 \le t < \pi \\ 2.5\cos t - 1, & \pi \le t < 2\pi. \end{cases}$$

On the interval $[0, 2\pi)$ the solution is

$$x(t) = \begin{cases} x_0 \cos\omega t + \dfrac{F}{\omega^2}(1 - \cos\omega t), & 0 \le t < T/2 \\ x_0 \cos\omega t + \dfrac{F}{\omega^2}(1 - \cos\omega t) - \dfrac{2F}{\omega^2}\left[1 - \cos\omega\left(t - \dfrac{T}{2}\right)\right], & T/2 \le t < T. \end{cases}$$

We let $m = 1$, $k = 1$, $f_k = 1$, and $x_0 = 5.5$. Then, when $T = 2\pi$, the foregoing becomes

$$x(t) = \begin{cases} 5.5\cos t + (1 - \cos t), & 0 \le t < \pi \\ 5.5\cos t + (1 - \cos t) - 2[1 - \cos(t - \pi)], & \pi \le t < 2\pi. \end{cases}$$

Simplifying this we get

$$x(t) = \begin{cases} 4.5\cos t + 1, & 0 \le t < \pi \\ 2.5\cos t - 1, & \pi \le t2\pi. \end{cases}$$

**(f)** At $t = T/2$, $x(T/2) = -x_0 + 2F/\omega^2$. At $t = T$,

$$x(T) = x_0 - \frac{4F}{\omega^2} = \left(x_0 - \frac{2F}{\omega^2}\right) - \frac{2F}{\omega^2} = -x\left(\frac{T}{2}\right) - \frac{2F}{\omega^2}.$$

At $t = 3T/2$, $x(3T/2) = -x_0 + 6F/\omega^2 = -x(T) + 2F/\omega^2$. We see in general that each successive oscillation is $2F/\omega^2$ shorter than the preceding one.

**(g)** The system will stay in motion until the oscillations bring the mass with the "dead zone" at which time the motion will cease.

**51. (a)** Rewriting the system as

$$\frac{d^2}{dt^2} = 0$$

$$\frac{d^2y}{dt^2} = -g.$$

Then, taking the Laplace of each equation, we have

$$s^2X(s) - sx(0) - x'(0) = 0$$

$$s^2Y(s) - sy(0) - y'(0) = \frac{g}{s}.$$

Using $x(0) = 0$, $x'(0) = v_0 \cos t$, $y(0) = 0$, $y'(0) = v_0 \sin\theta$ where $v_0 = |\mathbf{v}_0|$, we have

$$\begin{cases} s^2X(s) = v_0\cos\theta \\ s^2Y(s) = v_0\sin\theta - \dfrac{g}{s} \end{cases} \quad \text{or} \quad \begin{cases} X(s) = (v_0\cos\theta)\dfrac{1}{s^2} \\ Y(s) = (v_0\sin\theta)\dfrac{1}{s^2} - \dfrac{g}{s^3} \end{cases}$$

Then

$$\begin{cases} x(t) = (v_0\cos\theta)\,t \\ y(t) = (v_0\sin\theta)\,t - \dfrac{1}{2}gt^2 \end{cases}$$

**(b)** Substituting $t = \dfrac{x}{v_0\cos\theta}$ into the equation for $y(t)$ yields

$$y(x) = -\frac{1}{2}g\left(\frac{x}{v_0\cos\theta}\right)^2 + (v_0\sin\theta)\left(\frac{x}{v_0\cos\theta}\right) = \left(-\frac{1}{2}\left(\frac{g}{v_0^2\cos^2\theta}\right)x + \tan\theta\right)x$$

The projectile hits the ground when $y = 0$. This occurs when $x = 0$, which is the initial condition, or

$$0 = \frac{-g}{2v_0^2\cos^2\theta}\,x + \tan\theta$$

$$x = \frac{(\tan\theta)\,2v_0^2\cos^2\theta}{g} = \frac{v_0^2\sin 2\theta}{g}$$

which is the horizontal range.

**(c)** When $0 < \theta < \pi/2$ the complementary angle of $\theta$ is $\pi/2 - \theta$. Substituting this into the result of part (b) we have

$$R\left(\frac{\pi}{2} - \theta\right) = \frac{v_0^2\sin 2\left(\frac{\pi}{2} - \theta\right)}{g} = \frac{v_0^2\sin(\pi - 2\theta)}{g} = \frac{v_0^2(\sin\pi\cos 2\theta - \cos\pi\sin 2\theta)}{g}$$

$$= \frac{v_0^2\sin 2\theta}{g} = R(\theta).$$

**(d)** When $g = 32$, $\theta = 38°$ and $v_0 = 300$ w ehave from part (b) that

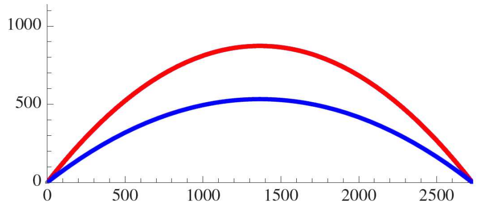
$$R = \frac{300^2}{32}\sin 76° \approx 2729 \text{ ft.}$$

Solving

$$x(t) = (300\cos 38°)t = 2729$$

for $t$, we see that the projectile hits the ground after about 11.54 sec.

**(e)** For $\theta = 38°$ the curve is shown in blue, while for $\theta = 52°$ the curve is shown in red.



**52.** **(a)** Taking the Laplace transform of the first equation, we obtain

$$m\left(s^2\mathscr{L}\left\{x(t)\right\} - v_0\cos\theta\right) = -\beta s\mathscr{L}\left\{x(t)\right\},$$

so that

$$\mathscr{L}\left\{x(t)\right\} = \frac{v_0\cos\theta}{s\left(s+\beta/m\right)} = \frac{mv_0\cos\theta}{\beta}\left(\frac{1}{s} - \frac{1}{s+\beta/m}\right)$$

and

$$x(t) = \frac{mv_0\cos\theta}{\beta}\left(1 - e^{-\beta t/m}\right)$$

Taking the Laplace transform of the second equation, we obtain

$$m\left(s^2\mathscr{L}\left\{x(t)\right\} - v_0\cos\theta\right) = -\beta s\mathscr{L}\left\{x(t)\right\},$$

so that

$$\begin{aligned}\mathscr{L}\left\{x(t)\right\} &= \frac{v_0\sin\theta}{s\left(s+\beta/m\right)} - \frac{g}{s^2\left(s^2+\beta/m\right)}\\ &= \frac{mv_0\sin\theta}{\beta}\left(\frac{1}{s} - \frac{1}{s+\beta/m}\right) - \left(\frac{mg}{\beta}\,\frac{1}{s^2} - \frac{m^2g}{\beta^2}\,\frac{1}{s} + \frac{m^2g}{\beta^2}\,\frac{1}{s+\beta/m}\right),\end{aligned}$$

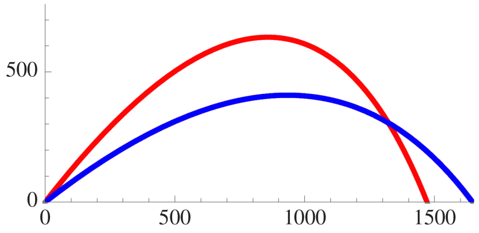
and

$$\begin{aligned}y(t) &= \frac{mv_0\sin\theta}{\beta} + \frac{m^2g}{\beta^2}\left(1 - e^{-\beta t/m}\right) - \left(\frac{mg}{\beta}\,t - \frac{m^2g}{\beta^2}\,e^{-\beta t/m}\right)\\ &= \frac{m}{\beta}\left(v_0\sin\theta + \frac{mg}{\beta}\right)\left(1 - e^{-\beta t/m}\right) - \frac{mg}{\beta}\,t.\end{aligned}$$

**(b)** To find when the projectile hits the ground, we use a CAS to solve $y(0) = 0$. This gives $t_0 = 10.18$ sec. The range is then approximately $x(t_0) = 1646$ ft.

**(c)** Solving $y(t) = 0$ when $\theta = 52°$ gives $t_1 = 12.67$, so the range in this case is $x(t_1) = 1470.0$ ft, which is considerably less than the range of 1646 ft when $\theta = 38°$.

**(d)** For $\theta = 38^\circ$ the curve is shown in blue, while for $\theta = 52^\circ$ the curve is shown in red. We see that the larger angle of elevation results in a smalled range for the projectile. Also, while the curves look at first glance like parabolas, a closer examination shows that they are not parabolas (at least not with vertical axes).



# Chapter 8

## Systems of Linear First-Order Differential Equations

### 8.1 Preliminary Theory – Linear Systems

**1.** Let $\mathbf{X} = \begin{pmatrix} x \\ y \end{pmatrix}$. Then $\mathbf{X}' = \begin{pmatrix} 3 & -5 \\ 4 & 8 \end{pmatrix} \mathbf{X}$.

**2.** Let $\mathbf{X} = \begin{pmatrix} x \\ y \end{pmatrix}$. Then $\mathbf{X}' = \begin{pmatrix} 4 & -7 \\ 5 & 0 \end{pmatrix} \mathbf{X}$.

**3.** Let $\mathbf{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. Then $\mathbf{X}' = \begin{pmatrix} -3 & 4 & -9 \\ 6 & -1 & 0 \\ 10 & 4 & 3 \end{pmatrix} \mathbf{X}$.

**4.** Let $\mathbf{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. Then $\mathbf{X}' = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & 2 \\ -1 & 0 & 1 \end{pmatrix} \mathbf{X}$.

**5.** Let $\mathbf{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. Then $\mathbf{X}' = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix} \mathbf{X} + \begin{pmatrix} 0 \\ -3t^2 \\ t^2 \end{pmatrix} + \begin{pmatrix} t \\ 0 \\ -t \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$.

**6.** Let $\mathbf{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. Then $\mathbf{X}' = \begin{pmatrix} -3 & 4 & 0 \\ 5 & 0 & 9 \\ 0 & 1 & 6 \end{pmatrix} \mathbf{X} + \begin{pmatrix} e^{-t}\sin 2t \\ 4e^{-t}\cos 2t \\ -e^{-t} \end{pmatrix}$.

**7.** $\dfrac{dx}{dt} = 4x + 2y + e^t; \quad \dfrac{dy}{dt} = -x + 3y - e^t$

**8.** $\dfrac{dx}{dt} = 7x + 5y - 9z - 8e^{-2t}; \quad \dfrac{dy}{dt} = 4x + y + z + 2e^{5t}; \quad \dfrac{dz}{dt} = -2y + 3z + e^{5t} - 3e^{-2t}$

**9.** $\dfrac{dx}{dt} = x - y + 2z + e^{-t} - 3t; \quad \dfrac{dy}{dt} = 3x - 4y + z + 2e^{-t} + t; \quad \dfrac{dz}{dt} = -2x + 5y + 6z + 2e^{-t} - t$

**10.** $\dfrac{dx}{dt} = 3x - 7y + 4\sin t + (t-4)e^{4t}; \quad \dfrac{dy}{dt} = x + y + 8\sin t + (2t+1)e^{4t}$

**11.** Since

$$\mathbf{X}' = \begin{pmatrix} -5 \\ -10 \end{pmatrix} e^{-5t} \qquad \text{and} \qquad \begin{pmatrix} 3 & -4 \\ 4 & -7 \end{pmatrix} \mathbf{X} = \begin{pmatrix} -5 \\ -10 \end{pmatrix} e^{-5t}$$

we see that

$$\mathbf{X}' = \begin{pmatrix} 3 & -4 \\ 4 & -7 \end{pmatrix} \mathbf{X}.$$

**12.** Since

$$\mathbf{X}' = \begin{pmatrix} 5\cos t - 5\sin t \\ 2\cos t - 4\sin t \end{pmatrix} e^{t} \quad \text{and} \quad \begin{pmatrix} -2 & 5 \\ -2 & 4 \end{pmatrix} \mathbf{X} = \begin{pmatrix} 5\cos t - 5\sin t \\ 2\cos t - 4\sin t \end{pmatrix} e^{t}$$

we see that

$$\mathbf{X}' = \begin{pmatrix} -2 & 5 \\ -2 & 4 \end{pmatrix} \mathbf{X}.$$

**13.** Since

$$\mathbf{X}' = \begin{pmatrix} \frac{3}{2} \\ -3 \end{pmatrix} e^{-3t/2} \quad \text{and} \quad \begin{pmatrix} -1 & \frac{1}{4} \\ 1 & -1 \end{pmatrix} \mathbf{X} = \begin{pmatrix} \frac{3}{2} \\ -3 \end{pmatrix} e^{-3t/2}$$

we see that

$$\mathbf{X}' = \begin{pmatrix} -1 & 1/4 \\ 1 & -1 \end{pmatrix} \mathbf{X}.$$

**14.** Since

$$\mathbf{X}' = \begin{pmatrix} 5 \\ -1 \end{pmatrix} e^{t} + \begin{pmatrix} 4 \\ -4 \end{pmatrix} te^{t} \qquad \text{and} \qquad \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{X} = \begin{pmatrix} 5 \\ -1 \end{pmatrix} e^{t} + \begin{pmatrix} 4 \\ -4 \end{pmatrix} te^{t}$$

we see that

$$\mathbf{X}' = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{X}.$$

**15.** Since

$$\mathbf{X}' = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix} \mathbf{X} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

we see that

$$\mathbf{X}' = \begin{pmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix} \mathbf{X}.$$

**16.** Since

$$\mathbf{X}' = \begin{pmatrix} \cos t \\ \frac{1}{2}\sin t - \frac{1}{2}\cos t \\ -\cos t - \sin t \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ -2 & 0 & -1 \end{pmatrix} \mathbf{X} = \begin{pmatrix} \cos t \\ \frac{1}{2}\sin t - \frac{1}{2}\cos t \\ -\cos t - \sin t \end{pmatrix}$$

we see that

$$\mathbf{X}' = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ -2 & 0 & -1 \end{pmatrix} \mathbf{X}.$$

**17.** Yes, since $W(\mathbf{X}_1, \mathbf{X}_2) = -2e^{-8t} \neq 0$ the set $\mathbf{X}_1$, $\mathbf{X}_2$ is linearly independent on the interval $(-\infty, \infty)$.

**18.** Yes, since $W(\mathbf{X}_1, \mathbf{X}_2) = 8e^{2t} \neq 0$ the set $\mathbf{X}_1$, $\mathbf{X}_2$ is linearly independent on the interval $(-\infty, \infty)$.

**19.** No, since $W(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3) = 0$ the set $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$ is linearly dependent on the interval $(-\infty, \infty)$.

**20.** Yes, since $W(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3) = -84e^{-t} \neq 0$ the set $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$ is linearly independent on the interval $(-\infty, \infty)$.

**21.** Since

$$\mathbf{X}_p' = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix} \mathbf{X}_p + \begin{pmatrix} 2 \\ -4 \end{pmatrix} t + \begin{pmatrix} -7 \\ -18 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

we see that

$$\mathbf{X}_p' = \begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix} \mathbf{X}_p + \begin{pmatrix} 2 \\ -4 \end{pmatrix} t + \begin{pmatrix} -7 \\ -18 \end{pmatrix}.$$

**22.** Since

$$\mathbf{X}_p' = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} \mathbf{X}_p + \begin{pmatrix} -5 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

we see that

$$\mathbf{X}_p' = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} \mathbf{X}_p + \begin{pmatrix} -5 \\ 2 \end{pmatrix}.$$

**23.** Since

$$\mathbf{X}_p' = \begin{pmatrix} 2 \\ 0 \end{pmatrix} e^t + \begin{pmatrix} 1 \\ -1 \end{pmatrix} te^t \quad \text{and} \quad \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix} \mathbf{X}_p - \begin{pmatrix} 1 \\ 7 \end{pmatrix} e^t = \begin{pmatrix} 2 \\ 0 \end{pmatrix} e^t + \begin{pmatrix} 1 \\ -1 \end{pmatrix} te^t$$

we see that

$$\mathbf{X}_p' = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix} \mathbf{X}_p - \begin{pmatrix} 1 \\ 7 \end{pmatrix} e^t.$$

**24.** Since

$$\mathbf{X}_p' = \begin{pmatrix} 3\cos 3t \\ 0 \\ -3\sin 3t \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 2 & 3 \\ -4 & 2 & 0 \\ -6 & 1 & 0 \end{pmatrix} \mathbf{X}_p + \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} \sin 3t = \begin{pmatrix} 3\cos 3t \\ 0 \\ -3\sin 3t \end{pmatrix}$$

we see that

$$\mathbf{X}_p' = \begin{pmatrix} 1 & 2 & 3 \\ -4 & 2 & 0 \\ -6 & 1 & 0 \end{pmatrix} \mathbf{X}_p + \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} \sin 3t.$$

**25.** Let

$$\mathbf{X}_1 = \begin{pmatrix} 6 \\ -1 \\ -5 \end{pmatrix} e^{-t}, \quad \mathbf{X}_2 = \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix} e^{-2t}, \quad \mathbf{X}_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} e^{3t}, \quad \text{and} \quad \mathbf{A} = \begin{pmatrix} 0 & 6 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Then

$$\mathbf{X}_1' = \begin{pmatrix} -6 \\ 1 \\ 5 \end{pmatrix} e^{-t} = \mathbf{A}\mathbf{X}_1, \qquad \mathbf{X}_2' = \begin{pmatrix} 6 \\ -2 \\ -2 \end{pmatrix} e^{-2t} = \mathbf{A}\mathbf{X}_2, \qquad \mathbf{X}_3' = \begin{pmatrix} 6 \\ 3 \\ 3 \end{pmatrix} e^{3t} = \mathbf{A}\mathbf{X}_3,$$

and $W(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3) = 20 \neq 0$ so that $\mathbf{X}_1$, $\mathbf{X}_2$, and $\mathbf{X}_3$ form a fundamental set for $\mathbf{X}' = \mathbf{A}\mathbf{X}$ on the interval $(-\infty, \infty)$.

**26.** Let

$$\mathbf{X}_1 = \begin{pmatrix} 1 \\ -1-\sqrt{2} \end{pmatrix} e^{\sqrt{2}\,t}, \quad \mathbf{X}_2 = \begin{pmatrix} 1 \\ -1+\sqrt{2} \end{pmatrix} e^{-\sqrt{2}\,t}, \quad \mathbf{X}_p = \begin{pmatrix} 1 \\ 0 \end{pmatrix} t^2 + \begin{pmatrix} -2 \\ 4 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

and

$$\mathbf{A} = \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}.$$

Then

$$\mathbf{X}_1' = \begin{pmatrix} \sqrt{2} \\ -2-\sqrt{2} \end{pmatrix} e^{\sqrt{2}\,t} = \mathbf{A}\mathbf{X}_1,$$

$$\mathbf{X}_2' = \begin{pmatrix} -\sqrt{2} \\ -2+\sqrt{2} \end{pmatrix} e^{-\sqrt{2}\,t} = \mathbf{A}\mathbf{X}_2,$$

$$\mathbf{X}_p' = \begin{pmatrix} 2 \\ 0 \end{pmatrix} t + \begin{pmatrix} -2 \\ 4 \end{pmatrix} = \mathbf{A}\mathbf{X}_p + \begin{pmatrix} 1 \\ 1 \end{pmatrix} t^2 + \begin{pmatrix} 4 \\ -6 \end{pmatrix} t + \begin{pmatrix} -1 \\ 5 \end{pmatrix},$$

and $W(\mathbf{X}_1, \mathbf{X}_2) = 2\sqrt{2} \neq 0$ so that $\mathbf{X}_p$ is a particular solution and $\mathbf{X}_1$ and $\mathbf{X}_2$ form a fundamental set on the interval $(-\infty, \infty)$.

## 8.2 Homogeneous Linear Systems

**1.** The system is

$$\mathbf{X}' = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \mathbf{X}$$

and $\det(\mathbf{A} - \lambda\mathbf{I}) = (\lambda - 5)(\lambda + 1) = 0$. For $\lambda_1 = 5$ we obtain

$$\left(\begin{array}{cc|c} -4 & 2 & 0 \\ 4 & -2 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right) \quad \text{so that} \quad \mathbf{K}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

For $\lambda_2 = -1$ we obtain

$$\left(\begin{array}{cc|c} 2 & 2 & 0 \\ 4 & 4 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad \text{so that} \quad \mathbf{K}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Then

$$\mathbf{X} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{5t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t}.$$

**2.** The system is

$$\mathbf{X}' = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix} \mathbf{X}$$

and $\det(\mathbf{A} - \lambda\mathbf{I}) = (\lambda - 1)(\lambda - 4) = 0$. For $\lambda_1 = 1$ we obtain

$$\left(\begin{array}{cc|c} 1 & 2 & 0 \\ 1 & 2 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad \text{so that} \quad \mathbf{K}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

For $\lambda_2 = 4$ we obtain

$$\left(\begin{array}{cc|c} -2 & 2 & 0 \\ 1 & -1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad \text{so that} \quad \mathbf{K}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Then

$$\mathbf{X} = c_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}.$$

**3.** The system is

$$\mathbf{X}' = \begin{pmatrix} -4 & 2 \\ -\frac{5}{2} & 2 \end{pmatrix} \mathbf{X}$$

and $\det(\mathbf{A} - \lambda\mathbf{I}) = (\lambda - 1)(\lambda + 3) = 0$. For $\lambda_1 = 1$ we obtain

$$\left(\begin{array}{cc|c} -5 & 2 & 0 \\ -\frac{5}{2} & 1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} -5 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad \text{so that} \quad \mathbf{K}_1 = \begin{pmatrix} 2 \\ 5 \end{pmatrix}.$$

For $\lambda_2 = -3$ we obtain

$$\left(\begin{array}{cc|c} -1 & 2 & 0 \\ -\frac{5}{2} & 5 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} -1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad \text{so that} \quad \mathbf{K}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Then

$$\mathbf{X} = c_1 \begin{pmatrix} 2 \\ 5 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-3t}.$$

**4.** The system is

$$\mathbf{X}' = \begin{pmatrix} -\frac{5}{2} & 2 \\ \frac{3}{4} & -2 \end{pmatrix} \mathbf{X}$$

and $\det(\mathbf{A} - \lambda\mathbf{I}) = \frac{1}{2}(\lambda+1)(2\lambda+7) = 0$. For $\lambda_1 = -7/2$ we obtain

$$\left(\begin{array}{cc|c} 1 & 2 & 0 \\ \frac{3}{4} & \frac{3}{2} & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad \text{so that} \quad \mathbf{K}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

For $\lambda_2 = -1$ we obtain

$$\left(\begin{array}{cc|c} -\frac{3}{2} & 2 & 0 \\ \frac{3}{4} & -1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} -3 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad \text{so that} \quad \mathbf{K}_2 = \begin{pmatrix} 4 \\ 3 \end{pmatrix}.$$

Then

$$\mathbf{X} = c_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{-7t/2} + c_2 \begin{pmatrix} 4 \\ 3 \end{pmatrix} e^{-t}.$$

**5.** The system is

$$\mathbf{X}' = \begin{pmatrix} 10 & -5 \\ 8 & -12 \end{pmatrix} \mathbf{X}$$

and $\det(\mathbf{A} - \lambda\mathbf{I}) = (\lambda - 8)(\lambda + 10) = 0$. For $\lambda_1 = 8$ we obtain

$$\left(\begin{array}{cc|c} 2 & -5 & 0 \\ 8 & -20 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & -\frac{5}{2} & 0 \\ 0 & 0 & 0 \end{array}\right) \quad \text{so that} \quad \mathbf{K}_1 = \begin{pmatrix} 5 \\ 2 \end{pmatrix}.$$

For $\lambda_2 = -10$ we obtain

$$\left(\begin{array}{cc|c} 20 & -5 & 0 \\ 8 & -2 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & -\frac{1}{4} & 0 \\ 0 & 0 & 0 \end{array}\right) \quad \text{so that} \quad \mathbf{K}_2 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}.$$

Then

$$\mathbf{X} = c_1 \begin{pmatrix} 5 \\ 2 \end{pmatrix} e^{8t} + c_2 \begin{pmatrix} 1 \\ 4 \end{pmatrix} e^{-10t}.$$

**6.** The system is

$$\mathbf{X}' = \begin{pmatrix} -6 & 2 \\ -3 & 1 \end{pmatrix} \mathbf{X}$$

and $\det(\mathbf{A} - \lambda\mathbf{I}) = \lambda(\lambda + 5) = 0$. For $\lambda_1 = 0$ we obtain

$$\left(\begin{array}{cc|c} -6 & 2 & 0 \\ -3 & 1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 \end{array}\right) \quad \text{so that} \quad \mathbf{K}_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

For $\lambda_2 = -5$ we obtain

$$\left(\begin{array}{cc|c} -1 & 2 & 0 \\ -3 & 6 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad \text{so that} \quad \mathbf{K}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Then

$$\mathbf{X} = c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-5t}.$$

**7.** The system is

$$\mathbf{X}' = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 0 \\ 0 & 1 & -1 \end{pmatrix} \mathbf{X}$$

and $\det(\mathbf{A} - \lambda\mathbf{I}) = (\lambda - 1)(2 - \lambda)(\lambda + 1) = 0$. For $\lambda_1 = 1$, $\lambda_2 = 2$, and $\lambda_3 = -1$ we obtain

$$\mathbf{K}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{K}_2 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}, \quad \text{and} \quad \mathbf{K}_3 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix},$$

so that

$$\mathbf{X} = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} e^{2t} + c_3 \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} e^{-t}.$$

**8.** The system is

$$\mathbf{X}' = \begin{pmatrix} 2 & -7 & 0 \\ 5 & 10 & 4 \\ 0 & 5 & 2 \end{pmatrix} \mathbf{X}$$

and $\det(\mathbf{A} - \lambda\mathbf{I}) = (2 - \lambda)(\lambda - 5)(\lambda - 7) = 0$. For $\lambda_1 = 2$, $\lambda_2 = 5$, and $\lambda_3 = 7$ we obtain

$$\mathbf{K}_1 = \begin{pmatrix} 4 \\ 0 \\ -5 \end{pmatrix}, \quad \mathbf{K}_2 = \begin{pmatrix} -7 \\ 3 \\ 5 \end{pmatrix}, \quad \text{and} \quad \mathbf{K}_3 = \begin{pmatrix} -7 \\ 5 \\ 5 \end{pmatrix},$$

so that

$$\mathbf{X} = c_1 \begin{pmatrix} 4 \\ 0 \\ -5 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} -7 \\ 3 \\ 5 \end{pmatrix} e^{5t} + c_3 \begin{pmatrix} -7 \\ 5 \\ 5 \end{pmatrix} e^{7t}.$$

**9.** We have $\det(\mathbf{A} - \lambda\mathbf{I}) = -(\lambda + 1)(\lambda - 3)(\lambda + 2) = 0$. For $\lambda_1 = -1$, $\lambda_2 = 3$, and $\lambda_3 = -2$ we obtain

$$\mathbf{K}_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{K}_2 = \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}, \quad \text{and} \quad \mathbf{K}_3 = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix},$$

so that

$$\mathbf{X} = c_1 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix} e^{3t} + c_3 \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} e^{-2t}.$$

**10.** We have $\det(\mathbf{A}-\lambda\mathbf{I}) = -\lambda(\lambda-1)(\lambda-2) = 0$. For $\lambda_1 = 0$, $\lambda_2 = 1$, and $\lambda_3 = 2$ we obtain

$$\mathbf{K}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \mathbf{K}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \text{and} \quad \mathbf{K}_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},$$

so that

$$\mathbf{X} = c_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{t} + c_3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{2t}.$$

**11.** We have $\det(\mathbf{A}-\lambda\mathbf{I}) = -(\lambda+1)(\lambda+1/2)(\lambda+3/2) = 0$. For $\lambda_1 = -1$, $\lambda_2 = -1/2$, and $\lambda_3 = -3/2$ we obtain

$$\mathbf{K}_1 = \begin{pmatrix} 4 \\ 0 \\ -1 \end{pmatrix}, \quad \mathbf{K}_2 = \begin{pmatrix} -12 \\ 6 \\ 5 \end{pmatrix}, \quad \text{and} \quad \mathbf{K}_3 = \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix},$$

so that

$$\mathbf{X} = c_1 \begin{pmatrix} 4 \\ 0 \\ -1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} -12 \\ 6 \\ 5 \end{pmatrix} e^{-t/2} + c_3 \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix} e^{-3t/2}.$$

**12.** We have $\det(\mathbf{A}-\lambda\mathbf{I}) = (\lambda-3)(\lambda+5)(6-\lambda) = 0$. For $\lambda_1 = 3$, $\lambda_2 = -5$, and $\lambda_3 = 6$ we obtain

$$\mathbf{K}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{K}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad \text{and} \quad \mathbf{K}_3 = \begin{pmatrix} 2 \\ -2 \\ 11 \end{pmatrix},$$

so that

$$\mathbf{X} = c_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} e^{-5t} + c_3 \begin{pmatrix} 2 \\ -2 \\ 11 \end{pmatrix} e^{6t}.$$

**13.** We have $\det(\mathbf{A}-\lambda\mathbf{I}) = (\lambda+1/2)(\lambda-1/2) = 0$. For $\lambda_1 = -1/2$ and $\lambda_2 = 1/2$ we obtain

$$\mathbf{K}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{K}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

so that

$$\mathbf{X} = c_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t/2} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t/2}.$$

If

$$\mathbf{X}(0) = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$

then $c_1 = 2$ and $c_2 = 3$. The solutions is

$$\mathbf{X} = 2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t/2} + 3 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t/2}.$$

**14.** We have $\det(\mathbf{A}-\lambda\mathbf{I}) = (2-\lambda)(\lambda-3)(\lambda+1) = 0$. For $\lambda_1 = 2$, $\lambda_2 = 3$, and $\lambda_3 = -1$ we obtain

$$\mathbf{K}_1 = \begin{pmatrix} 5 \\ -3 \\ 2 \end{pmatrix}, \quad \mathbf{K}_2 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \quad \text{and} \quad \mathbf{K}_3 = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix},$$

so that

$$\mathbf{X} = c_1 \begin{pmatrix} 5 \\ -3 \\ 2 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} e^{3t} + c_3 \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} e^{-t}.$$

If

$$\mathbf{X}(0) = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$$

then $c_1 = -1$, $c_2 = 5/2$, and $c_3 = -1/2$. The solutions is

$$\mathbf{X} = -1 \begin{pmatrix} 5 \\ -3 \\ 2 \end{pmatrix} e^{2t} + \frac{5}{2} \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} e^{3t} - \frac{1}{2} \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} e^{-t}.$$

**15.** **(a)** From the discussion in Section 2.9 we get the system

$$\begin{cases} \dfrac{dx_1}{dt} = -\dfrac{3}{100}x_1 + \dfrac{1}{100}x_2 \\[2ex] \dfrac{dx_2}{dt} = \dfrac{1}{50}x_1 - \dfrac{1}{50}x_2 \end{cases} \qquad \text{Thus} \qquad \mathbf{X}' = \begin{pmatrix} -\frac{3}{100} & \frac{1}{100} \\ \frac{1}{50} & -\frac{1}{50} \end{pmatrix} \mathbf{X}$$

**(b)** The eigenvalues and eigenvectors of the coefficient matrix are found by solving $\det(\mathbf{A}-\lambda\mathbf{I}) = 0$ to get

$$\lambda_1 = -\frac{1}{25} \quad \text{so} \quad \mathbf{K}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \qquad \text{and} \qquad \lambda_2 = -\frac{1}{100} \quad \text{so} \quad \mathbf{K}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

The general solution is then

$$\mathbf{X}(t) = c_1\mathbf{K}_1 e^{\lambda_1 t} + c_2\mathbf{K}_2 e^{\lambda_2 t} = c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t/25} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t/100}$$

Using the initial conditions, we get

$$\mathbf{X}(0) = c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -c_1 + c_2 \\ c_1 + c_2 \end{pmatrix} = \begin{pmatrix} 20 \\ 5 \end{pmatrix} \quad \text{so} \quad c_1 = -\frac{35}{3} \quad \text{and} \quad c_2 = \frac{25}{3}$$

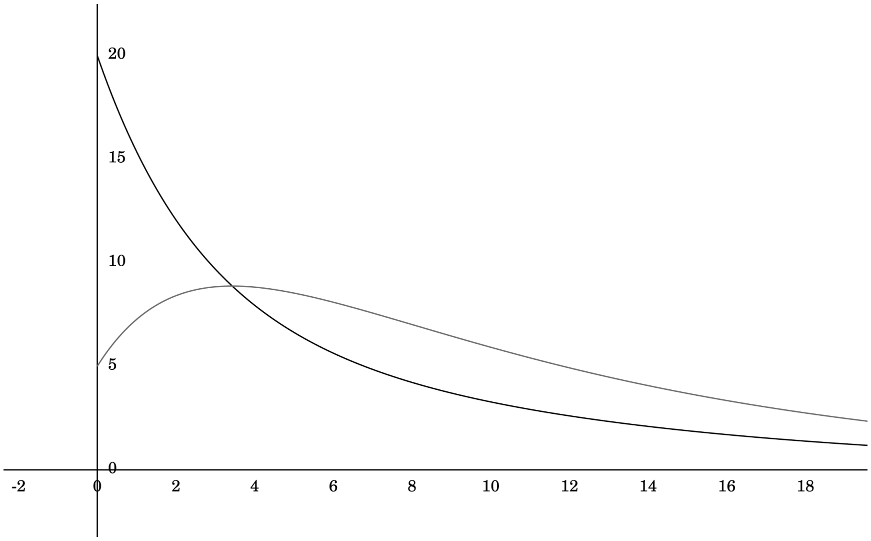
Thus

$$\mathbf{X}(t) = -\frac{35}{3} \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t/25} + \frac{25}{3} \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t/100}$$

Hence

$$x_1(t) = \frac{35}{3}e^{-t/25} + \frac{25}{3}e^{-t/100} \qquad \text{and} \qquad x_2(t) = -\frac{35}{3}e^{-t/25} + \frac{50}{3}e^{-t/100}$$

**(c)**



**(d)** Set $x_1(t) = x_2(t)$ and, using a scientific calculator, we find that $t \approx 34.3277$ min.

**16. (a)** The system to solve is

$$\begin{cases} \dfrac{dx_1}{dt} = -\dfrac{1}{50}x_1 \\ \dfrac{dx_2}{dt} = \dfrac{1}{50}x_1 - \dfrac{2}{75}x_2 \\ \dfrac{dx_3}{dt} = \dfrac{2}{75}x_2 - \dfrac{1}{25}x_3 \end{cases} \qquad \text{Thus} \qquad \mathbf{X}' = \begin{pmatrix} -\frac{1}{50} & 0 & 0 \\ \frac{1}{50} & -\frac{2}{75} & 0 \\ 0 & \frac{2}{75} & -\frac{1}{25} \end{pmatrix} \mathbf{X}$$

The eigenvalues and eigenvectors of the coefficient matrix are found by solving $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ to get

$$\lambda_1 = -\frac{1}{25} \quad \text{so} \quad \mathbf{K}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \lambda_2 = -\frac{1}{50} \quad \text{so} \quad \mathbf{K}_2 = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix},$$

$$\text{and} \qquad \lambda_3 = -\frac{2}{75} \quad \text{so} \quad \mathbf{K}_3 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

The general solution is then

$$\mathbf{X}(t) = c_1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{-t/25} + c_2 \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} e^{-t/50} + c_3 \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} e^{-2t/75}.$$

Using the initial conditions, we get

$$\mathbf{X}(0) = c_1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} c_2 \\ 3c_2 + c_3 \\ c_1 + 4c_2 + 2c_3 \end{pmatrix} = \begin{pmatrix} 15 \\ 10 \\ 5 \end{pmatrix}$$

$$c_1 = 15, \quad c_2 = 15, \quad \text{and} \quad c_3 = -35$$

$$\mathbf{X}(t) = 15 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{-t/25} + 15 \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} e^{-t/50} - 35 \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} e^{-2t/75}$$

Therefore,

$$x_1(t) = 15e^{-t/50} \quad \text{and} \quad x_2(t) = 45e^{-t/50} - 35e^{-2t/75}$$

$$\text{and} \quad x_3(t) = 15e^{-t/25} + 60e^{-t/50} - 70e^{-2t/75}$$

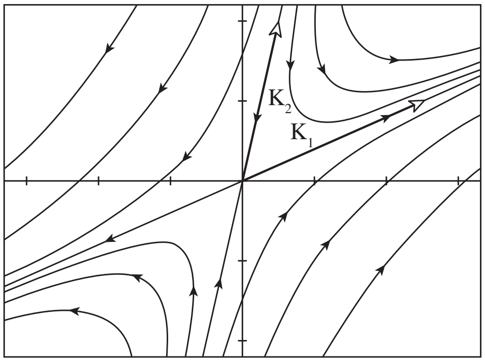
**(b)** All three solutions tend to zero as $t \to \infty$ which means eventually the tanks will contain pure water.

**17.** $\mathbf{X} = c_1 \begin{pmatrix} 0.382175 \\ 0.851161 \\ 0.359815 \end{pmatrix} e^{8.58979t} + c_2 \begin{pmatrix} 0.405188 \\ -0.676043 \\ 0.615458 \end{pmatrix} e^{2.25684t} + c_3 \begin{pmatrix} -0.923562 \\ -0.132174 \\ 0.35995 \end{pmatrix} e^{-0.0466321t}$

**18.** $\mathbf{X} = c_1 \begin{pmatrix} 0.0312209 \\ 0.949058 \\ 0.239535 \\ 0.195825 \\ 0.0508861 \end{pmatrix} e^{5.05452t} + c_2 \begin{pmatrix} -0.280232 \\ -0.836611 \\ -0.275304 \\ 0.176045 \\ 0.338775 \end{pmatrix} e^{4.09561t} + c_3 \begin{pmatrix} 0.262219 \\ -0.162664 \\ -0.826218 \\ -0.346439 \\ 0.31957 \end{pmatrix} e^{-2.92362t}$

$$+ c_4 \begin{pmatrix} 0.313235 \\ 0.64181 \\ 0.31754 \\ 0.173787 \\ -0.599108 \end{pmatrix} e^{2.02882t} + c_5 \begin{pmatrix} -0.301294 \\ 0.466599 \\ 0.222136 \\ 0.0534311 \\ -0.799567 \end{pmatrix} e^{-0.155338t}$$

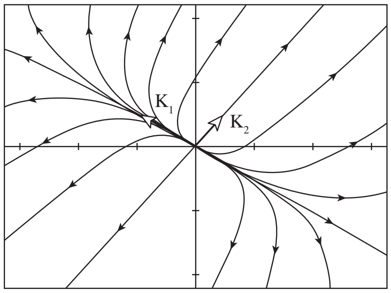
**19. (a)**



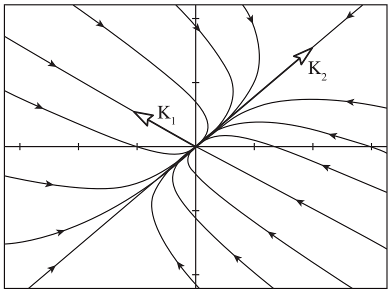
**(b)** Letting $c_1 = 1$ and $c_2 = 0$ we get $x = 5e^{8t}$, $y = 2e^{8t}$. Eliminating the parameter we find $y = \frac{2}{5}x$, $x > 0$. When $c_1 = -1$ and $c_2 = 0$ we find $y = \frac{2}{5}x$, $x < 0$. Letting $c_1 = 0$ and $c_2 = 1$ we get $x = e^{-10t}$, $y = 4e^{-10t}$. Eliminating the parameter we find $y = 4x$, $x > 0$. Letting $c_1 = 0$ and $c_2 = -1$ we find $y = 4x$, $x < 0$.

**(c)** The eigenvectors $\mathbf{K}_1 = (5, 2)$ and $\mathbf{K}_2 = (1, 4)$ are shown in the figure in part (**A**).

**20.** In Problem 2, letting $c_1 = 1$ and $c_2 = 0$ we get $x = -2e^{t}$, $y = e^{t}$. Eliminating the parameter we find $y = -\frac{1}{2}x$, $x < 0$. When $c_1 = -1$ and $c_2 = 0$ we find $y = -\frac{1}{2}x$, $x > 0$. Letting $c_1 = 0$ and $c_2 = 1$ we get $x = e^{4t}$, $y = e^{4t}$. Eliminating the parameter we find $y = x$, $x > 0$. When $c_1 = 0$ and $c_2 = -1$ we find $y = x$, $x < 0$.



In Problem 4, letting $c_1 = 1$ and $c_2 = 0$ we get $x = -2e^{-7t/2}$, $y = e^{-7t/2}$. Eliminating the parameter we find $y = -\frac{1}{2}x$, $x < 0$. When $c_1 = -1$ and $c_2 = 0$ we find $y = -\frac{1}{2}x$, $x > 0$. Letting $c_1 = 0$ and $c_2 = 1$ we get $x = 4e^{-t}$, $y = 3e^{-t}$. Eliminating the parameter we find $y = \frac{3}{4}x$, $x > 0$. When $c_1 = 0$ and $c_2 = -1$ we find $y = \frac{3}{4}x$, $x < 0$.



**21.** We have $\det(\mathbf{A} - \lambda\mathbf{I}) = \lambda^2 = 0$. For $\lambda_1 = 0$ we obtain

$$\mathbf{K} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

A solution of $(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{P} = \mathbf{K}$ is

$$\mathbf{P} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

so that

$$\mathbf{X} = c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} + c_2 \left[ \begin{pmatrix} 1 \\ 3 \end{pmatrix} t + \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right].$$

**22.** We have $\det(\mathbf{A} - \lambda\mathbf{I}) = (\lambda + 1)^2 = 0$. For $\lambda_1 = -1$ we obtain

$$\mathbf{K} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

A solution of $(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{P} = \mathbf{K}$ is

$$\mathbf{P} = \begin{pmatrix} 0 \\ \frac{1}{5} \end{pmatrix}$$

so that

$$\mathbf{X} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_2 \left[ \begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-t} + \begin{pmatrix} 0 \\ \frac{1}{5} \end{pmatrix} e^{-t} \right].$$

**23.** We have $\det(\mathbf{A} - \lambda\mathbf{I}) = (\lambda - 2)^2 = 0$. For $\lambda_1 = 2$ we obtain

$$\mathbf{K} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

A solution of $(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{P} = \mathbf{K}$ is

$$\mathbf{P} = \begin{pmatrix} -\frac{1}{3} \\ 0 \end{pmatrix}$$

so that

$$\mathbf{X} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + c_2 \left[ \begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{2t} + \begin{pmatrix} -\frac{1}{3} \\ 0 \end{pmatrix} e^{2t} \right].$$

**24.** We have $\det(\mathbf{A} - \lambda\mathbf{I}) = (\lambda - 6)^2 = 0$. For $\lambda_1 = 6$ we obtain

$$\mathbf{K} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

A solution of $(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{P} = \mathbf{K}$ is

$$\mathbf{P} = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}$$

so that

$$\mathbf{X} = c_1 \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{6t} + c_2 \left[ \begin{pmatrix} 3 \\ 2 \end{pmatrix} te^{6t} + \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} e^{6t} \right].$$

**25.** We have $\det(\mathbf{A} - \lambda\mathbf{I}) = (1 - \lambda)(\lambda - 2)^2 = 0$. For $\lambda_1 = 1$ we obtain

$$\mathbf{K}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

For $\lambda_2 = 2$ we obtain

$$\mathbf{K}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{K}_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

Then

$$\mathbf{X} = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{2t} + c_3 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{2t}.$$

**26.** We have $\det(\mathbf{A}-\lambda\mathbf{I}) = (\lambda-8)(\lambda+1)^2 = 0$. For $\lambda_1 = 8$ we obtain

$$\mathbf{K}_1 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}.$$

For $\lambda_2 = -1$ we obtain

$$\mathbf{K}_2 = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{K}_3 = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}.$$

Then

$$\mathbf{X} = c_1 \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} e^{8t} + c_2 \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} e^{-t}.$$

**27.** We have $\det(\mathbf{A}-\lambda\mathbf{I}) = -\lambda(5-\lambda)^2 = 0$. For $\lambda_1 = 0$ we obtain

$$\mathbf{K}_1 = \begin{pmatrix} -4 \\ -5 \\ 2 \end{pmatrix}.$$

For $\lambda_2 = 5$ we obtain

$$\mathbf{K} = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}.$$

A solution of $(\mathbf{A}-\lambda_1\mathbf{I})\mathbf{P} = \mathbf{K}$ is

$$\mathbf{P} = \begin{pmatrix} \frac{5}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}$$

so that

$$\mathbf{X} = c_1 \begin{pmatrix} -4 \\ -5 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} e^{5t} + c_3 \left[ \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} te^{5t} + \begin{pmatrix} \frac{5}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix} e^{5t} \right].$$

**28.** We have $\det(\mathbf{A}-\lambda\mathbf{I}) = (1-\lambda)(\lambda-2)^2 = 0$. For $\lambda_1 = 1$ we obtain

$$\mathbf{K}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

For $\lambda_2 = 2$ we obtain

$$\mathbf{K} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}.$$

A solution of $(\mathbf{A} - \lambda_2\mathbf{I})\mathbf{P} = \mathbf{K}$ is

$$\mathbf{P} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$$

so that

$$\mathbf{X} = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^t + c_2 \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} e^{2t} + c_3 \left[ \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} te^{2t} + \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} e^{2t} \right].$$

**29.** We have $\det(\mathbf{A} - \lambda\mathbf{I}) = -(\lambda - 1)^3 = 0$. For $\lambda_1 = 1$ we obtain

$$\mathbf{K} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

Solutions of $(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{P} = \mathbf{K}$ and $(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{Q} = \mathbf{P}$ are

$$\mathbf{P} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{Q} = \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \end{pmatrix}$$

so that

$$\mathbf{X} = c_1 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} e^t + c_2 \left[ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} te^t + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^t \right] + c_3 \left[ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \frac{t^2}{2} e^t + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} te^t + \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} e^t \right].$$

**30.** We have $\det(\mathbf{A} - \lambda\mathbf{I}) = (\lambda - 4)^3 = 0$. For $\lambda_1 = 4$ we obtain

$$\mathbf{K} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Solutions of $(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{P} = \mathbf{K}$ and $(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{Q} = \mathbf{P}$ are

$$\mathbf{P} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{Q} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

so that

$$\mathbf{X} = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{4t} + c_2 \left[ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} te^{4t} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{4t} \right] + c_3 \left[ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \frac{t^2}{2} e^{4t} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} te^{4t} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{4t} \right].$$

**31.** We have $\det(\mathbf{A} - \lambda\mathbf{I}) = (\lambda - 4)^2 = 0$. For $\lambda_1 = 4$ we obtain

$$\mathbf{K} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

A solution of $(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{P} = \mathbf{K}$ is

$$\mathbf{P} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

so that

$$\mathbf{X} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{4t} + c_2 \left[ \begin{pmatrix} 2 \\ 1 \end{pmatrix} te^{4t} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} \right].$$

If

$$\mathbf{X}(0) = \begin{pmatrix} -1 \\ 6 \end{pmatrix}$$

then $c_1 = -7$ and $c_2 = 13$. The solutions is

$$\mathbf{X} = -7 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{4t} + 13 \begin{pmatrix} 2t + 1 \\ t + 1 \end{pmatrix} e^{4t}.$$

**32.** We have $\det(\mathbf{A} - \lambda\mathbf{I}) = -(\lambda + 1)(\lambda - 1)^2 = 0$. For $\lambda_1 = -1$ we obtain

$$\mathbf{K}_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

For $\lambda_2 = 1$ we obtain

$$\mathbf{K}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{K}_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

so that

$$\mathbf{X} = c_1 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{t} + c_3 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{t}.$$

If

$$\mathbf{X}(0) = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$$

then $c_1 = 2$, $c_2 = 3$, and $c_3 = 2$. The solution is

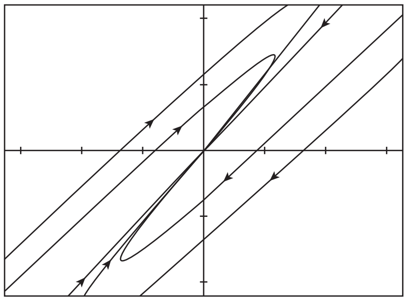
$$\mathbf{X} = 2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} e^{-t} + 3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{t} + 2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{t}.$$

**33.** In this case $\det(\mathbf{A}-\lambda\mathbf{I}) = (2-\lambda)^5$, and $\lambda_1 = 2$ is an eigenvalue of multiplicity 5. Linearly independent eigenvectors are

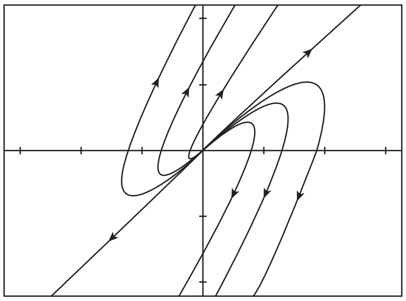
$$\mathbf{K}_1 = \begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}, \qquad \mathbf{K}_2 = \begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix}, \qquad \text{and} \qquad \mathbf{K}_3 = \begin{pmatrix}0\\0\\0\\1\\0\end{pmatrix}.$$

**34.** In Problem 22 letting $c_1 = 1$ and $c_2 = 0$ we get $x = e^t$, $y = e^t$. Eliminating the parameter we find $y = x$, $x > 0$. When $c_1 = -1$ and $c_2 = 0$ we find $y = x$, $x < 0$.

In Problem 23 letting $c_1 = 1$ and $c_2 = 0$ we get $x = e^{2t}$, $y = e^{2t}$. Eliminating the parameter we find $y = x$, $x > 0$. When $c_1 = -1$ and $c_2 = 0$ we find $y = x$, $x < 0$.



Phase portarit for Problem 20



Phase portarit for Problem 21

*In Problems 35-46 the form of the answer will vary according to the choice of eigenvector. For example, in Problem 35, if $\mathbf{K}_1$ is chosen to be $\begin{pmatrix}1\\2-i\end{pmatrix}$ the solution has the form*

$$\mathbf{X} = c_1\begin{pmatrix}\cos t\\2\cos t+\sin t\end{pmatrix}e^{4t} + c_2\begin{pmatrix}\sin t\\2\sin t-\cos t\end{pmatrix}e^{4t}.$$

**35.** We have $\det(\mathbf{A}-\lambda\mathbf{I}) = \lambda^2 - 8\lambda + 17 = 0$. For $\lambda_1 = 4+i$ we obtain

$$\mathbf{K}_1 = \begin{pmatrix}2+i\\5\end{pmatrix}$$

so that

$$\mathbf{X}_1 = \begin{pmatrix}2+i\\5\end{pmatrix}e^{(4+i)t} = \begin{pmatrix}2\cos t-\sin t\\5\cos t\end{pmatrix}e^{4t} + i\begin{pmatrix}\cos t+2\sin t\\5\sin t\end{pmatrix}e^{4t}.$$

Then

$$\mathbf{X} = c_1\begin{pmatrix}2\cos t-\sin t\\5\cos t\end{pmatrix}e^{4t} + c_2\begin{pmatrix}\cos t+2\sin t\\5\sin t\end{pmatrix}e^{4t}.$$

**36.** We have $\det(\mathbf{A}-\lambda\mathbf{I})=\lambda^2+1=0$. For $\lambda_1=i$ we obtain

$$\mathbf{K}_1=\begin{pmatrix}-1-i\\2\end{pmatrix}$$

so that

$$\mathbf{X}_1=\begin{pmatrix}-1-i\\2\end{pmatrix}e^{it}=\begin{pmatrix}\sin t-\cos t\\2\cos t\end{pmatrix}+i\begin{pmatrix}-\cos t-\sin t\\2\sin t\end{pmatrix}.$$

Then

$$\mathbf{X}=c_1\begin{pmatrix}\sin t-\cos t\\2\cos t\end{pmatrix}+c_2\begin{pmatrix}-\cos t-\sin t\\2\sin t\end{pmatrix}.$$

**37.** We have $\det(\mathbf{A}-\lambda\mathbf{I})=\lambda^2-8\lambda+17=0$. For $\lambda_1=4+i$ we obtain

$$\mathbf{K}_1=\begin{pmatrix}-1-i\\2\end{pmatrix}$$

so that

$$\mathbf{X}_1=\begin{pmatrix}-1-i\\2\end{pmatrix}e^{(4+i)t}=\begin{pmatrix}\sin t-\cos t\\2\cos t\end{pmatrix}e^{4t}+i\begin{pmatrix}-\sin t-\cos t\\2\sin t\end{pmatrix}e^{4t}.$$

Then

$$\mathbf{X}=c_1\begin{pmatrix}\sin t-\cos t\\2\cos t\end{pmatrix}e^{4t}+c_2\begin{pmatrix}-\sin t-\cos t\\2\sin t\end{pmatrix}e^{4t}.$$

**38.** We have $\det(\mathbf{A}-\lambda\mathbf{I})=\lambda^2-10\lambda+34=0$. For $\lambda_1=5+3i$ we obtain

$$\mathbf{K}_1=\begin{pmatrix}1-3i\\2\end{pmatrix}$$

so that

$$\mathbf{X}_1=\begin{pmatrix}1-3i\\2\end{pmatrix}e^{(5+3i)t}=\begin{pmatrix}\cos 3t+3\sin 3t\\2\cos 3t\end{pmatrix}e^{5t}+i\begin{pmatrix}\sin 3t-3\cos 3t\\2\sin 3t\end{pmatrix}e^{5t}.$$

Then

$$\mathbf{X}=c_1\begin{pmatrix}\cos 3t+3\sin 3t\\2\cos 3t\end{pmatrix}e^{5t}+c_2\begin{pmatrix}\sin 3t-3\cos 3t\\2\sin 3t\end{pmatrix}e^{5t}.$$

**39.** We have $\det(\mathbf{A}-\lambda\mathbf{I})=\lambda^2+9=0$. For $\lambda_1=3i$ we obtain

$$\mathbf{K}_1=\begin{pmatrix}4+3i\\5\end{pmatrix}$$

so that

$$\mathbf{X}_1=\begin{pmatrix}4+3i\\5\end{pmatrix}e^{3it}=\begin{pmatrix}4\cos 3t-3\sin 3t\\5\cos 3t\end{pmatrix}+i\begin{pmatrix}4\sin 3t+3\cos 3t\\5\sin 3t\end{pmatrix}.$$

Then

$$\mathbf{X}=c_1\begin{pmatrix}4\cos 3t-3\sin 3t\\5\cos 3t\end{pmatrix}+c_2\begin{pmatrix}4\sin 3t+3\cos 3t\\5\sin 3t\end{pmatrix}.$$

**40.** We have $\det(\mathbf{A}-\lambda\mathbf{I}) = \lambda^2+2\lambda+5=0$. For $\lambda_1=-1+2i$ we obtain

$$\mathbf{K}_1=\begin{pmatrix}2+2i\\1\end{pmatrix}$$

so that

$$\mathbf{X}_1=\begin{pmatrix}2+2i\\1\end{pmatrix}e^{(-1+2i)t}=\begin{pmatrix}2\cos 2t-2\sin 2t\\ \cos 2t\end{pmatrix}e^{-t}+i\begin{pmatrix}2\cos 2t+2\sin 2t\\ \sin 2t\end{pmatrix}e^{-t}.$$

Then

$$\mathbf{X}=c_1\begin{pmatrix}2\cos 2t-2\sin 2t\\ \cos 2t\end{pmatrix}e^{-t}+c_2\begin{pmatrix}2\cos 2t+2\sin 2t\\ \sin 2t\end{pmatrix}e^{-t}.$$

**41.** We have $\det(\mathbf{A}-\lambda\mathbf{I}) = -\lambda\left(\lambda^2+1\right)=0$. For $\lambda_1=0$ we obtain

$$\mathbf{K}_1=\begin{pmatrix}1\\0\\0\end{pmatrix}.$$

For $\lambda_2=i$ we obtain

$$\mathbf{K}_2=\begin{pmatrix}-i\\i\\1\end{pmatrix}$$

so that

$$\mathbf{X}_2=\begin{pmatrix}-i\\i\\1\end{pmatrix}e^{it}=\begin{pmatrix}\sin t\\-\sin t\\ \cos t\end{pmatrix}+i\begin{pmatrix}-\cos t\\ \cos t\\ \sin t\end{pmatrix}.$$

Then

$$\mathbf{X}=c_1\begin{pmatrix}1\\0\\0\end{pmatrix}+c_2\begin{pmatrix}\sin t\\-\sin t\\ \cos t\end{pmatrix}+c_3\begin{pmatrix}-\cos t\\ \cos t\\ \sin t\end{pmatrix}.$$

**42.** We have $\det(\mathbf{A}-\lambda\mathbf{I}) = -(\lambda+3)(\lambda^2-2\lambda+5)=0$. For $\lambda_1=-3$ we obtain

$$\mathbf{K}_1=\begin{pmatrix}0\\-2\\1\end{pmatrix}.$$

For $\lambda_2=1+2i$ we obtain

$$\mathbf{K}_2=\begin{pmatrix}-2-i\\-3i\\2\end{pmatrix}$$

so that

$$\mathbf{X}_2=\begin{pmatrix}-2\cos 2t+\sin 2t\\3\sin 2t\\2\cos 2t\end{pmatrix}e^{t}+i\begin{pmatrix}-\cos 2t-2\sin 2t\\-3\cos 2t\\2\sin 2t\end{pmatrix}e^{t}.$$

Then

$$\mathbf{X} = c_1 \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} -2\cos 2t + \sin 2t \\ 3\sin 2t \\ 2\cos 2t \end{pmatrix} e^{t} + c_3 \begin{pmatrix} -\cos 2t - 2\sin 2t \\ -3\cos 2t \\ 2\sin 2t \end{pmatrix} e^{t}.$$

**43.** We have $\det(\mathbf{A} - \lambda\mathbf{I}) = (1-\lambda)(\lambda^2 - 2\lambda + 2) = 0$. For $\lambda_1 = 1$ we obtain

$$\mathbf{K}_1 = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}.$$

For $\lambda_2 = 1 + i$ we obtain

$$\mathbf{K}_2 = \begin{pmatrix} 1 \\ i \\ i \end{pmatrix}$$

so that

$$\mathbf{X}_2 = \begin{pmatrix} 1 \\ i \\ i \end{pmatrix} e^{(1+i)t} = \begin{pmatrix} \cos t \\ -\sin t \\ -\sin t \end{pmatrix} e^{t} + i \begin{pmatrix} \sin t \\ \cos t \\ \cos t \end{pmatrix} e^{t}.$$

Then

$$\mathbf{X} = c_1 \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} \cos t \\ -\sin t \\ -\sin t \end{pmatrix} e^{t} + c_3 \begin{pmatrix} \sin t \\ \cos t \\ \cos t \end{pmatrix} e^{t}.$$

**44.** We have $\det(\mathbf{A} - \lambda\mathbf{I}) = -(\lambda - 6)(\lambda^2 - 8\lambda + 20) = 0$. For $\lambda_1 = 6$ we obtain

$$\mathbf{K}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

For $\lambda_2 = 4 + 2i$ we obtain

$$\mathbf{K}_2 = \begin{pmatrix} -i \\ 0 \\ 2 \end{pmatrix}$$

so that

$$\mathbf{X}_2 = \begin{pmatrix} -i \\ 0 \\ 2 \end{pmatrix} e^{(4+2i)t} = \begin{pmatrix} \sin 2t \\ 0 \\ 2\cos 2t \end{pmatrix} e^{4t} + i \begin{pmatrix} -\cos 2t \\ 0 \\ 2\sin 2t \end{pmatrix} e^{4t}.$$

Then

$$\mathbf{X} = c_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{6t} + c_2 \begin{pmatrix} \sin 2t \\ 0 \\ 2\cos 2t \end{pmatrix} e^{4t} + c_3 \begin{pmatrix} -\cos 2t \\ 0 \\ 2\sin 2t \end{pmatrix} e^{4t}.$$

**45.** We have $\det(\mathbf{A}-\lambda\mathbf{I}) = (2-\lambda)(\lambda^2+4\lambda+13) = 0$. For $\lambda_1 = 2$ we obtain

$$\mathbf{K}_1 = \begin{pmatrix} 28 \\ -5 \\ 25 \end{pmatrix}.$$

For $\lambda_2 = -2+3i$ we obtain

$$\mathbf{K}_2 = \begin{pmatrix} 4+3i \\ -5 \\ 0 \end{pmatrix}$$

so that

$$\mathbf{X}_2 = \begin{pmatrix} 4+3i \\ -5 \\ 0 \end{pmatrix} e^{(-2+3i)t} = \begin{pmatrix} 4\cos 3t - 3\sin 3t \\ -5\cos 3t \\ 0 \end{pmatrix} e^{-2t} + i \begin{pmatrix} 4\sin 3t + 3\cos 3t \\ -5\sin 3t \\ 0 \end{pmatrix} e^{-2t}.$$

Then

$$\mathbf{X} = c_1 \begin{pmatrix} 28 \\ -5 \\ 25 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 4\cos 3t - 3\sin 3t \\ -5\cos 3t \\ 0 \end{pmatrix} e^{-2t} + c_3 \begin{pmatrix} 4\sin 3t + 3\cos 3t \\ -5\sin 3t \\ 0 \end{pmatrix} e^{-2t}.$$

**46.** We have $\det(\mathbf{A}-\lambda\mathbf{I}) = -(\lambda+2)(\lambda^2+4) = 0$. For $\lambda_1 = -2$ we obtain

$$\mathbf{K}_1 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}.$$

For $\lambda_2 = 2i$ we obtain

$$\mathbf{K}_2 = \begin{pmatrix} -2-2i \\ 1 \\ 1 \end{pmatrix}$$

so that

$$\mathbf{X}_2 = \begin{pmatrix} -2-2i \\ 1 \\ 1 \end{pmatrix} e^{2it} = \begin{pmatrix} -2\cos 2t + 2\sin 2t \\ \cos 2t \\ \cos 2t \end{pmatrix} + i \begin{pmatrix} -2\cos 2t - 2\sin 2t \\ \sin 2t \\ \sin 2t \end{pmatrix}.$$

Then

$$\mathbf{X} = c_1 \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} -2\cos 2t + 2\sin 2t \\ \cos 2t \\ \cos 2t \end{pmatrix} + c_3 \begin{pmatrix} -2\cos 2t - 2\sin 2t \\ \sin 2t \\ \sin 2t \end{pmatrix}.$$

**47.** We have $\det(\mathbf{A}-\lambda\mathbf{I}) = (1-\lambda)(\lambda^2+25) = 0$. For $\lambda_1 = 1$ we obtain

$$\mathbf{K}_1 = \begin{pmatrix} 25 \\ -7 \\ 6 \end{pmatrix}.$$

For $\lambda_2 = 5i$ we obtain

$$\mathbf{K}_2 = \begin{pmatrix} 1+5i \\ 1 \\ 1 \end{pmatrix}$$

so that

$$\mathbf{X}_2 = \begin{pmatrix} 1+5i \\ 1 \\ 1 \end{pmatrix} e^{5it} = \begin{pmatrix} \cos 5t - 5\sin 5t \\ \cos 5t \\ \cos 5t \end{pmatrix} + i \begin{pmatrix} \sin 5t + 5\cos 5t \\ \sin 5t \\ \sin 5t \end{pmatrix}.$$

Then

$$\mathbf{X} = c_1 \begin{pmatrix} 25 \\ -7 \\ 6 \end{pmatrix} e^t + c_2 \begin{pmatrix} \cos 5t - 5\sin 5t \\ \cos 5t \\ \cos 5t \end{pmatrix} + c_3 \begin{pmatrix} \sin 5t + 5\cos 5t \\ \sin 5t \\ \sin 5t \end{pmatrix}.$$

If

$$\mathbf{X}(0) = \begin{pmatrix} 4 \\ 6 \\ -7 \end{pmatrix}$$

then $c_1 = c_2 = -1$ and $c_3 = 6$. The solution is

$$\mathbf{X} = -1 \begin{pmatrix} 25 \\ -7 \\ 6 \end{pmatrix} e^t - \begin{pmatrix} \cos 5t - 5\sin 5t \\ \cos 5t \\ \cos 5t \end{pmatrix} + 6 \begin{pmatrix} \sin 5t + 5\cos 5t \\ \sin 5t \\ \sin 5t \end{pmatrix}.$$

**48.** We have $\det(\mathbf{A}-\lambda\mathbf{I}) = \lambda^2 - 10\lambda + 29 = 0$. For $\lambda_1 = 5+2i$ we obtain

$$\mathbf{K}_1 = \begin{pmatrix} 1 \\ 1-2i \end{pmatrix}$$

so that

$$\mathbf{X}_1 = \begin{pmatrix} 1 \\ 1-2i \end{pmatrix} e^{(5+2i)t} = \begin{pmatrix} \cos 2t \\ \cos 2t + 2\sin 2t \end{pmatrix} e^{5t} + i \begin{pmatrix} \sin 2t \\ \sin 2t - 2\cos 2t \end{pmatrix} e^{5t}.$$

and

$$\mathbf{X} = c_1 \begin{pmatrix} \cos 2t \\ \cos 2t + 2\sin 2t \end{pmatrix} e^{5t} + c_2 \begin{pmatrix} \sin 2t \\ \sin 2t - 2\cos 2t \end{pmatrix} e^{5t}.$$

If $\mathbf{X}(0) = \begin{pmatrix} -2 \\ 8 \end{pmatrix}$, then $c_1 = -2$ and $c_2 = -5$. The solution is

$$\mathbf{X} = -2 \begin{pmatrix} \cos 2t \\ \cos 2t + 2\sin 2t \end{pmatrix} e^{5t} - 5 \begin{pmatrix} \sin 2t \\ \sin 2t - 2\cos 2t \end{pmatrix} e^{5t}.$$

**49. (a)** The system to solve is

$$\begin{cases} \dfrac{dx_1}{dt} = -\dfrac{1}{20}x_1 + \dfrac{1}{10}x_3 \\ \dfrac{dx_2}{dt} = \dfrac{1}{20}x_1 - \dfrac{1}{20}x_2 \\ \dfrac{dx_3}{dt} = \dfrac{1}{20}x_2 - \dfrac{1}{10}x_3 \end{cases} \qquad \text{Thus} \qquad \mathbf{X}' = \begin{pmatrix} -\frac{1}{20} & 0 & \frac{1}{10} \\ \frac{1}{20} & -\frac{1}{20} & 0 \\ 0 & \frac{1}{20} & -\frac{1}{10} \end{pmatrix}\mathbf{X}$$

**(b)** The eigenvalues and eigenvectors of the coefficient matrix are found by solving $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ to get

$$\lambda_1 = 0 \quad \text{so} \quad \mathbf{K}_1 = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}, \qquad \lambda_2 = -\frac{1}{10} + \frac{1}{20}i \quad \text{so} \quad \mathbf{K}_2 = \begin{pmatrix} -1-i \\ i \\ 1 \end{pmatrix},$$

$$\text{and} \qquad \lambda_3 = -\frac{1}{10} - \frac{1}{20}i \quad \text{so} \quad \mathbf{K}_3 = \begin{pmatrix} -1+i \\ -i \\ 1 \end{pmatrix}$$

The general solution is then

$$\mathbf{X}(t) = c_1\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} -\cos\frac{1}{20}t + \sin\frac{1}{20}t \\ -\sin\frac{1}{20}t \\ \cos\frac{1}{20}t \end{pmatrix}e^{-t/10} + c_3\begin{pmatrix} -\cos\frac{1}{20}t - \sin\frac{1}{20}t \\ \cos\frac{1}{20}t \\ \sin\frac{1}{20}t \end{pmatrix}e^{-t/10}$$

Using the initial conditions, we get

$$\mathbf{X}(0) = c_1\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + c_3\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 30 \\ 20 \\ 5 \end{pmatrix}$$

$$c_1 = 11, \quad c_2 = -6, \quad \text{and} \quad c_3 = -2$$

$$\mathbf{X}(t) = 11\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} - 6\begin{pmatrix} -\cos\frac{1}{20}t + \sin\frac{1}{20}t \\ -\sin\frac{1}{20}t \\ \cos\frac{1}{20}t \end{pmatrix}e^{-t/10} - 2\begin{pmatrix} -\cos\frac{1}{20}t - \sin\frac{1}{20}t \\ \cos\frac{1}{20}t \\ \sin\frac{1}{20}t \end{pmatrix}e^{-t/10}$$

**50. (a)** From Problem 49,

$$\mathbf{X}(t) = 11\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} - 6\begin{pmatrix} -\cos\frac{1}{20}t + \sin\frac{1}{20}t \\ -\sin\frac{1}{20}t \\ \cos\frac{1}{20}t \end{pmatrix}e^{-t/10} - 2\begin{pmatrix} \cos\frac{1}{20}t - \sin\frac{1}{20}t \\ -\cos\frac{1}{20}t \\ \sin\frac{1}{20}t \end{pmatrix}e^{-t/10}$$

The sum of the three individual solutions is

$$\left[22+\left(6\cos\frac{1}{20}t-6\sin\frac{1}{20}t\right)e^{-t/10}+\left(-2\cos\frac{1}{20}t+2\sin\frac{1}{20}t\right)e^{-t/10}\right]+$$
$$\left[22+\left(6\sin\frac{1}{20}t\right)e^{-t/10}+\left(2\cos\frac{1}{20}t\right)e^{-t/10}\right]+$$
$$\left[11+\left(-6\cos\frac{1}{20}t\right)e^{-t/10}+\left(-2\sin\frac{1}{20}t\right)e^{-t/10}\right]=55$$

This shows that the total amount of salt in the system remains constant over time. The fact that the sum $x_1(t)+x_2(t)+x_3(t)$ is a constant can also be seen from the fact that

$$\frac{dx_1}{dt}+\frac{dx_2}{dt}+\frac{dx_3}{dt}=\frac{d}{dt}\left(x_1+x_2+x_3\right)=0\,.$$

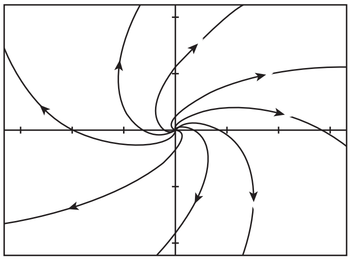
**(b)** From the last three equation in part (b) of Problem 49 we see that $\begin{pmatrix}x_1(t)\\x_2(t)\\x_3(t)\end{pmatrix}\to\begin{pmatrix}22\\22\\11\end{pmatrix}$ as $t\to\infty$. These are steady state solutions and represent the fact that regardless of the initial amount of salt in each tank (see note below) the salt attains a uniform distribution in the system over time, that is, $22/55$ or 40% of salt in each of the tanks $A$ and $B$ and $11/55$ or 20% in tank $C$.

(Note: Regardless of intial conditions note that from the equations in part (b) of Problem 49,

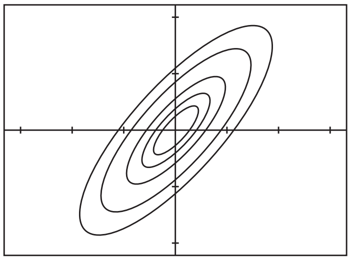
$$\begin{pmatrix}x_1(t)\\x_2(t)\\x_3(t)\end{pmatrix}\to\begin{pmatrix}2c_3\\2c_3\\c_3\end{pmatrix}\quad\text{as } t\to\infty$$

Moreover, $x_1(t)+x_2(t)+x_3(t)=5c_3$. So over time there is $2c_3/5c_3=0.4$ or 40% in tank $A$, $2c+3/5c_3=0.4$ or 40% in tank $B$, and $c_3/5c_3=0.2$ or 20% in tank $C$.)

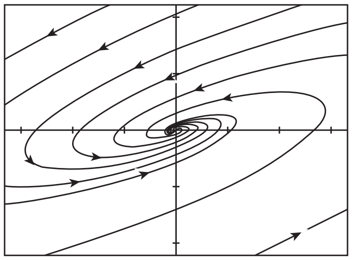
**51.**



Phase portrait for Problem 38



Phase portrait for Problem 39



Phase portrait for Problem 40

**52.** Letting $x_1=y_1$, $x_1'=y_2$, $x_2=y_3$, and $x_2'=y_4$ we have

$$y_2'=x_1''=-10x_1+4x_2=-10y_1+4y_3$$
$$y_4'=x_2''=4x_1-4x_2=4y_1-4y_3.$$

The corresponding linear system is

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= -10y_1 + 4y_3 \\ y_3' &= y_4 \\ y_4' &= 4y_1 - 4y_3 \end{aligned}$$

or

$$\mathbf{Y}' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -10 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 \\ 4 & 0 & -4 & 0 \end{pmatrix} \mathbf{Y}.$$

Using a CAS, we find eigenvalues $\pm\sqrt{2}i$ and $\pm 2\sqrt{3}i$ with corresponding eigenvectors

$$\begin{pmatrix} \mp\sqrt{2}\,i/4 \\ 1/2 \\ \mp\sqrt{2}\,i/2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1/2 \\ 0 \\ 1 \end{pmatrix} + i \begin{pmatrix} \mp\sqrt{2}/4 \\ 0 \\ \mp\sqrt{2}/2 \\ 0 \end{pmatrix}$$

and

$$\begin{pmatrix} \pm\sqrt{3}\,i/3 \\ -2 \\ \mp\sqrt{3}\,i/6 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ 0 \\ 1 \end{pmatrix} + i \begin{pmatrix} \pm\sqrt{3}/3 \\ 0 \\ \mp\sqrt{3}/6 \\ 0 \end{pmatrix}.$$

Thus

$$\mathbf{Y}(t) = c_1\left[\begin{pmatrix}0\\1/2\\0\\1\end{pmatrix}\cos\sqrt{2}\,t - \begin{pmatrix}-\sqrt{2}/4\\0\\-\sqrt{2}/2\\0\end{pmatrix}\sin\sqrt{2}\,t\right]$$

$$+ c_2\left[\begin{pmatrix}-\sqrt{2}/4\\0\\-\sqrt{2}/2\\0\end{pmatrix}\cos\sqrt{2}\,t + \begin{pmatrix}0\\1/2\\0\\1\end{pmatrix}\sin\sqrt{2}\,t\right]$$

$$+ c_3\left[\begin{pmatrix}0\\-2\\0\\1\end{pmatrix}\cos 2\sqrt{3}\,t - \begin{pmatrix}\sqrt{3}/3\\0\\-\sqrt{3}/6\\0\end{pmatrix}\sin 2\sqrt{3}\,t\right]$$

$$+ c_4\left[\begin{pmatrix}\sqrt{3}/3\\0\\-\sqrt{3}/6\\0\end{pmatrix}\cos 2\sqrt{3}\,t + \begin{pmatrix}0\\-2\\0\\1\end{pmatrix}\sin 2\sqrt{3}\,t\right].$$

The initial conditions $y_1(0) = 0$, $y_2(0) = 1$, $y_3(0) = 0$, and $y_4(0) = -1$ imply $c_1 = -\frac{2}{5}$, $c_2 = 0$, $c_3 = -\frac{3}{5}$, and $c_4 = 0$. Thus,

$$x_1(t) = y_1(t) = -\frac{\sqrt{2}}{10}\sin\sqrt{2}\,t + \frac{\sqrt{3}}{5}\sin 2\sqrt{3}\,t$$

$$x_2(t) = y_3(t) = -\frac{\sqrt{2}}{5}\sin\sqrt{2}\,t - \frac{\sqrt{3}}{10}\sin 2\sqrt{3}\,t.$$

**53. (a)** From $\det(\mathbf{A} - \lambda\mathbf{I}) = \lambda(\lambda - 2) = 0$ we get $\lambda_1 = 0$ and $\lambda_2 = 2$. For $\lambda_1 = 0$ we obtain

$$\left(\begin{array}{cc|c}1 & 1 & 0\\1 & 1 & 0\end{array}\right) \longrightarrow \left(\begin{array}{cc|c}1 & 1 & 0\\0 & 0 & 0\end{array}\right) \quad \text{so that} \quad \mathbf{K}_1 = \begin{pmatrix}-1\\1\end{pmatrix}.$$

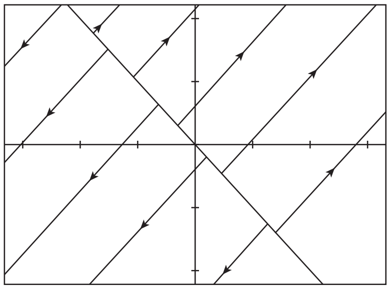
For $\lambda_2 = 2$ we obtain

$$\left(\begin{array}{cc|c}-1 & 1 & 0\\1 & -1 & 0\end{array}\right) \longrightarrow \left(\begin{array}{cc|c}-1 & 1 & 0\\0 & 0 & 0\end{array}\right) \quad \text{so that} \quad \mathbf{K}_2 = \begin{pmatrix}1\\1\end{pmatrix}.$$

Then

$$\mathbf{X} = c_1\begin{pmatrix}-1\\1\end{pmatrix} + c_2\begin{pmatrix}1\\1\end{pmatrix}e^{2t}.$$

The line $y = -x$ is not a trajectory of the system. Trajectories are $x = -c_1 + c_2e^{2t}$, $y = c_1 + c_2e^{2t}$ or $y = x + 2c_1$. This is a family of lines perpendicular to the line $y = -x$. All of the constant solutions of the system do, however, lie on the line $y = -x$.



**(b)** From $\det(\mathbf{A} - \lambda\mathbf{I}) = \lambda^2 = 0$ we get $\lambda_1 = 0$ and

$$\mathbf{K} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

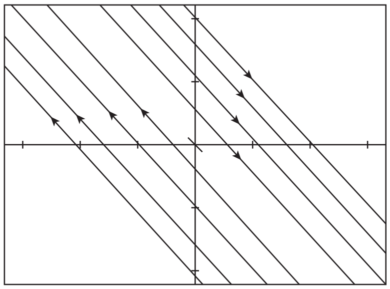
A solution of $(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{P} = \mathbf{K}$ is

$$\mathbf{P} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

so that

$$\mathbf{X} = c_1\begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2\left[\begin{pmatrix} -1 \\ 1 \end{pmatrix} t + \begin{pmatrix} -1 \\ 0 \end{pmatrix}\right].$$

All trajectories are parallel to $y = -x$, but $y = -x$ is not a trajectory. There are constant solutions of the system, however, that do lie on the line $y = -x$.



**54.** The system of differential equations is

$$x_1' = 2x_1 + x_2$$

$$x_2' = 2x_2$$

$$x_3' = 2x_3$$

$$x_4' = 2x_4 + x_5$$

$$x_5' = 2x_5.$$

We see immediately that $x_2 = c_2e^{2t}$, $x_3 = c_3e^{2t}$, and $x_5 = c_5e^{2t}$. Then

$$x_1' = 2x_1 + c_2e^{2t} \qquad \text{so} \qquad x_1 = c_2te^{2t} + c_1e^{2t},$$

and

$$x_4' = 2x_4 + c_5e^{2t} \qquad \text{so} \qquad x_4 = c_5te^{2t} + c_4e^{2t}.$$

The general solution of the system is

$$\mathbf{X}=\begin{pmatrix}c_2te^{2t}+c_1e^{2t}\\ c_2e^{2t}\\ c_3e^{2t}\\ c_5te^{2t}+c_4e^{2t}\\ c_5e^{2t}\end{pmatrix}=c_1\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}e^{2t}+c_2\left[\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}te^{2t}+\begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix}e^{2t}\right]$$

$$+c_3\begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix}e^{2t}+c_4\begin{pmatrix}0\\0\\0\\1\\0\end{pmatrix}e^{2t}+c_5\left[\begin{pmatrix}0\\0\\0\\1\\0\end{pmatrix}te^{2t}+\begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix}e^{2t}\right]$$

$$=c_1\mathbf{K}_1e^{2t}+c_2\left[\mathbf{K}_1te^{2t}+\begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix}e^{2t}\right]+c_3\mathbf{K}_2e^{2t}+c_4\mathbf{K}_3e^{2t}+c_5\left[\mathbf{K}_3te^{2t}+\begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix}e^{2t}\right].$$

There are three solutions of the form $\mathbf{X}=\mathbf{K}e^{2t}$, where $\mathbf{K}$ is an eigenvector, and two solutions of the form $\mathbf{X}=\mathbf{K}te^{2t}+\mathbf{P}e^{2t}$. See (12) in the text. From (13) and (14) in the text

$$(\mathbf{A}-2\mathbf{I})\mathbf{K}_1=\mathbf{0}$$

and

$$(\mathbf{A}-2\mathbf{I})\mathbf{P}_1=\mathbf{K}_1.$$

This implies

$$\begin{pmatrix}0&1&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&1\\0&0&0&0&0\end{pmatrix}\begin{pmatrix}p_1\\p_2\\p_3\\p_4\\p_5\end{pmatrix}=\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix},$$

so $p_2=1$ and $p_5=0$, while $p_1$, $p_3$, and $p_4$ are arbitrary. Choosing $p_1=p_3=p_4=0$ we have

$$\mathbf{P}_1=\begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix}.$$

Therefore a solution is

$$\mathbf{X} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} te^{2t} + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} e^{2t}.$$

Repeating for $\mathbf{K}_3$ we find

$$\mathbf{P}_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

so another solution is

$$\mathbf{X} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} te^{2t} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} e^{2t}.$$

**55.** From $x = 2\cos 2t - 2\sin 2t$, $y = -\cos 2t$ we find $x + 2y = -2\sin 2t$. Then

$$(x + 2y)^2 = 4\sin^2 2t = 4(1 - \cos^2 2t) = 4 - 4\cos^2 2t = 4 - 4y^2$$

and

$$x^2 + 4xy + 4y^2 = 4 - 4y^2 \qquad \text{or} \qquad x^2 + 4xy + 8y^2 = 4.$$

This is a rotated conic section and, from the discriminant $b^2 - 4ac = 16 - 32 < 0$, we see that the curve is an ellipse.

**56.** Suppose the eigenvalues are $\alpha \pm i\beta$, $\beta > 0$. In Problem 38 the eigenvalues are $5 \pm 3i$, in Problem 39 they are $\pm 3i$, and in Problem 40 they are $-1 \pm 2i$. From Problem 51 we deduce that the phase portrait will consist of a family of closed curves when $\alpha = 0$ and spirals when $\alpha \neq 0$. The origin will be a repellor when $\alpha > 0$, and an attractor when $\alpha < 0$.

## 8.3 Nonhomogeneous Linear Systems

**1.** Solving

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 2-\lambda & 3 \\ -1 & -2-\lambda \end{vmatrix} = \lambda^2 - 1 = (\lambda - 1)(\lambda + 1) = 0$$

we obtain eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 1$. Corresponding eigenvectors are

$$\mathbf{K}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{K}_2 = \begin{pmatrix} -3 \\ 1 \end{pmatrix}.$$

Thus

$$\mathbf{X}_c = c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^{t}.$$

Substituting

$$\mathbf{X}_p = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$$

into the system yields

$$\begin{aligned} 2a_1 + 3b_1 &= 7 \\ -a_1 - 2b_1 &= -5, \end{aligned}$$

from which we obtain $a_1 = -1$ and $b_1 = 3$. Then

$$\mathbf{X}(t) = c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^{t} + \begin{pmatrix} -1 \\ 3 \end{pmatrix}.$$

**2.** Solving

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 5-\lambda & 9 \\ -1 & 11-\lambda \end{vmatrix} = \lambda^2 - 16\lambda + 64 = (\lambda - 8)^2 = 0$$

we obtain the eigenvalue $\lambda = 8$. A corresponding eigenvector is

$$\mathbf{K} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

Solving $(\mathbf{A} - 8\mathbf{I})\mathbf{P} = \mathbf{K}$ we obtain

$$\mathbf{P} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Thus

$$\mathbf{X}_c = c_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{8t} + c_2 \left[ \begin{pmatrix} 3 \\ 1 \end{pmatrix} te^{8t} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{8t} \right].$$

Substituting

$$\mathbf{X}_p = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$$

into the system yields

$$\begin{aligned} 5a_1 + 9b_1 &= -2 \\ -a_1 + 11b_1 &= -6, \end{aligned}$$

from which we obtain $a_1 = 1/2$ and $b_1 = -1/2$. Then

$$\mathbf{X}(t) = c_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{8t} + c_2 \left[ \begin{pmatrix} 3 \\ 1 \end{pmatrix} te^{8t} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{8t} \right] + \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}.$$

**3.** Solving

$$\det(\mathbf{A}-\lambda\mathbf{I}) = \begin{vmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda - 8 = (\lambda-4)(\lambda+2) = 0$$

we obtain eigenvalues $\lambda_1 = -2$ and $\lambda_2 = 4$. Corresponding eigenvectors are

$$\mathbf{K}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{and} \quad \mathbf{K}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Thus

$$\mathbf{X}_c = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}.$$

Substituting

$$\mathbf{X}_p = \begin{pmatrix} a_3 \\ b_3 \end{pmatrix} t^2 + \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} t + \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$$

into the system yields

$$\begin{aligned} a_3 + 3b_3 &= 2 & \qquad a_2 + 3b_2 &= 2a_3 & \qquad a_1 + 3b_1 &= a_2 \\ 3a_3 + b_3 &= 0 & 3a_2 + b_2 + 1 &= 2b_3 & 3a_1 + b_1 + 5 &= b_2 \end{aligned}$$

from which we obtain $a_3 = -1/4$, $b_3 = 3/4$, $a_2 = 1/4$, $b_2 = -1/4$, $a_1 = -2$, and $b_1 = 3/4$. Then

$$\mathbf{X}(t) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} + \begin{pmatrix} -\frac{1}{4} \\ \frac{3}{4} \end{pmatrix} t^2 + \begin{pmatrix} \frac{1}{4} \\ -\frac{1}{4} \end{pmatrix} t + \begin{pmatrix} -2 \\ \frac{3}{4} \end{pmatrix}.$$

**4.** Solving

$$\det(\mathbf{A}-\lambda\mathbf{I}) = \begin{vmatrix} 1-\lambda & -4 \\ 4 & 1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda + 17 = 0$$

we obtain eigenvalues $\lambda_1 = 1+4i$ and $\lambda_2 = 1-4i$. Corresponding eigenvectors are

$$\mathbf{K}_1 = \begin{pmatrix} i \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{K}_2 = \begin{pmatrix} -i \\ 1 \end{pmatrix}.$$

Thus

$$\begin{aligned} \mathbf{X}_c &= c_1 \left[ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos 4t + \begin{pmatrix} -1 \\ 0 \end{pmatrix} \sin 4t \right] e^t + c_2 \left[ \begin{pmatrix} -1 \\ 0 \end{pmatrix} \cos 4t - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin 4t \right] e^t \\ &= c_1 \begin{pmatrix} -\sin 4t \\ \cos 4t \end{pmatrix} e^t + c_2 \begin{pmatrix} -\cos 4t \\ -\sin 4t \end{pmatrix} e^t. \end{aligned}$$

Substituting

$$\mathbf{X}_p = \begin{pmatrix} a_3 \\ b_3 \end{pmatrix} t + \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} + \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} e^{6t}$$

into the system yields

$$a_3 - 4b_3 = -4 \qquad a_2 - 4b_2 = a_3 \qquad -5a_1 - 4b_1 = -9$$
$$4a_3 + b_3 = 1 \qquad 4a_2 + b_2 = b_3 \qquad 4a_1 - 5b_1 = -1$$

from which we obtain $a_3 = 0$, $b_3 = 1$, $a_2 = 4/17$, $b_2 = 1/17$, $a_1 = 1$, and $b_1 = 1$. Then

$$\mathbf{X}(t) = c_1 \begin{pmatrix} -\sin 4t \\ \cos 4t \end{pmatrix} e^t + c_2 \begin{pmatrix} -\cos 4t \\ -\sin 4t \end{pmatrix} e^t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} t + \begin{pmatrix} \frac{4}{17} \\ \frac{1}{17} \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{6t}.$$

**5.** Solving

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 4-\lambda & \frac{1}{3} \\ 9 & 6-\lambda \end{vmatrix} = \lambda^2 - 10\lambda + 21 = (\lambda - 3)(\lambda - 7) = 0$$

we obtain the eigenvalues $\lambda_1 = 3$ and $\lambda_2 = 7$. Corresponding eigenvectors are

$$\mathbf{K}_1 = \begin{pmatrix} 1 \\ -3 \end{pmatrix} \quad \text{and} \quad \mathbf{K}_2 = \begin{pmatrix} 1 \\ 9 \end{pmatrix}.$$

Thus

$$\mathbf{X}_c = c_1 \begin{pmatrix} 1 \\ -3 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ 9 \end{pmatrix} e^{7t}.$$

Substituting

$$\mathbf{X}_p = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} e^t$$

into the system yields

$$3a_1 + \frac{1}{3}b_1 = 3$$
$$9a_1 + 5b_1 = -10$$

from which we obtain $a_1 = 55/36$ and $b_1 = -19/4$. Then

$$\mathbf{X}(t) = c_1 \begin{pmatrix} 1 \\ -3 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ 9 \end{pmatrix} e^{7t} + \begin{pmatrix} \frac{55}{36} \\ -\frac{19}{4} \end{pmatrix} e^t.$$

**6.** Solving

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} -1-\lambda & 5 \\ -1 & 1-\lambda \end{vmatrix} = \lambda^2 + 4 = 0$$

we obtain the eigenvalues $\lambda_1 = 2i$ and $\lambda_2 = -2i$. Corresponding eigenvectors are

$$\mathbf{K}_1 = \begin{pmatrix} 5 \\ 1+2i \end{pmatrix} \quad \text{and} \quad \mathbf{K}_2 = \begin{pmatrix} 5 \\ 1-2i \end{pmatrix}.$$

Thus

$$\mathbf{X}_c = c_1 \begin{pmatrix} 5\cos 2t \\ \cos 2t - 2\sin 2t \end{pmatrix} + c_2 \begin{pmatrix} 5\sin 2t \\ 2\cos 2t + \sin 2t \end{pmatrix}.$$

Substituting

$$\mathbf{X}_p = \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} \cos t + \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} \sin t$$

into the system yields

$$\begin{aligned} -a_2 + 5b_2 - a_1 &= 0 \\ -a_2 + b_2 - b_1 - 2 &= 0 \\ -a_1 + 5b_1 + a_2 + 1 &= 0 \\ -a_1 + b_1 + b_2 &= 0 \end{aligned}$$

from which we obtain $a_2 = -3$, $b_2 = -2/3$, $a_1 = -1/3$, and $b_1 = 1/3$. Then

$$\mathbf{X}(t) = c_1 \begin{pmatrix} 5\cos 2t \\ \cos 2t - 2\sin 2t \end{pmatrix} + c_2 \begin{pmatrix} 5\sin 2t \\ 2\cos 2t + \sin 2t \end{pmatrix} + \begin{pmatrix} -3 \\ -\frac{2}{3} \end{pmatrix} \cos t + \begin{pmatrix} -\frac{1}{3} \\ \frac{1}{3} \end{pmatrix} \sin t.$$

**7.** Solving

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 1-\lambda & 1 & 1 \\ 0 & 2-\lambda & 3 \\ 0 & 0 & 5-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda)(5-\lambda) = 0$$

we obtain the eigenvalues $\lambda_1 = 1$, $\lambda_2 = 2$, and $\lambda_3 = 5$. Corresponding eigenvectors are

$$\mathbf{K}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{K}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{K}_3 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}.$$

Thus

$$\mathbf{X}_c = C_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{t} + C_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{2t} + C_3 \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} e^{5t}.$$

Substituting

$$\mathbf{X}_p = \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} e^{4t}$$

into the system yields

$$\begin{aligned} -3a_1 + b_1 + c_1 &= -1 \\ -2b_1 + 3c_1 &= 1 \\ c_1 &= -2 \end{aligned}$$

from which we obtain $c_1 = -2$, $b_1 = -7/2$, and $a_1 = -3/2$. Then

$$\mathbf{X}(t) = C_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{t} + C_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{2t} + C_3 \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} e^{5t} + \begin{pmatrix} -\frac{3}{2} \\ -\frac{7}{2} \\ -2 \end{pmatrix} e^{4t}.$$

**8.** Solving

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} -\lambda & 0 & 5 \\ 0 & 5-\lambda & 0 \\ 5 & 0 & -\lambda \end{vmatrix} = -(\lambda-5)^2(\lambda+5) = 0$$

we obtain the eigenvalues $\lambda_1 = 5$, $\lambda_2 = 5$, and $\lambda_3 = -5$. Corresponding eigenvectors are

$$\mathbf{K}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{K}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{K}_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

Thus

$$\mathbf{X}_c = C_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{5t} + C_2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{5t} + C_3 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{-5t}.$$

Substituting

$$\mathbf{X}_p = \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix}$$

into the system yields

$$\begin{aligned} 5c_1 &= -5 \\ 5b_1 &= 10 \\ 5a_1 &= -40 \end{aligned}$$

from which we obtain $c_1 = -1$, $b_1 = 2$, and $a_1 = -8$. Then

$$\mathbf{X}(t) = C_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{5t} + C_2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{5t} + C_3 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{-5t} + \begin{pmatrix} -8 \\ 2 \\ -1 \end{pmatrix}.$$

**9.** First solve the associated homogeneous system

$$\mathbf{X}' = \begin{pmatrix} -1 & -2 \\ 3 & 4 \end{pmatrix} \mathbf{X}$$

The eigenvalues and eigenvectors of the coefficient matrix are found by solving $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ to get

$$\lambda_1 = 1 \quad \text{so} \quad \mathbf{K}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \text{and} \qquad \lambda_2 = 2 \quad \text{so} \quad \mathbf{K}_2 = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

The complementary solution is then

$$\mathbf{X}_c(t) = c_1\mathbf{K}_1 e^{\lambda_1 t} + c_2\mathbf{K}_2 e^{\lambda_2 t} = c_1\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t + c_2\begin{pmatrix} 2 \\ -3 \end{pmatrix} e^{2t}$$

Based on the form of $\mathbf{F}(t)$, guess $\mathbf{X}_p = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$ and force it into the original system to get $\mathbf{X}_p = \begin{pmatrix} -9 \\ 6 \end{pmatrix}$. The general solution is then

$$\mathbf{X} = \mathbf{X}_c + \mathbf{X}_p = c_1\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t + c_2\begin{pmatrix} 2 \\ -3 \end{pmatrix} e^{2t} + \begin{pmatrix} -9 \\ 6 \end{pmatrix}$$

Next use the initial condition to solve for $c_1$ and $c_2$:

$$\mathbf{X}(0) = c_1\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2\begin{pmatrix} 2 \\ -3 \end{pmatrix} + \begin{pmatrix} -9 \\ 6 \end{pmatrix} = \begin{pmatrix} -4 \\ 5 \end{pmatrix}$$

$$c_1 = 13 \quad \text{and} \quad c_2 = -4$$

$$\mathbf{X} = 13\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t - 4\begin{pmatrix} 2 \\ -3 \end{pmatrix} e^{2t} + \begin{pmatrix} -9 \\ 6 \end{pmatrix}$$

**10.** First solve the associated homogeneous system

$$\mathbf{X}' = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}\mathbf{X}$$

The eigenvalues of the coefficient matrix are found by solving $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ to get $\lambda_1 = \lambda_2 = 2$ so we get

$$\mathbf{K}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \qquad \text{Thus} \quad \mathbf{X}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{2t}$$

A second solution is found to be $\mathbf{X}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} te^{2t} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t}$, and so we have

$$\mathbf{X}_c(t) = c_1\begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{2t} + c_2\left[\begin{pmatrix} -1 \\ 1 \end{pmatrix} te^{2t} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t}\right]$$

Based on the form of $\mathbf{F}(t)$, guess $\mathbf{X}_p = \begin{pmatrix} a_2 t + a_1 \\ b_2 t + b_1 \end{pmatrix}$ and force it into the original system to get $\mathbf{X}_p = \begin{pmatrix} -t-1 \\ 0 \end{pmatrix}$. The general solution is then

$$\mathbf{X}(t) = c_1\begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{2t} + c_2\left[\begin{pmatrix} -1 \\ 1 \end{pmatrix} te^{2t} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t}\right] + \begin{pmatrix} -t-1 \\ 0 \end{pmatrix}$$

Next use the initial condition to solve for $c_1$ and $c_2$:

$$\mathbf{X}(0) = c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 \left[ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] + \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$c_1 = 2 \quad \text{and} \quad c_2 = 6$$

$$\mathbf{X}(t) = 2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{2t} + 6 \left[ \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{2t} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t} \right] + \begin{pmatrix} -t-1 \\ 0 \end{pmatrix}$$

**11. (a)** From the discussion in Section 2.9 we get the system

$$\begin{cases} \dfrac{dx_1}{dt} = -\dfrac{3}{100}x_1 + \dfrac{1}{100}x_2 \\ \dfrac{dx_2}{dt} = \dfrac{1}{50}x_1 - \dfrac{1}{25}x_2 + 1 \end{cases} \qquad \text{Thus} \quad \mathbf{X}' = \begin{pmatrix} -\frac{3}{100} & \frac{1}{100} \\ \frac{1}{50} & -\frac{1}{25} \end{pmatrix} \mathbf{X} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

**(b)** First solve the associated homogeneous system

$$\mathbf{X}' = \begin{pmatrix} -\frac{3}{100} & \frac{1}{100} \\ \frac{1}{50} & -\frac{1}{25} \end{pmatrix} \mathbf{X}$$

The eigenvalues and eigenvectors of the coefficient matrix are found by solving $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ to get

$$\lambda_1 = -\frac{1}{20} \quad \text{so} \quad \mathbf{K}_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \qquad \text{and} \qquad \lambda_2 = -\frac{1}{50} \quad \text{so} \quad \mathbf{K}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The complementary solution is then

$$\mathbf{X}_c(t) = c_1\mathbf{K}_1 e^{\lambda_1 t} + c_2\mathbf{K}_2 e^{\lambda_2 t} = c_1 \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{-t/20} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t/50}$$

Based on the form of $\mathbf{F}(t)$, guess $\mathbf{X}_p = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$ and force it into the original system to get $\mathbf{X}_p = \begin{pmatrix} 10 \\ 30 \end{pmatrix}$. The general solution is then

$$\mathbf{X} = \mathbf{X}_c + \mathbf{X}_p = c_1 \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{-t/20} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t/50} + \begin{pmatrix} 10 \\ 30 \end{pmatrix}$$

Next use the initial condition to solve for $c_1$ and $c_2$:

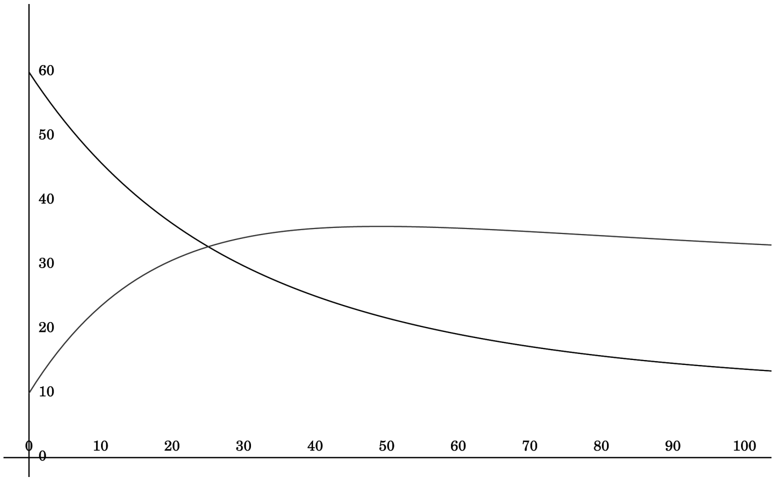
$$\mathbf{X}(0) = c_1 \begin{pmatrix} -1 \\ 2 \end{pmatrix} + c_2 \left[ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right] + \begin{pmatrix} 10 \\ 30 \end{pmatrix} = \begin{pmatrix} 60 \\ 10 \end{pmatrix}$$

$$c_1 = -\frac{70}{3} \quad \text{and} \quad c_2 = \frac{80}{3}$$

$$\mathbf{X} = -\frac{70}{3} \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{-t/20} + \frac{80}{3} \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t/50} + \begin{pmatrix} 10 \\ 30 \end{pmatrix}$$

**(c)** The solution $\mathbf{X}(t) \to \begin{pmatrix} 10 \\ 30 \end{pmatrix}$ as $t \to \infty$. Over a long period of time the total amount of salt in the system of tanks approaches 40 lb.

**(d)**



**12. (a)** Let $\mathbf{X} = \begin{pmatrix} i_2 \\ i_3 \end{pmatrix}$ so that

$$\mathbf{X}' = \begin{pmatrix} -2 & -2 \\ -2 & -5 \end{pmatrix} \mathbf{X} + \begin{pmatrix} 60 \\ 60 \end{pmatrix} \qquad \text{and} \qquad \mathbf{X}_c = c_1 \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-6t}.$$

If $\mathbf{X}_p = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$ then $\mathbf{X}_p = \begin{pmatrix} 30 \\ 0 \end{pmatrix}$ so that

$$\mathbf{X} = c_1 \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-6t} + \begin{pmatrix} 30 \\ 0 \end{pmatrix}.$$

For $\mathbf{X}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ we find $c_1 = -12$ and $c_2 = -6$.

**(b)** $i_1(t) = i_2(t) + i_3(t) = -12e^{-t} - 18e^{-6t} + 30.$

**13.** From

$$\mathbf{X}' = \begin{pmatrix} 3 & -3 \\ 2 & -2 \end{pmatrix} \mathbf{X} + \begin{pmatrix} 4 \\ -1 \end{pmatrix}$$

we obtain

$$\mathbf{X}_c = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{t}.$$

Then

$$\boldsymbol{\Phi} = \begin{pmatrix} 1 & 3e^{t} \\ 1 & 2e^{t} \end{pmatrix} \quad \text{and} \quad \boldsymbol{\Phi}^{-1} = \begin{pmatrix} -2 & 3 \\ e^{-t} & -e^{-t} \end{pmatrix}$$

so that

$$\mathbf{U}=\int\mathbf{\Phi}^{-1}\mathbf{F}\,dt=\int\begin{pmatrix}-11\\5e^{-t}\end{pmatrix}dt=\begin{pmatrix}-11t\\-5e^{-t}\end{pmatrix}$$

and

$$\mathbf{X}_p=\mathbf{\Phi}\mathbf{U}=\begin{pmatrix}-11\\-11\end{pmatrix}t+\begin{pmatrix}-15\\-10\end{pmatrix}.$$

The solution is

$$\mathbf{X}=\mathbf{X}_c+\mathbf{X}_p=c_1\begin{pmatrix}1\\1\end{pmatrix}+c_2\begin{pmatrix}3\\2\end{pmatrix}e^t-\begin{pmatrix}11\\11\end{pmatrix}t-\begin{pmatrix}15\\10\end{pmatrix}$$

**14.** From

$$\mathbf{X}'=\begin{pmatrix}2&-1\\3&-2\end{pmatrix}\mathbf{X}+\begin{pmatrix}0\\4\end{pmatrix}t$$

we obtain

$$\mathbf{X}_c=c_1\begin{pmatrix}1\\1\end{pmatrix}e^t+c_2\begin{pmatrix}1\\3\end{pmatrix}e^{-t}.$$

Then

$$\mathbf{\Phi}=\begin{pmatrix}e^t&e^{-t}\\e^t&3e^{-t}\end{pmatrix}\quad\text{and}\quad\mathbf{\Phi}^{-1}=\begin{pmatrix}\frac{3}{2}e^{-t}&-\frac{1}{2}e^{-t}\\-\frac{1}{2}e^t&\frac{1}{2}e^t\end{pmatrix}$$

so that

$$\mathbf{U}=\int\mathbf{\Phi}^{-1}\mathbf{F}\,dt=\int\begin{pmatrix}-2te^{-t}\\2te^t\end{pmatrix}dt=\begin{pmatrix}2te^{-t}+2e^{-t}\\2te^t-2e^t\end{pmatrix}$$

and

$$\mathbf{X}_p=\mathbf{\Phi}\mathbf{U}=\begin{pmatrix}4\\8\end{pmatrix}t+\begin{pmatrix}0\\-4\end{pmatrix}.$$

The solution is

$$\mathbf{X}=\mathbf{X}_c+\mathbf{X}_p=c_1\begin{pmatrix}1\\1\end{pmatrix}e^t+c_2\begin{pmatrix}1\\3\end{pmatrix}e^{-t}+\begin{pmatrix}4\\8\end{pmatrix}t-\begin{pmatrix}0\\4\end{pmatrix}$$

**15.** From

$$\mathbf{X}'=\begin{pmatrix}3&-5\\\frac{3}{4}&-1\end{pmatrix}\mathbf{X}+\begin{pmatrix}1\\-1\end{pmatrix}e^{t/2}$$

we obtain

$$\mathbf{X}_c=c_1\begin{pmatrix}10\\3\end{pmatrix}e^{3t/2}+c_2\begin{pmatrix}2\\1\end{pmatrix}e^{t/2}.$$

Then

$$\mathbf{\Phi}=\begin{pmatrix}10e^{3t/2}&2e^{t/2}\\3e^{3t/2}&e^{t/2}\end{pmatrix}\quad\text{and}\quad\mathbf{\Phi}^{-1}=\begin{pmatrix}\frac{1}{4}e^{-3t/2}&-\frac{1}{2}e^{-3t/2}\\-\frac{3}{4}e^{-t/2}&\frac{5}{2}e^{-t/2}\end{pmatrix}$$

so that

$$\mathbf{U} = \int \mathbf{\Phi}^{-1}\mathbf{F}\,dt = \int \begin{pmatrix} \frac{3}{4}e^{-t} \\ -\frac{13}{4} \end{pmatrix} dt = \begin{pmatrix} -\frac{3}{4}e^{-t} \\ -\frac{13}{4}t \end{pmatrix}$$

and

$$\mathbf{X}_p = \mathbf{\Phi}\mathbf{U} = \begin{pmatrix} -\frac{13}{2} \\ -\frac{13}{4} \end{pmatrix} te^{t/2} + \begin{pmatrix} -\frac{15}{2} \\ -\frac{9}{4} \end{pmatrix} e^{t/2}.$$

The solution is

$$\mathbf{X} = \mathbf{X}_c + \mathbf{X}_p = c_1 \begin{pmatrix} 10 \\ 3 \end{pmatrix} e^{3t/2} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{t/2} - \begin{pmatrix} \frac{13}{2} \\ \frac{13}{4} \end{pmatrix} te^{t/2} - \begin{pmatrix} \frac{15}{2} \\ \frac{9}{4} \end{pmatrix} e^{t/2}$$

**16.** From

$$\mathbf{X}' = \begin{pmatrix} 2 & -1 \\ 4 & 2 \end{pmatrix} \mathbf{X} + \begin{pmatrix} \sin 2t \\ 2\cos 2t \end{pmatrix}$$

we obtain

$$\mathbf{X}_c = c_1 \begin{pmatrix} -\sin 2t \\ 2\cos 2t \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} \cos 2t \\ 2\sin 2t \end{pmatrix} e^{2t}.$$

Then

$$\mathbf{\Phi} = \begin{pmatrix} -e^{2t}\sin 2t & e^{2t}\cos 2t \\ 2e^{2t}\cos 2t & 2e^{2t}\sin 2t \end{pmatrix} \quad \text{and} \quad \mathbf{\Phi}^{-1} = \begin{pmatrix} -e^{-2t}\sin 2t & \frac{1}{2}\,e^{-2t}\cos 2t \\ e^{-2t}\cos 2t & \frac{1}{2}\,e^{-2t}\sin 2t \end{pmatrix}$$

so that

$$\mathbf{U} = \int \mathbf{\Phi}^{-1}\mathbf{F}\,dt = \int \begin{pmatrix} \cos 4t \\ \sin 4t \end{pmatrix} dt = \begin{pmatrix} \frac{1}{4}\sin 4t \\ -\frac{1}{4}\cos 4t \end{pmatrix}$$

and

$$\mathbf{X}_p = \mathbf{\Phi}\mathbf{U} = \begin{pmatrix} -\frac{1}{4}\sin 2t\sin 4t - \frac{1}{4}\cos 2t\cos 4t \\ \frac{1}{2}\cos 2t\sin 4t - \frac{1}{2}\sin 2t\cos 4t \end{pmatrix} e^{2t}.$$

The solution is

$$\mathbf{X} = \mathbf{X}_c + \mathbf{X}_p = c_1 \begin{pmatrix} -\sin 2t \\ 2\cos 2t \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} \cos 2t \\ 2\sin 2t \end{pmatrix} e^{2t} + \begin{pmatrix} -\frac{1}{4}\sin 2t\sin 4t - \frac{1}{4}\cos 2t\cos 4t \\ \frac{1}{2}\cos 2t\sin 4t - \frac{1}{2}\sin 2t\cos 4t \end{pmatrix} e^{2t}$$

**17.** From

$$\mathbf{X}' = \begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix} \mathbf{X} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{t}$$

we obtain

$$\mathbf{X}_c = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}.$$

Then

$$\mathbf{\Phi} = \begin{pmatrix} 2e^{t} & e^{2t} \\ e^{t} & e^{2t} \end{pmatrix} \quad \text{and} \quad \mathbf{\Phi}^{-1} = \begin{pmatrix} e^{-t} & -e^{-t} \\ -e^{-2t} & 2e^{-2t} \end{pmatrix}$$

so that

$$\mathbf{U} = \int \mathbf{\Phi}^{-1}\mathbf{F}\,dt = \int \begin{pmatrix} 2 \\ -3e^{-t} \end{pmatrix} dt = \begin{pmatrix} 2t \\ 3e^{-t} \end{pmatrix}$$

and

$$\mathbf{X}_p = \mathbf{\Phi}\mathbf{U} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} te^t + \begin{pmatrix} 3 \\ 3 \end{pmatrix} e^t.$$

The solution is

$$\mathbf{X} = \mathbf{X}_c + \mathbf{X}_p = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + \begin{pmatrix} 4 \\ 2 \end{pmatrix} te^t + \begin{pmatrix} 3 \\ 3 \end{pmatrix} e^t$$

**18.** From

$$\mathbf{X}' = \begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix} \mathbf{X} + \begin{pmatrix} 2 \\ e^{-3t} \end{pmatrix}$$

we obtain

$$\mathbf{X}_c = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}.$$

Then

$$\mathbf{\Phi} = \begin{pmatrix} 2e^t & e^{2t} \\ e^t & e^{2t} \end{pmatrix} \quad \text{and} \quad \mathbf{\Phi}^{-1} = \begin{pmatrix} e^{-t} & -e^{-t} \\ -e^{-2t} & 2e^{-2t} \end{pmatrix}$$

so that

$$\mathbf{U} = \int \mathbf{\Phi}^{-1}\mathbf{F}\,dt = \int \begin{pmatrix} 2e^{-t} - e^{-4t} \\ -2e^{-2t} + 2e^{-5t} \end{pmatrix} dt = \begin{pmatrix} -2e^{-t} + \frac{1}{4}e^{-4t} \\ e^{-2t} - \frac{2}{5}e^{-5t} \end{pmatrix}$$

and

$$\mathbf{X}_p = \mathbf{\Phi}\mathbf{U} = \begin{pmatrix} \frac{1}{10}e^{-3t} - 3 \\ -\frac{3}{20}e^{-3t} - 1 \end{pmatrix}.$$

The solution is

$$\mathbf{X} = \mathbf{X}_c + \mathbf{X}_p = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + \begin{pmatrix} \frac{1}{10}e^{-3t} - 3 \\ -\frac{3}{20}e^{-3t} - 1 \end{pmatrix}$$

**19.** From

$$\mathbf{X}' = \begin{pmatrix} 1 & 8 \\ 1 & -1 \end{pmatrix} \mathbf{X} + \begin{pmatrix} 12 \\ 12 \end{pmatrix} t$$

we obtain

$$\mathbf{X}_c = c_1 \begin{pmatrix} 4 \\ 1 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{-3t}.$$

Then

$$\mathbf{\Phi} = \begin{pmatrix} 4e^{3t} & -2e^{-3t} \\ e^{3t} & e^{-3t} \end{pmatrix} \quad \text{and} \quad \mathbf{\Phi}^{-1} = \begin{pmatrix} \frac{1}{6}e^{-3t} & \frac{1}{3}e^{-3t} \\ -\frac{1}{6}e^{3t} & \frac{2}{3}e^{3t} \end{pmatrix}$$

so that

$$\mathbf{U} = \int \mathbf{\Phi}^{-1}\mathbf{F}\,dt = \int \begin{pmatrix} 6te^{-3t} \\ 6te^{3t} \end{pmatrix} dt = \begin{pmatrix} -2te^{-3t} - \frac{2}{3}e^{-3t} \\ 2te^{3t} - \frac{2}{3}e^{3t} \end{pmatrix}$$

and

$$\mathbf{X}_p = \mathbf{\Phi}\mathbf{U} = \begin{pmatrix} -12 \\ 0 \end{pmatrix} t + \begin{pmatrix} -\frac{4}{3} \\ -\frac{4}{3} \end{pmatrix}.$$

The solution is

$$\mathbf{X} = \mathbf{X}_c + \mathbf{X}_p = c_1 \begin{pmatrix} 4 \\ 1 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{-3t} + \begin{pmatrix} -12 \\ 0 \end{pmatrix} t - \begin{pmatrix} \frac{4}{3} \\ \frac{4}{3} \end{pmatrix}$$

**20.** From

$$\mathbf{X}' = \begin{pmatrix} 1 & 8 \\ 1 & -1 \end{pmatrix} \mathbf{X} + \begin{pmatrix} e^{-t} \\ te^{t} \end{pmatrix}$$

we obtain

$$\mathbf{X}_c = c_1 \begin{pmatrix} 4 \\ 1 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{-3t}.$$

Then

$$\mathbf{\Phi} = \begin{pmatrix} 4e^{3t} & -2e^{-3t} \\ e^{3t} & e^{-3t} \end{pmatrix} \quad \text{and} \quad \mathbf{\Phi}^{-1} = \begin{pmatrix} \frac{1}{6}e^{-3t} & \frac{1}{3}e^{-3t} \\ -\frac{1}{6}e^{3t} & \frac{2}{3}e^{3t} \end{pmatrix}$$

so that

$$\mathbf{U} = \int \mathbf{\Phi}^{-1}\mathbf{F}\,dt = \int \begin{pmatrix} \frac{1}{6}e^{-4t} + \frac{1}{3}te^{-2t} \\ -\frac{1}{6}e^{2t} + \frac{2}{3}te^{4t} \end{pmatrix} dt = \begin{pmatrix} -\frac{1}{24}e^{-4t} - \frac{1}{6}te^{-2t} - \frac{1}{12}e^{-2t} \\ -\frac{1}{12}e^{2t} + \frac{1}{6}te^{4t} - \frac{1}{24}e^{4t} \end{pmatrix}$$

and

$$\mathbf{X}_p = \mathbf{\Phi}\mathbf{U} = \begin{pmatrix} -te^{t} - \frac{1}{4}e^{t} \\ -\frac{1}{8}e^{-t} - \frac{1}{8}e^{t} \end{pmatrix}.$$

The solution is

$$\mathbf{X} = \mathbf{X}_c + \mathbf{X}_p = c_1 \begin{pmatrix} 4 \\ 1 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{-3t} - \begin{pmatrix} te^{t} + \frac{1}{4}e^{t} \\ \frac{1}{8}e^{-t} + \frac{1}{8}e^{t} \end{pmatrix}$$

**21.** From

$$\mathbf{X}' = \begin{pmatrix} 3 & 2 \\ -2 & -1 \end{pmatrix} \mathbf{X} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-t}$$

we obtain

$$\mathbf{X}_c = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{t} + c_2 \left[ \begin{pmatrix} 1 \\ -1 \end{pmatrix} te^{t} + \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix} e^{t} \right].$$

Then

$$\mathbf{\Phi} = \begin{pmatrix} e^{t} & te^{t} \\ -e^{t} & \frac{1}{2}e^{t} - te^{t} \end{pmatrix} \quad \text{and} \quad \mathbf{\Phi}^{-1} = \begin{pmatrix} e^{-t} - 2te^{-t} & -2te^{-t} \\ 2e^{-t} & 2e^{-t} \end{pmatrix}$$

so that

$$\mathbf{U} = \int \mathbf{\Phi}^{-1}\mathbf{F}\,dt = \int \begin{pmatrix} 2e^{-2t} - 6te^{-2t} \\ 6e^{-2t} \end{pmatrix} dt = \begin{pmatrix} \frac{1}{2}e^{-2t} + 3te^{-2t} \\ -3e^{-2t} \end{pmatrix}$$

and

$$\mathbf{X}_p = \mathbf{\Phi}\mathbf{U} = \begin{pmatrix} \frac{1}{2} \\ -2 \end{pmatrix} e^{-t}.$$

The solution is

$$\mathbf{X} = \mathbf{X}_c + \mathbf{X}_p = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t + c_2 \begin{pmatrix} t \\ \frac{1}{2} - t \end{pmatrix} e^t + \begin{pmatrix} \frac{1}{2} \\ -2 \end{pmatrix} e^{-t}$$

**22.** From

$$\mathbf{X}' = \begin{pmatrix} 3 & 2 \\ -2 & -1 \end{pmatrix} \mathbf{X} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

we obtain

$$\mathbf{X}_c = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t + c_2 \left[ \begin{pmatrix} 1 \\ -1 \end{pmatrix} te^t + \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix} e^t \right].$$

Then

$$\mathbf{\Phi} = \begin{pmatrix} e^t & te^t \\ -e^t & \frac{1}{2}e^t - te^t \end{pmatrix} \quad \text{and} \quad \mathbf{\Phi}^{-1} = \begin{pmatrix} e^{-t} - 2te^{-t} & -2te^{-t} \\ 2e^{-t} & 2e^{-t} \end{pmatrix}$$

so that

$$\mathbf{U} = \int \mathbf{\Phi}^{-1}\mathbf{F}\,dt = \int \begin{pmatrix} e^{-t} - 4te^{-t} \\ 2e^{-t} \end{pmatrix} dt = \begin{pmatrix} 3e^{-t} + 4te^{-t} \\ -2e^{-t} \end{pmatrix}$$

and

$$\mathbf{X}_p = \mathbf{\Phi}\mathbf{U} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}.$$

The solution is

$$\mathbf{X} = \mathbf{X}_c + \mathbf{X}_p = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t + c_2 \begin{pmatrix} t \\ \frac{1}{2} - t \end{pmatrix} e^t + \begin{pmatrix} 3 \\ -5 \end{pmatrix}$$

**23.** From

$$\mathbf{X}' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{X} + \begin{pmatrix} \sec t \\ 0 \end{pmatrix}$$

we obtain

$$\mathbf{X}_c = c_1 \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} + c_2 \begin{pmatrix} \sin t \\ -\cos t \end{pmatrix}.$$

Then

$$\mathbf{\Phi} = \begin{pmatrix} \cos t & \sin t \\ \sin t & -\cos t \end{pmatrix} \quad \text{and} \quad \mathbf{\Phi}^{-1} = \begin{pmatrix} \cos t & \sin t \\ \sin t & -\cos t \end{pmatrix}$$

so that

$$\mathbf{U} = \int \mathbf{\Phi}^{-1}\mathbf{F}\,dt = \int \begin{pmatrix} 1 \\ \tan t \end{pmatrix} dt = \begin{pmatrix} t \\ -\ln|\cos t| \end{pmatrix}$$

and

$$\mathbf{X}_p = \mathbf{\Phi}\mathbf{U} = \begin{pmatrix} t\cos t - \sin t\ln|\cos t| \\ t\sin t + \cos t\ln|\cos t| \end{pmatrix}.$$

The solution is

$$\mathbf{X} = \mathbf{X}_c + \mathbf{X}_p = c_1\begin{pmatrix} \cos t \\ \sin t \end{pmatrix} + c_2\begin{pmatrix} \sin t \\ -\cos t \end{pmatrix} + \begin{pmatrix} t\cos t - \sin t\ln|\cos t| \\ t\sin t + \cos t\ln|\cos t| \end{pmatrix}$$

**24.** From

$$\mathbf{X}' = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\mathbf{X} + \begin{pmatrix} 3 \\ 3 \end{pmatrix} e^t$$

we obtain

$$\mathbf{X}_c = c_1\begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} e^t + c_2\begin{pmatrix} \cos t \\ \sin t \end{pmatrix} e^t.$$

Then

$$\mathbf{\Phi} = \begin{pmatrix} -\sin t & \cos t \\ \cos t & \sin t \end{pmatrix} e^t \quad \text{and} \quad \mathbf{\Phi}^{-1} = \begin{pmatrix} -\sin t & \cos t \\ \cos t & \sin t \end{pmatrix} e^{-t}$$

so that

$$\mathbf{U} = \int \mathbf{\Phi}^{-1}\mathbf{F}\,dt = \int \begin{pmatrix} -3\sin t + 3\cos t \\ 3\cos t + 3\sin t \end{pmatrix} dt = \begin{pmatrix} 3\cos t + 3\sin t \\ 3\sin t - 3\cos t \end{pmatrix}$$

and

$$\mathbf{X}_p = \mathbf{\Phi}\mathbf{U} = \begin{pmatrix} -3 \\ 3 \end{pmatrix} e^t.$$

The solution is

$$\mathbf{X} = \mathbf{X}_c + \mathbf{X}_p = c_1\begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} e^t + c_2\begin{pmatrix} \cos t \\ \sin t \end{pmatrix} e^t + \begin{pmatrix} -3 \\ 3 \end{pmatrix} e^t$$

**25.** From

$$\mathbf{X}' = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\mathbf{X} + \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} e^t$$

we obtain

$$\mathbf{X}_c = c_1\begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} e^t + c_2\begin{pmatrix} \cos t \\ \sin t \end{pmatrix} e^t.$$

Then

$$\mathbf{\Phi} = \begin{pmatrix} -\sin t & \cos t \\ \cos t & \sin t \end{pmatrix} e^t \quad \text{and} \quad \mathbf{\Phi}^{-1} = \begin{pmatrix} -\sin t & \cos t \\ \cos t & \sin t \end{pmatrix} e^{-t}$$

so that

$$\mathbf{U} = \int \boldsymbol{\Phi}^{-1}\mathbf{F}\,dt = \int \begin{pmatrix} 0 \\ 1 \end{pmatrix} dt = \begin{pmatrix} 0 \\ t \end{pmatrix}$$

and

$$\mathbf{X}_p = \boldsymbol{\Phi}\mathbf{U} = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} te^t.$$

The solution is

$$\mathbf{X} = \mathbf{X}_c + \mathbf{X}_p = c_1 \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} e^t + c_2 \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} e^t + \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} te^t$$

**26.** From

$$\mathbf{X}' = \begin{pmatrix} 2 & -2 \\ 8 & -6 \end{pmatrix} \mathbf{X} + \begin{pmatrix} 1 \\ 3 \end{pmatrix} \frac{1}{t} e^{-2t}$$

we obtain

$$\mathbf{X}_c = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-2t} + c_2 \left[ \begin{pmatrix} 1 \\ 2 \end{pmatrix} te^{-2t} + \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} e^{-2t} \right].$$

Then

$$\boldsymbol{\Phi} = \begin{pmatrix} 1 & t + \frac{1}{2} \\ 2 & 2t + \frac{1}{2} \end{pmatrix} e^{-2t} \quad \text{and} \quad \boldsymbol{\Phi}^{-1} = \begin{pmatrix} -4t - 1 & 2t + 1 \\ 4 & -2 \end{pmatrix} e^{2t}$$

so that

$$\mathbf{U} = \int \boldsymbol{\Phi}^{-1}\mathbf{F}\,dt = \int \begin{pmatrix} 2 + 2/t \\ -2/t \end{pmatrix} dt = \begin{pmatrix} 2t + 2\ln t \\ -2\ln t \end{pmatrix}$$

and

$$\mathbf{X}_p = \boldsymbol{\Phi}\mathbf{U} = \begin{pmatrix} 2t + \ln t - 2t\ln t \\ 4t + 3\ln t - 4t\ln t \end{pmatrix} e^{-2t}.$$

The solution is

$$\mathbf{X} = \mathbf{X}_c + \mathbf{X}_p = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} \frac{1}{2} + t \\ \frac{1}{2} + 2t \end{pmatrix} e^{-2t} + \begin{pmatrix} 2t + \ln t - 2t\ln t \\ 4t + 3\ln t - 4t\ln t \end{pmatrix} e^{-2t}$$

**27.** From

$$\mathbf{X}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{X} + \begin{pmatrix} 0 \\ \sec t \tan t \end{pmatrix}$$

we obtain

$$\mathbf{X}_c = c_1 \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + c_2 \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}.$$

Then

$$\boldsymbol{\Phi} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} t \quad \text{and} \quad \boldsymbol{\Phi}^{-1} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

so that

$$\mathbf{U} = \int \mathbf{\Phi}^{-1}\mathbf{F}\,dt = \int \begin{pmatrix} -\tan^2 t \\ \tan t \end{pmatrix} dt = \begin{pmatrix} t - \tan t \\ -\ln|\cos t| \end{pmatrix}$$

and

$$\mathbf{X}_p = \mathbf{\Phi}\mathbf{U} = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} t + \begin{pmatrix} -\sin t \\ \sin t \tan t \end{pmatrix} - \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} \ln|\cos t|.$$

The solution is

$$\mathbf{X} = \mathbf{X}_c + \mathbf{X}_p = c_1 \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + c_2 \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} + \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} t + \begin{pmatrix} -\sin t \\ \sin t \tan t \end{pmatrix} - \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} \ln|\cos t|$$

**28.** From

$$\mathbf{X}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{X} + \begin{pmatrix} 1 \\ \cot t \end{pmatrix}$$

we obtain

$$\mathbf{X}_c = c_1 \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + c_2 \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}.$$

Then

$$\mathbf{\Phi} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \quad \text{and} \quad \mathbf{\Phi}^{-1} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

so that

$$\mathbf{U} = \int \mathbf{\Phi}^{-1}\mathbf{F}\,dt = \int \begin{pmatrix} 0 \\ \csc t \end{pmatrix} dt = \begin{pmatrix} 0 \\ \ln|\csc t - \cot t| \end{pmatrix}$$

and

$$\mathbf{X}_p = \mathbf{\Phi}\mathbf{U} = \begin{pmatrix} \sin t \ln|\csc t - \cot t| \\ \cos t \ln|\csc t - \cot t| \end{pmatrix}.$$

The solution is

$$\mathbf{X} = \mathbf{X}_c + \mathbf{X}_p = c_1 \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + c_2 \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} + \begin{pmatrix} \sin t \ln|\csc t - \cot t| \\ \cos t \ln|\csc t - \cot t| \end{pmatrix}$$

**29.** From

$$\mathbf{X}' = \begin{pmatrix} 1 & 2 \\ -\frac{1}{2} & 1 \end{pmatrix} \mathbf{X} + \begin{pmatrix} \csc t \\ \sec t \end{pmatrix} e^t$$

we obtain

$$\mathbf{X}_c = c_1 \begin{pmatrix} 2\sin t \\ \cos t \end{pmatrix} e^t + c_2 \begin{pmatrix} 2\cos t \\ -\sin t \end{pmatrix} e^t.$$

Then

$$\mathbf{\Phi} = \begin{pmatrix} 2\sin t & 2\cos t \\ \cos t & -\sin t \end{pmatrix} e^t \quad \text{and} \quad \mathbf{\Phi}^{-1} = \begin{pmatrix} \frac{1}{2}\sin t & \cos t \\ \frac{1}{2}\cos t & -\sin t \end{pmatrix} e^{-t}$$

so that

$$\mathbf{U} = \int \mathbf{\Phi}^{-1}\mathbf{F}\,dt = \int \begin{pmatrix} \frac{3}{2} \\ \frac{1}{2}\cot t - \tan t \end{pmatrix} dt = \begin{pmatrix} \frac{3}{2}t \\ \frac{1}{2}\ln|\sin t| + \ln|\cos t| \end{pmatrix}$$

and

$$\mathbf{X}_p = \mathbf{\Phi}\mathbf{U} = \begin{pmatrix} 3\sin t \\ \frac{3}{2}\cos t \end{pmatrix} te^t + \begin{pmatrix} \cos t \\ -\frac{1}{2}\sin t \end{pmatrix} e^t \ln|\sin t| + \begin{pmatrix} 2\cos t \\ -\sin t \end{pmatrix} e^t \ln|\cos t|.$$

The solution is

$$\mathbf{X} = \mathbf{X}_c + \mathbf{X}_p = c_1 \begin{pmatrix} 2\sin t \\ \cos t \end{pmatrix} e^t + c_2 \begin{pmatrix} 2\cos t \\ -\sin t \end{pmatrix} e^t + \begin{pmatrix} 3\sin t \\ \frac{3}{2}\cos t \end{pmatrix} te^t$$

$$+ \begin{pmatrix} \cos t \\ -\frac{1}{2}\sin t \end{pmatrix} e^t \ln|\sin t| + \begin{pmatrix} 2\cos t \\ -\sin t \end{pmatrix} e^t \ln|\cos t|$$

**30.** From

$$\mathbf{X}' = \begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix} \mathbf{X} + \begin{pmatrix} \tan t \\ 1 \end{pmatrix}$$

we obtain

$$\mathbf{X}_c = c_1 \begin{pmatrix} \cos t - \sin t \\ \cos t \end{pmatrix} + c_2 \begin{pmatrix} \cos t + \sin t \\ \sin t \end{pmatrix}.$$

Then

$$\mathbf{\Phi} = \begin{pmatrix} \cos t - \sin t & \cos t + \sin t \\ \cos t & \sin t \end{pmatrix} \quad \text{and} \quad \mathbf{\Phi}^{-1} = \begin{pmatrix} -\sin t & \cos t + \sin t \\ \cos t & \sin t - \cos t \end{pmatrix}$$

so that

$$\mathbf{U} = \int \mathbf{\Phi}^{-1}\mathbf{F}\,dt = \int \begin{pmatrix} 2\cos t + \sin t - \sec t \\ 2\sin t - \cos t \end{pmatrix} dt = \begin{pmatrix} 2\sin t - \cos t - \ln|\sec t + \tan t| \\ -2\cos t - \sin t \end{pmatrix}$$

and

$$\mathbf{X}_p = \mathbf{\Phi}\mathbf{U} = \begin{pmatrix} -3 - \cos t \ln|\sec t + \tan t| + \sin t \ln|\sec t + \tan t| \\ -1 - \cos t \ln|\sec t + \tan t| \end{pmatrix}.$$

The solution is

$$\mathbf{X} = \mathbf{X}_c + \mathbf{X}_p = c_1 \begin{pmatrix} \cos t - \sin t \\ \cos t \end{pmatrix} + c_2 \begin{pmatrix} \cos t + \sin t \\ \sin t \end{pmatrix}$$

$$+ \begin{pmatrix} -3 - \cos t \ln|\sec t + \tan t| + \sin t \ln|\sec t + \tan t| \\ -1 - \cos t \ln|\sec t + \tan t| \end{pmatrix}$$

**31.** From

$$\mathbf{X}' = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \mathbf{X} + \begin{pmatrix} e^t \\ e^{2t} \\ te^{3t} \end{pmatrix}$$

we obtain

$$\mathbf{X}_c = c_1\begin{pmatrix}1\\-1\\0\end{pmatrix} + c_2\begin{pmatrix}1\\1\\0\end{pmatrix}e^{2t} + c_3\begin{pmatrix}0\\0\\1\end{pmatrix}e^{3t}.$$

Then

$$\mathbf{\Phi} = \begin{pmatrix}1 & e^{2t} & 0\\ -1 & e^{2t} & 0\\ 0 & 0 & e^{3t}\end{pmatrix} \quad\text{and}\quad \mathbf{\Phi}^{-1} = \begin{pmatrix}\frac{1}{2} & -\frac{1}{2} & 0\\ \frac{1}{2}e^{-2t} & \frac{1}{2}e^{-2t} & 0\\ 0 & 0 & e^{-3t}\end{pmatrix}$$

so that

$$\mathbf{U} = \int \mathbf{\Phi}^{-1}\mathbf{F}\,dt = \int \begin{pmatrix}\frac{1}{2}e^{t} - \frac{1}{2}e^{2t}\\ \frac{1}{2}e^{-t} + \frac{1}{2}\\ t\end{pmatrix} dt = \begin{pmatrix}\frac{1}{2}e^{t} - \frac{1}{4}e^{2t}\\ -\frac{1}{2}e^{-t} + \frac{1}{2}t\\ \frac{1}{2}t^2\end{pmatrix}$$

and

$$\mathbf{X}_p = \mathbf{\Phi}\mathbf{U} = \begin{pmatrix}-\frac{1}{4}e^{2t} + \frac{1}{2}te^{2t}\\ -e^{t} + \frac{1}{4}e^{2t} + \frac{1}{2}te^{2t}\\ \frac{1}{2}t^2e^{3t}\end{pmatrix}.$$

The solution is

$$\mathbf{X} = \mathbf{X}_c + \mathbf{X}_p = c_1\begin{pmatrix}1\\-1\\0\end{pmatrix} + c_2\begin{pmatrix}1\\1\\0\end{pmatrix}e^{2t} + c_3\begin{pmatrix}0\\0\\1\end{pmatrix}e^{3t} + \begin{pmatrix}-\frac{1}{4}e^{2t} + \frac{1}{2}te^{2t}\\ -e^{t} + \frac{1}{4}e^{2t} + \frac{1}{2}te^{2t}\\ \frac{1}{2}t^2e^{3t}\end{pmatrix}$$

**32.** From

$$\mathbf{X}' = \begin{pmatrix}3 & -1 & -1\\ 1 & 1 & -1\\ 1 & -1 & 1\end{pmatrix}\mathbf{X} + \begin{pmatrix}0\\ t\\ 2e^{t}\end{pmatrix}$$

we obtain

$$\mathbf{X}_c = c_1\begin{pmatrix}1\\1\\1\end{pmatrix}e^{t} + c_2\begin{pmatrix}1\\1\\0\end{pmatrix}e^{2t} + c_3\begin{pmatrix}1\\0\\1\end{pmatrix}e^{2t}.$$

Then

$$\mathbf{\Phi} = \begin{pmatrix}e^{t} & e^{2t} & e^{2t}\\ e^{t} & e^{2t} & 0\\ e^{t} & 0 & e^{2t}\end{pmatrix} \quad\text{and}\quad \mathbf{\Phi}^{-1} = \begin{pmatrix}-e^{-t} & e^{-t} & e^{-t}\\ e^{-2t} & 0 & -e^{-2t}\\ e^{-2t} & -e^{-2t} & 0\end{pmatrix}$$

so that

$$\mathbf{U} = \int \mathbf{\Phi}^{-1}\mathbf{F}\,dt = \int \begin{pmatrix}te^{-t} + 2\\ -2e^{-t}\\ -te^{-2t}\end{pmatrix} dt = \begin{pmatrix}-te^{-t} - e^{-t} + 2t\\ 2e^{-t}\\ \frac{1}{2}te^{-2t} + \frac{1}{4}e^{-2t}\end{pmatrix}$$

and

$$\mathbf{X}_p = \mathbf{\Phi}\mathbf{U} = \begin{pmatrix} -\frac{1}{2} \\ -1 \\ -\frac{1}{2} \end{pmatrix} t + \begin{pmatrix} -\frac{3}{4} \\ -1 \\ -\frac{3}{4} \end{pmatrix} + \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} e^t + \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} te^t.$$

The solution is

$$\mathbf{X} = \mathbf{X}_c + \mathbf{X}_p = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{2t} + c_3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{2t} + \begin{pmatrix} -\frac{1}{2} \\ -1 \\ -\frac{1}{2} \end{pmatrix} t + \begin{pmatrix} -\frac{3}{4} \\ -1 \\ -\frac{3}{4} \end{pmatrix} + \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} e^t + \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} te^t$$

**33.** From

$$\mathbf{X}' = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix} \mathbf{X} + \begin{pmatrix} 4e^{2t} \\ 4e^{4t} \end{pmatrix}$$

we obtain

$$\mathbf{\Phi} = \begin{pmatrix} -e^{4t} & e^{2t} \\ e^{4t} & e^{2t} \end{pmatrix}, \quad \mathbf{\Phi}^{-1} = \begin{pmatrix} -\frac{1}{2}e^{-4t} & \frac{1}{2}e^{-4t} \\ \frac{1}{2}e^{-2t} & \frac{1}{2}e^{-2t} \end{pmatrix},$$

and

$$\mathbf{X} = \mathbf{\Phi}\mathbf{\Phi}^{-1}(0)\mathbf{X}(0) + \mathbf{\Phi} \int_0^t \mathbf{\Phi}^{-1}\mathbf{F}\, ds = \mathbf{\Phi} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \mathbf{\Phi} \cdot \begin{pmatrix} e^{-2t} + 2t - 1 \\ e^{2t} + 2t - 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 2 \end{pmatrix} te^{2t} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{2t} + \begin{pmatrix} -2 \\ 2 \end{pmatrix} te^{4t} + \begin{pmatrix} 2 \\ 0 \end{pmatrix} e^{4t}.$$

**34.** From

$$\mathbf{X}' = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \mathbf{X} + \begin{pmatrix} 1/t \\ 1/t \end{pmatrix}$$

we obtain

$$\mathbf{\Phi} = \begin{pmatrix} 1 & 1+t \\ 1 & t \end{pmatrix}, \quad \mathbf{\Phi}^{-1} = \begin{pmatrix} -t & 1+t \\ 1 & -1 \end{pmatrix},$$

and

$$\mathbf{X} = \mathbf{\Phi}\mathbf{\Phi}^{-1}(1)\mathbf{X}(1) + \mathbf{\Phi} \int_1^t \mathbf{\Phi}^{-1}\mathbf{F}\, ds = \mathbf{\Phi} \cdot \begin{pmatrix} -4 \\ 3 \end{pmatrix} + \mathbf{\Phi} \cdot \begin{pmatrix} \ln t \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} t - \begin{pmatrix} 1 \\ 4 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \ln t.$$

**35.** Let $\mathbf{X} = \begin{pmatrix} i_1 \\ i_2 \end{pmatrix}$ so that

$$\mathbf{X}' = \begin{pmatrix} -11 & 3 \\ 3 & -3 \end{pmatrix} \mathbf{X} + \begin{pmatrix} 100 \sin t \\ 0 \end{pmatrix}$$

and

$$\mathbf{X}_C = c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 3 \\ -1 \end{pmatrix} e^{-12t}.$$

Then

$$\mathbf{\Phi} = \begin{pmatrix} e^{-2t} & 3e^{-12t} \\ 3e^{-2t} & -e^{-12t} \end{pmatrix}, \qquad \mathbf{\Phi}^{-1} = \begin{pmatrix} \frac{1}{10}e^{2t} & \frac{3}{0}e^{2t} \\ \frac{3}{10}e^{12t} & -\frac{1}{10}e^{12t} \end{pmatrix},$$

$$\mathbf{U} = \int \mathbf{\Phi}^{-1}\mathbf{F}\,dt = \int \begin{pmatrix} 10e^{2t}\sin t \\ 30e^{12t}\sin t \end{pmatrix} dt = \begin{pmatrix} 2e^{2t}(2\sin t - \cos t) \\ \frac{6}{29}e^{12t}(12\sin t - \cos t) \end{pmatrix},$$

and

$$\mathbf{X}_p = \mathbf{\Phi}\mathbf{U} = \begin{pmatrix} \frac{332}{29}\sin t - \frac{76}{29}\cos t \\ \frac{276}{29}\sin t - \frac{168}{29}\cos t \end{pmatrix}$$

so that

$$\mathbf{X} = c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 3 \\ -1 \end{pmatrix} e^{-12t} + \mathbf{X}_p.$$

If $\mathbf{X}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ then $c_1 = 2$ and $c_2 = \frac{6}{29}$.

The solutions is

$$\mathbf{X} = 2\begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-2t} + \frac{6}{29}\begin{pmatrix} 3 \\ -1 \end{pmatrix} e^{-12t} - \frac{4}{29}\begin{pmatrix} 19 \\ 42 \end{pmatrix} \cos t + \frac{4}{29}\begin{pmatrix} 83 \\ 69 \end{pmatrix} \sin t.$$

**36.** Write the differential equation as a system

$$\begin{aligned} y' &= v \\ v' &= -Qy - Pv + f \end{aligned} \qquad \text{or} \qquad \begin{pmatrix} y \\ v \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -Q & -P \end{pmatrix} \begin{pmatrix} y \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ f \end{pmatrix}.$$

From (9) in the text of this section, a particular solution is then $\mathbf{X}_p = \mathbf{\Phi}(x)\int \mathbf{\Phi}^{-1}(x)\mathbf{F}(x)\,dx$ where

$$\mathbf{\Phi}(x) = \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \qquad \text{and} \qquad \mathbf{X}_p = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

Then

$$\mathbf{\Phi}^{-1}(x) = \frac{1}{y_1y_2' - y_2y_1'} \begin{pmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{pmatrix},$$

so

$$\mathbf{X}_p = \int \frac{1}{W} \begin{pmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{pmatrix} \begin{pmatrix} 0 \\ f \end{pmatrix} dx$$

and $W = y_1y_2' - y_2y_1'$. Thus

$$u_1 = \int \frac{-y_2 f(x)}{W}\,dx \qquad \text{and} \qquad u_2 = \int \frac{y_1 f(x)}{W}\,dx,$$

which are the antiderivative forms of the equations in (5) of Section 4.6 in the text.

**37. (a)** The eigenvalues are 0, 1, 3, and 4, with corresponding eigenvectors

$$\begin{pmatrix}-6\\-4\\1\\2\end{pmatrix}, \quad \begin{pmatrix}2\\1\\0\\0\end{pmatrix}, \quad \begin{pmatrix}3\\1\\2\\1\end{pmatrix}, \quad \text{and} \quad \begin{pmatrix}-1\\1\\0\\0\end{pmatrix}.$$

**(b)** $$\mathbf{\Phi} = \begin{pmatrix}-6 & 2e^{t} & 3e^{3t} & -e^{4t}\\ -4 & e^{t} & e^{3t} & e^{4t}\\ 1 & 0 & 2e^{3t} & 0\\ 2 & 0 & e^{3t} & 0\end{pmatrix}, \qquad \mathbf{\Phi}^{-1} = \begin{pmatrix}0 & 0 & -\frac{1}{3} & \frac{2}{3}\\ \frac{1}{3}\,e^{-t} & \frac{1}{3}\,e^{-t} & -2e^{-t} & \frac{8}{3}\,e^{-t}\\ 0 & 0 & \frac{2}{3}\,e^{-3t} & -\frac{1}{3}\,e^{-3t}\\ -\frac{1}{3}\,e^{-4t} & \frac{2}{3}\,e^{-4t} & 0 & \frac{1}{3}\,e^{-4t}\end{pmatrix}$$

**(c)** $$\mathbf{\Phi}^{-1}(t)\mathbf{F}(t) = \begin{pmatrix}\frac{2}{3} - \frac{1}{3}\,e^{2t}\\ \frac{1}{3}\,e^{-2t} + \frac{8}{3}\,e^{-t} - 2e^{t} + \frac{1}{3}\,t\\ -\frac{1}{3}\,e^{-3t} + \frac{2}{3}e^{-t}\\ \frac{2}{3}\,e^{-5t} + \frac{1}{3}\,e^{-4t} - \frac{1}{3}\,te^{-3t}\end{pmatrix},$$

$$\int \mathbf{\Phi}^{-1}(t)\mathbf{F}(t)\,dt = \begin{pmatrix}-\frac{1}{6}\,e^{2t} + \frac{2}{3}\,t\\ -\frac{1}{6}\,e^{-2t} - \frac{8}{3}\,e^{-t} - 2e^{t} + \frac{1}{6}\,t^2\\ \frac{1}{9}\,e^{-3t} - \frac{2}{3}\,e^{-t}\\ -\frac{2}{15}e^{-5t} - \frac{1}{12}\,e^{-4t} + \frac{1}{27}\,e^{-3t} + \frac{1}{9}\,te^{-3t}\end{pmatrix},$$

$$\begin{aligned}\mathbf{X}_p(t) &= \mathbf{\Phi}(t)\int \mathbf{\Phi}^{-1}(t)\mathbf{F}(t)\,dt\\ &= \begin{pmatrix}-5e^{2t} - \frac{1}{5}\,e^{-t} - \frac{1}{27}\,e^{t} - \frac{1}{9}\,te^{t} + \frac{1}{3}\,t^2e^{t} - 4t - \frac{59}{12}\\ -2e^{2t} - \frac{3}{10}\,e^{-t} + \frac{1}{27}e^{t} + \frac{1}{9}\,te^{t} + \frac{1}{6}\,t^2e^{t} - \frac{8}{3}t - \frac{95}{36}\\ -\frac{3}{2}\,e^{2t} + \frac{2}{3}\,t + \frac{2}{9}\\ -e^{2t} + \frac{4}{3}\,t - \frac{1}{9}\end{pmatrix},\end{aligned}$$

$$\mathbf{X}_c(t) = \mathbf{\Phi}(t)\mathbf{C} = \begin{pmatrix}-6c_1 + 2c_2e^{t} + 3c_3e^{3t} - c_4e^{4t}\\ -4c_1 + c_2e^{t} + c_3e^{3t} + c_4e^{4t}\\ c_1 + 2c_3e^{3t}\\ 2c_1 + c_3e^{3t}\end{pmatrix},$$

$$\mathbf{X}(t) = \mathbf{\Phi}(t)\mathbf{C} + \mathbf{\Phi}(t)\int \mathbf{\Phi}^{-1}(t)\mathbf{F}(t)\,dt = \begin{pmatrix} -6c_1 + 2c_2e^t + 3c_3e^{3t} - c_4e^{4t} \\ -4c_1 + c_2e^t + c_3e^{3t} + c_4e^{4t} \\ c_1 + 2c_3e^{3t} \\ 2c_1 + c_3e^{3t} \end{pmatrix}$$

$$+ \begin{pmatrix} -5e^{2t} - \frac{1}{5}\,e^{-t} - \frac{1}{27}\,e^t - \frac{1}{9}\,te^t + \frac{1}{3}\,t^2e^t - 4t - \frac{59}{12} \\ -2e^{2t} - \frac{3}{10}\,e^{-t} + \frac{1}{27}e^t + \frac{1}{9}\,te^t + \frac{1}{6}\,t^2e^t - \frac{8}{3}t - \frac{95}{36} \\ -\frac{3}{2}\,e^{2t} + \frac{2}{3}\,t + \frac{2}{9} \\ -e^{2t} + \frac{4}{3}\,t - \frac{1}{9} \end{pmatrix}$$

**(d)** $$\mathbf{X}(t) = c_1\begin{pmatrix} -6 \\ -4 \\ 1 \\ 2 \end{pmatrix} + c_2\begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}e^t + c_3\begin{pmatrix} 3 \\ 1 \\ 2 \\ 1 \end{pmatrix}e^{3t} + c_4\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}e^{4t}$$

$$+ \begin{pmatrix} -5e^{2t} - \frac{1}{5}\,e^{-t} - \frac{1}{27}\,e^t - \frac{1}{9}\,te^t + \frac{1}{3}\,t^2e^t - 4t - \frac{59}{12} \\ -2e^{2t} - \frac{3}{10}\,e^{-t} + \frac{1}{27}e^t + \frac{1}{9}\,te^t + \frac{1}{6}\,t^2e^t - \frac{8}{3}t - \frac{95}{36} \\ -\frac{3}{2}\,e^{2t} + \frac{2}{3}\,t + \frac{2}{9} \\ -e^{2t} + \frac{4}{3}\,t - \frac{1}{9} \end{pmatrix}$$

## 8.4 Matrix Exponential

**1.** For $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ we have

$$\mathbf{A}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix},$$

$$\mathbf{A}^3 = \mathbf{A}\mathbf{A}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 8 \end{pmatrix},$$

$$\mathbf{A}^4 = \mathbf{A}\mathbf{A}^3 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 8 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 16 \end{pmatrix},$$

and so on. In general

$$\mathbf{A}^k = \begin{pmatrix} 1 & 0 \\ 0 & 2^k \end{pmatrix} \quad \text{for} \quad k = 1,\, 2,\, 3,\, \ldots\,.$$

Thus

$$\begin{aligned} e^{\mathbf{A}t} &= \mathbf{I} + \frac{\mathbf{A}}{1!}t + \frac{\mathbf{A}^2}{2!}t^2 + \frac{\mathbf{A}^3}{3!}t^3 + \cdots \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{1!}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} t + \frac{1}{2!}\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} t^2 + \frac{1}{3!}\begin{pmatrix} 1 & 0 \\ 0 & 8 \end{pmatrix} t^3 + \cdots \\ &= \begin{pmatrix} 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots & 0 \\ 0 & 1 + t + \frac{(2t)^2}{2!} + \frac{(2t)^3}{3!} + \cdots \end{pmatrix} = \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix} \end{aligned}$$

and

$$e^{-\mathbf{A}t} = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{pmatrix}.$$

**2.** For $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ we have

$$\begin{aligned} \mathbf{A}^2 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I} \\ \mathbf{A}^3 &= \mathbf{A}\mathbf{A}^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\mathbf{I} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \mathbf{A} \\ \mathbf{A}^4 &= (\mathbf{A}^2)^2 = \mathbf{I} \\ \mathbf{A}^5 &= \mathbf{A}\mathbf{A}^4 = \mathbf{A}\mathbf{I} = \mathbf{A}, \end{aligned}$$

and so on. In general,

$$\mathbf{A}^k = \begin{cases} \mathbf{A}, & k = 1, 3, 5, \ldots \\ \mathbf{I}, & k = 2, 4, 6, \ldots. \end{cases}$$

Thus

$$\begin{aligned} e^{\mathbf{A}t} &= \mathbf{I} + \frac{\mathbf{A}}{1!}t + \frac{\mathbf{A}^2}{2!}t^2 + \frac{\mathbf{A}^3}{3!}t^3 + \cdots \\ &= \mathbf{I} + \mathbf{A}t + \frac{1}{2!}\mathbf{I}t^2 + \frac{1}{3!}\mathbf{A}t^3 + \cdots \\ &= \mathbf{I}\left(1 + \frac{1}{2!}t^2 + \frac{1}{4!}t^4 + \cdots\right) + \mathbf{A}\left(t + \frac{1}{3!}t^3 + \frac{1}{5!}t^5 + \cdots\right) \\ &= \mathbf{I}\cosh t + \mathbf{A}\sinh t = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \end{aligned}$$

and

$$e^{-\mathbf{A}t} = \begin{pmatrix} \cosh(-t) & \sinh(-t) \\ \sinh(-t) & \cosh(-t) \end{pmatrix} = \begin{pmatrix} \cosh t & -\sinh t \\ -\sinh t & \cosh t \end{pmatrix}.$$

**3.** For

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ -2 & -2 & -2 \end{pmatrix}$$

we have

$$\mathbf{A}^2 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ -2 & -2 & -2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ -2 & -2 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus, $\mathbf{A}^3 = \mathbf{A}^4 = \mathbf{A}^5 = \cdots = \mathbf{0}$ and

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} t & t & t \\ t & t & t \\ -2t & -2t & -2t \end{pmatrix} = \begin{pmatrix} t+1 & t & t \\ t & t+1 & t \\ -2t & -2t & -2t+1 \end{pmatrix}.$$

**4.** For

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 5 & 1 & 0 \end{pmatrix}$$

we have

$$\mathbf{A}^2 = \begin{pmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 5 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 5 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 3 & 0 & 0 \end{pmatrix}$$

$$\mathbf{A}^3 = \mathbf{A}\mathbf{A}^2 = \begin{pmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 5 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 3 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus, $\mathbf{A}^4 = \mathbf{A}^5 = \mathbf{A}^6 = \cdots = \mathbf{0}$ and

$$\begin{aligned} e^{\mathbf{A}t} &= \mathbf{I} + \mathbf{A}t + \frac{1}{2}\mathbf{A}^2 t^2 \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 3t & 0 & 0 \\ 5t & t & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{3}{2}t^2 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 3t & 1 & 0 \\ \frac{3}{2}t^2 + 5t & t & 1 \end{pmatrix}. \end{aligned}$$

**5.** Using the result of Problem 1,

$$\mathbf{X} = \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = c_1 \begin{pmatrix} e^t \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ e^{2t} \end{pmatrix}.$$

**6.** Using the result of Problem 2,

$$\mathbf{X} = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = c_1 \begin{pmatrix} \cosh t \\ \sinh t \end{pmatrix} + c_2 \begin{pmatrix} \sinh t \\ \cosh t \end{pmatrix}.$$

**7.** Using the result of Problem 3,

$$\mathbf{X}=\begin{pmatrix}t+1&t&t\\t&t+1&t\\-2t&-2t&-2t+1\end{pmatrix}\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}=c_1\begin{pmatrix}t+1\\t\\-2t\end{pmatrix}+c_2\begin{pmatrix}t\\t+1\\-2t\end{pmatrix}+c_3\begin{pmatrix}t\\t\\-2t+1\end{pmatrix}.$$

**8.** Using the result of Problem 4,

$$\mathbf{X}=\begin{pmatrix}1&0&0\\3t&1&0\\\frac{3}{2}t^2+5t&t&1\end{pmatrix}\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}=c_1\begin{pmatrix}1\\3t\\\frac{3}{2}t^2+5t\end{pmatrix}+c_2\begin{pmatrix}0\\1\\t\end{pmatrix}+c_3\begin{pmatrix}0\\0\\1\end{pmatrix}.$$

**9.** To solve

$$\mathbf{X}'=\begin{pmatrix}1&0\\0&2\end{pmatrix}\mathbf{X}+\begin{pmatrix}3\\-1\end{pmatrix}$$

we identify $t_0=0$, $\mathbf{F}(t)=\begin{pmatrix}3\\-1\end{pmatrix}$, and use the results of Problem 1 and Equation (6) in the text.

$$\begin{aligned}\mathbf{X}(t)&=e^{\mathbf{A}t}\mathbf{C}+e^{\mathbf{A}t}\int_{t_0}^{t}e^{-\mathbf{A}s}\mathbf{F}(s)\,ds\\&=\begin{pmatrix}e^t&0\\0&e^{2t}\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix}+\begin{pmatrix}e^t&0\\0&e^{2t}\end{pmatrix}\int_0^t\begin{pmatrix}e^{-s}&0\\0&e^{-2s}\end{pmatrix}\begin{pmatrix}3\\-1\end{pmatrix}ds\\&=\begin{pmatrix}c_1e^t\\c_2e^{2t}\end{pmatrix}+\begin{pmatrix}e^t&0\\0&e^{2t}\end{pmatrix}\int_0^t\begin{pmatrix}3e^{-s}\\-e^{-2s}\end{pmatrix}ds\\&=\begin{pmatrix}c_1e^t\\c_2e^{2t}\end{pmatrix}+\begin{pmatrix}e^t&0\\0&e^{2t}\end{pmatrix}\begin{pmatrix}-3e^{-s}\\\frac{1}{2}e^{-2s}\end{pmatrix}\Bigg|_0^t\\&=\begin{pmatrix}c_1e^t\\c_2e^{2t}\end{pmatrix}+\begin{pmatrix}e^t&0\\0&e^{2t}\end{pmatrix}\begin{pmatrix}-3e^{-t}+3\\\frac{1}{2}e^{-2t}-\frac{1}{2}\end{pmatrix}\\&=\begin{pmatrix}c_1e^t\\c_2e^{2t}\end{pmatrix}+\begin{pmatrix}-3+3e^t\\\frac{1}{2}-\frac{1}{2}e^{2t}\end{pmatrix}=c_3\begin{pmatrix}1\\0\end{pmatrix}e^t+c_4\begin{pmatrix}0\\1\end{pmatrix}e^{2t}+\begin{pmatrix}-3\\\frac{1}{2}\end{pmatrix}.\end{aligned}$$

**10.** To solve

$$\mathbf{X}'=\begin{pmatrix}1&0\\0&2\end{pmatrix}\mathbf{X}+\begin{pmatrix}t\\e^{4t}\end{pmatrix}$$

we identify $t_0=0$, $\mathbf{F}(t)=\begin{pmatrix}t\\e^{4t}\end{pmatrix}$, and use the results of Problem 1 and Equation (6) in the text.

$$\begin{aligned}\mathbf{X}(t) &= e^{\mathbf{A}t}\mathbf{C} + e^{\mathbf{A}t}\int_{t_0}^{t} e^{-\mathbf{A}s}\mathbf{F}(s)\,ds\\ &= \begin{pmatrix} e^t & 0\\ 0 & e^{2t}\end{pmatrix}\begin{pmatrix} c_1\\ c_2\end{pmatrix} + \begin{pmatrix} e^t & 0\\ 0 & e^{2t}\end{pmatrix}\int_0^t \begin{pmatrix} e^{-s} & 0\\ 0 & e^{-2s}\end{pmatrix}\begin{pmatrix} s\\ e^{4s}\end{pmatrix} ds\\ &= \begin{pmatrix} c_1e^t\\ c_2e^{2t}\end{pmatrix} + \begin{pmatrix} e^t & 0\\ 0 & e^{2t}\end{pmatrix}\int_0^t \begin{pmatrix} se^{-s}\\ e^{2s}\end{pmatrix} ds\\ &= \begin{pmatrix} c_1e^t\\ c_2e^{2t}\end{pmatrix} + \begin{pmatrix} e^t & 0\\ 0 & e^{2t}\end{pmatrix}\begin{pmatrix} -se^{-s} - e^{-s}\\ \frac{1}{2}e^{2s}\end{pmatrix}\Bigg|_0^t\end{aligned}$$

$$\begin{aligned}&= \begin{pmatrix} c_1e^t\\ c_2e^{2t}\end{pmatrix} + \begin{pmatrix} e^t & 0\\ 0 & e^{2t}\end{pmatrix}\begin{pmatrix} -te^{-t} - e^{-t} + 1\\ \frac{1}{2}e^{2t} - \frac{1}{2}\end{pmatrix}\\ &= \begin{pmatrix} c_1e^t\\ c_2e^{2t}\end{pmatrix} + \begin{pmatrix} -t - 1 + e^t\\ \frac{1}{2}e^{4t} - \frac{1}{2}e^{2t}\end{pmatrix} = c_3\begin{pmatrix} 1\\ 0\end{pmatrix}e^t + c_4\begin{pmatrix} 0\\ 1\end{pmatrix}e^{2t} + \begin{pmatrix} -t - 1\\ \frac{1}{2}e^{4t}\end{pmatrix}.\end{aligned}$$

**11.** To solve

$$\mathbf{X}' = \begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}\mathbf{X} + \begin{pmatrix} 1\\ 1\end{pmatrix}$$

we identify $t_0 = 0$, $\mathbf{F}(t) = \begin{pmatrix} 1\\ 1\end{pmatrix}$, and use the results of Problem 2 and Equation (6) in the text.

$$\begin{aligned}
\mathbf{X}(t) &= e^{\mathbf{A}t}\mathbf{C} + e^{\mathbf{A}t}\int_{t_0}^{t} e^{-\mathbf{A}s}\mathbf{F}(s)\,ds \\
&= \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}\int_0^t \begin{pmatrix} \cosh s & -\sinh s \\ -\sinh s & \cosh s \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} ds \\
&= \begin{pmatrix} c_1 \cosh t + c_2 \sinh t \\ c_1 \sinh t + c_2 \cosh t \end{pmatrix} + \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}\int_0^t \begin{pmatrix} \cosh s - \sinh s \\ -\sinh s + \cosh s \end{pmatrix} ds \\
&= \begin{pmatrix} c_1 \cosh t + c_2 \sinh t \\ c_1 \sinh t + c_2 \cosh t \end{pmatrix} + \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}\begin{pmatrix} \sinh s - \cosh s \\ -\cosh s + \sinh s \end{pmatrix}\Bigg|_0^t \\
&= \begin{pmatrix} c_1 \cosh t + c_2 \sinh t \\ c_1 \sinh t + c_2 \cosh t \end{pmatrix} + \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}\begin{pmatrix} \sinh t - \cosh t + 1 \\ -\cosh t + \sinh t + 1 \end{pmatrix} \\
&= \begin{pmatrix} c_1 \cosh t + c_2 \sinh t \\ c_1 \sinh t + c_2 \cosh t \end{pmatrix} + \begin{pmatrix} \sinh^2 t - \cosh^2 t + \cosh t + \sinh t \\ \sinh^2 t - \cosh^2 t + \sinh t + \cosh t \end{pmatrix} \\
&= c_1\begin{pmatrix} \cosh t \\ \sinh t \end{pmatrix} + c_2\begin{pmatrix} \sinh t \\ \cosh t \end{pmatrix} + \begin{pmatrix} \cosh t \\ \sinh t \end{pmatrix} + \begin{pmatrix} \sinh t \\ \cosh t \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
&= c_3\begin{pmatrix} \cosh t \\ \sinh t \end{pmatrix} + c_4\begin{pmatrix} \sinh t \\ \cosh t \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix}.
\end{aligned}$$

**12.** To solve

$$\mathbf{X}' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\mathbf{X} + \begin{pmatrix} \cosh t \\ \sinh t \end{pmatrix}$$

we identify $t_0 = 0$, $\mathbf{F}(t) = \begin{pmatrix} \cosh t \\ \sinh t \end{pmatrix}$, and use the results of Problem 2 and Equation (6) in the text.

$$\begin{aligned}
\mathbf{X}(t) &= e^{\mathbf{A}t}\mathbf{C} + e^{\mathbf{A}t}\int_{t_0}^{t} e^{-\mathbf{A}s}\mathbf{F}(s)\,ds \\
&= \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}\int_0^t \begin{pmatrix} \cosh s & -\sinh s \\ -\sinh s & \cosh s \end{pmatrix}\begin{pmatrix} \cosh s \\ \sinh s \end{pmatrix} ds \\
&= \begin{pmatrix} c_1\cosh t + c_2\sinh t \\ c_1\sinh t + c_2\cosh t \end{pmatrix} + \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}\int_0^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} ds \\
&= \begin{pmatrix} c_1\cosh t + c_2\sinh t \\ c_1\sinh t + c_2\cosh t \end{pmatrix} + \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}\begin{pmatrix} s \\ 0 \end{pmatrix}\Bigg|_0^t \\
&= \begin{pmatrix} c_1\cosh t + c_2\sinh t \\ c_1\sinh t + c_2\cosh t \end{pmatrix} + \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}\begin{pmatrix} t \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} c_1\cosh t + c_2\sinh t \\ c_1\sinh t + c_2\cosh t \end{pmatrix} + \begin{pmatrix} t\cosh t \\ t\sinh t \end{pmatrix} = c_1\begin{pmatrix} \cosh t \\ \sinh t \end{pmatrix} + c_2\begin{pmatrix} \sinh t \\ \cosh t \end{pmatrix} + t\begin{pmatrix} \cosh t \\ \sinh t \end{pmatrix}.
\end{aligned}$$

**13.** We have

$$\mathbf{X}(0) = c_1\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \\ 6 \end{pmatrix}.$$

Thus, the solution of the initial-value problem is

$$\mathbf{X} = \begin{pmatrix} t+1 \\ t \\ -2t \end{pmatrix} - 4\begin{pmatrix} t \\ t+1 \\ -2t \end{pmatrix} + 6\begin{pmatrix} t \\ t \\ -2t+1 \end{pmatrix}.$$

**14.** We have

$$\mathbf{X}(0) = c_3\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_4\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -3 \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} c_3 - 3 \\ c_4 + \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}.$$

Thus, $c_3 = 7$ and $c_4 = \frac{5}{2}$, so

$$\mathbf{X} = 7\begin{pmatrix} 1 \\ 0 \end{pmatrix}e^t + \frac{5}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix}e^{2t} + \begin{pmatrix} -3 \\ \frac{1}{2} \end{pmatrix}.$$

**15.** From $s\mathbf{I} - \mathbf{A} = \begin{pmatrix} s-4 & -3 \\ 4 & s+4 \end{pmatrix}$ we find

$$(s\mathbf{I} - \mathbf{A})^{-1} = \begin{pmatrix} \dfrac{3/2}{s-2} - \dfrac{1/2}{s+2} & \dfrac{3/4}{s-2} - \dfrac{3/4}{s+2} \\[2ex] \dfrac{-1}{s-2} + \dfrac{1}{s+2} & \dfrac{-1/2}{s-2} + \dfrac{3/2}{s+2} \end{pmatrix}$$

and

$$e^{\mathbf{A}t} = \begin{pmatrix} \frac{3}{2}e^{2t} - \frac{1}{2}e^{-2t} & \frac{3}{4}e^{2t} - \frac{3}{4}e^{-2t} \\ -e^{2t} + e^{-2t} & -\frac{1}{2}e^{2t} + \frac{3}{2}e^{-2t} \end{pmatrix}.$$

The general solution of the system is then

$$\begin{aligned} \mathbf{X} = e^{\mathbf{A}t}\mathbf{C} &= \begin{pmatrix} \frac{3}{2}e^{2t} - \frac{1}{2}e^{-2t} & \frac{3}{4}e^{2t} - \frac{3}{4}e^{-2t} \\ -e^{2t} + e^{-2t} & -\frac{1}{2}e^{2t} + \frac{3}{2}e^{-2t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\ &= c_1 \begin{pmatrix} \frac{3}{2} \\ -1 \end{pmatrix} e^{2t} + c_1 \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} \frac{3}{4} \\ -\frac{1}{2} \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} -\frac{3}{4} \\ \frac{3}{2} \end{pmatrix} e^{-2t} \\ &= \left(\frac{1}{2}c_1 + \frac{1}{4}c_2\right) \begin{pmatrix} 3 \\ -2 \end{pmatrix} e^{2t} + \left(-\frac{1}{2}c_1 - \frac{3}{4}c_2\right) \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-2t} \\ &= c_3 \begin{pmatrix} 3 \\ -2 \end{pmatrix} e^{2t} + c_4 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-2t}. \end{aligned}$$

**16.** From $s\mathbf{I} - \mathbf{A} = \begin{pmatrix} s-4 & 2 \\ -1 & s-1 \end{pmatrix}$ we find

$$(s\mathbf{I} - \mathbf{A})^{-1} = \begin{pmatrix} \dfrac{2}{s-3} - \dfrac{1}{s-2} & -\dfrac{2}{s-3} + \dfrac{2}{s-2} \\[2ex] \dfrac{1}{s-3} - \dfrac{1}{s-2} & \dfrac{-1}{s-3} + \dfrac{2}{s-2} \end{pmatrix}$$

and

$$e^{\mathbf{A}t} = \begin{pmatrix} 2e^{3t} - e^{2t} & -2e^{3t} + 2e^{2t} \\ e^{3t} - e^{2t} & -e^{3t} + 2e^{2t} \end{pmatrix}.$$

The general solution of the system is then

$$\begin{aligned} \mathbf{X} = e^{\mathbf{A}t}\mathbf{C} &= \begin{pmatrix} 2e^{3t} - e^{2t} & -2e^{3t} + 2e^{2t} \\ e^{3t} - e^{2t} & -e^{3t} + 2e^{2t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\ &= c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{3t} + c_1 \begin{pmatrix} -1 \\ -1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} -2 \\ -1 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 2 \\ 2 \end{pmatrix} e^{2t} \\ &= (c_1 - c_2) \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{3t} + (-c_1 + 2c_2) \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} \\ &= c_3 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{3t} + c_4 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}. \end{aligned}$$

**17.** From $s\mathbf{I}-\mathbf{A}=\begin{pmatrix} s-5 & 9 \\ -1 & s+1 \end{pmatrix}$ we find

$$(s\mathbf{I}-\mathbf{A})^{-1}=\begin{pmatrix} \dfrac{1}{s-2}+\dfrac{3}{(s-2)^2} & -\dfrac{9}{(s-2)^2} \\ \dfrac{1}{(s-2)^2} & \dfrac{1}{s-2}-\dfrac{3}{(s-2)^2} \end{pmatrix}$$

and

$$e^{\mathbf{A}t}=\begin{pmatrix} e^{2t}+3te^{2t} & -9te^{2t} \\ te^{2t} & e^{2t}-3te^{2t} \end{pmatrix}.$$

The general solution of the system is then

$$\begin{aligned}\mathbf{X}=e^{\mathbf{A}t}\mathbf{C}&=\begin{pmatrix} e^{2t}+3te^{2t} & -9te^{2t} \\ te^{2t} & e^{2t}-3te^{2t} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}\\ &=c_1\begin{pmatrix} 1 \\ 0 \end{pmatrix}e^{2t}+c_1\begin{pmatrix} 3 \\ 1 \end{pmatrix}te^{2t}+c_2\begin{pmatrix} 0 \\ 1 \end{pmatrix}e^{2t}+c_2\begin{pmatrix} -9 \\ -3 \end{pmatrix}te^{2t}\\ &=c_1\begin{pmatrix} 1+3t \\ t \end{pmatrix}e^{2t}+c_2\begin{pmatrix} -9t \\ 1-3t \end{pmatrix}e^{2t}.\end{aligned}$$

**18.** From $s\mathbf{I}-\mathbf{A}=\begin{pmatrix} s & -1 \\ 2 & s+2 \end{pmatrix}$ we find

$$(s\mathbf{I}-\mathbf{A})^{-1}=\begin{pmatrix} \dfrac{s+1+1}{(s+1)^2+1} & \dfrac{1}{(s+1)^2+1} \\ \dfrac{-2}{(s+1)^2+1} & \dfrac{s+1-1}{(s+1)^2+1} \end{pmatrix}$$

and

$$e^{\mathbf{A}t}=\begin{pmatrix} e^{-t}\cos t+e^{-t}\sin t & e^{-t}\sin t \\ -2e^{-t}\sin t & e^{-t}\cos t-e^{-t}\sin t \end{pmatrix}.$$

The general solution of the system is then

$$\begin{aligned}\mathbf{X}=e^{\mathbf{A}t}\mathbf{C}&=\begin{pmatrix} e^{-t}\cos t+e^{-t}\sin t & e^{-t}\sin t \\ -2e^{-t}\sin t & e^{-t}\cos t-e^{-t}\sin t \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}\\ &=c_1\begin{pmatrix} 1 \\ 0 \end{pmatrix}e^{-t}\cos t+c_1\begin{pmatrix} 1 \\ -2 \end{pmatrix}e^{-t}\sin t+c_2\begin{pmatrix} 0 \\ 1 \end{pmatrix}e^{-t}\cos t+c_2\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{-t}\sin t\\ &=c_1\begin{pmatrix} \cos t+\sin t \\ -2\sin t \end{pmatrix}e^{-t}+c_2\begin{pmatrix} \sin t \\ \cos t-\sin t \end{pmatrix}e^{-t}.\end{aligned}$$

**19.** Solving

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix}2-\lambda & 1\\ -3 & 6-\lambda\end{vmatrix}=\lambda^2-8\lambda+15=(\lambda-3)(\lambda-5)=0$$

we find eigenvalues $\lambda_1=3$ and $\lambda_2=5$. Corresponding eigenvectors are

$$\mathbf{K}_1=\begin{pmatrix}1\\1\end{pmatrix}\quad\text{and}\quad\mathbf{K}_2=\begin{pmatrix}1\\3\end{pmatrix}.$$

Then

$$\mathbf{P}=\begin{pmatrix}1&1\\1&3\end{pmatrix},\quad\mathbf{P}^{-1}=\begin{pmatrix}3/2&-1/2\\-1/2&1/2\end{pmatrix},\quad\text{and}\quad\mathbf{D}=\begin{pmatrix}3&0\\0&5\end{pmatrix},$$

so that

$$\mathbf{PDP}^{-1}=\begin{pmatrix}2&1\\-3&6\end{pmatrix}.$$

**20.** Solving

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix}2-\lambda & 1\\ 1 & 2-\lambda\end{vmatrix}=\lambda^2-4\lambda+3=(\lambda-1)(\lambda-3)=0$$

we find eigenvalues $\lambda_1=1$ and $\lambda_2=3$. Corresponding eigenvectors are

$$\mathbf{K}_1=\begin{pmatrix}-1\\1\end{pmatrix}\quad\text{and}\quad\mathbf{K}_2=\begin{pmatrix}1\\1\end{pmatrix}.$$

Then

$$\mathbf{P}=\begin{pmatrix}-1&1\\1&1\end{pmatrix},\quad\mathbf{P}^{-1}=\begin{pmatrix}-1/2&1/2\\1/2&1/2\end{pmatrix},\quad\text{and}\quad\mathbf{D}=\begin{pmatrix}1&0\\0&3\end{pmatrix},$$

so that

$$\mathbf{PDP}^{-1}=\begin{pmatrix}2&1\\1&2\end{pmatrix}.$$

**21.** From equation (3) in the text

$$\begin{aligned}e^{t\mathbf{A}}=e^{t\mathbf{PDP}^{-1}}&=\mathbf{I}+t(\mathbf{PDP}^{-1})+\frac{1}{2!}t^2(\mathbf{PDP}^{-1})^2+\frac{1}{3!}t^3(\mathbf{PDP}^{-1})^3+\cdots\\&=\mathbf{P}\left[\mathbf{I}+t\mathbf{D}+\frac{1}{2!}(t\mathbf{D})^2+\frac{1}{3!}(t\mathbf{D})^3+\cdots\right]\mathbf{P}^{-1}=\mathbf{P}e^{t\mathbf{D}}\mathbf{P}^{-1}.\end{aligned}$$

**22.** From Equation (3) in the text

$$e^{\mathbf{D}t} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} + \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} + \frac{1}{2!}t^2 \begin{pmatrix} \lambda_1^2 & 0 & \cdots & 0 \\ 0 & \lambda_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n^2 \end{pmatrix}$$

$$+ \frac{1}{3!}t^3 \begin{pmatrix} \lambda_1^3 & 0 & \cdots & 0 \\ 0 & \lambda_2^3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n^3 \end{pmatrix} + \cdots$$

$$= \begin{pmatrix} 1+\lambda_1 t+\frac{1}{2!}(\lambda_1 t)^2+\cdots & 0 & \cdots & 0 \\ 0 & 1+\lambda_2 t+\frac{1}{2!}(\lambda_2 t)^2+\cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1+\lambda_n t+\frac{1}{2!}(\lambda_n t)^2+\cdots \end{pmatrix}$$

$$= \begin{pmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n t} \end{pmatrix}$$

**23.** From Problems 19, 21, and 22 and Equation (1) in the text

$$\mathbf{X} = e^{\mathbf{A}t}\mathbf{C} = \mathbf{P}e^{\mathbf{D}t}\mathbf{P}^{-1}\mathbf{C} = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} e^{3t} & 0 \\ 0 & e^{5t} \end{pmatrix} \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{3}{2}e^{3t} - \frac{1}{2}e^{5t} & -\frac{1}{2}e^{3t} + \frac{1}{2}e^{5t} \\ \frac{3}{2}e^{3t} - \frac{3}{2}e^{5t} & -\frac{1}{2}e^{3t} + \frac{3}{2}e^{5t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

**24.** From Problems 20, 21, and 22 and Equation (1) in the text

$$\mathbf{X} = e^{\mathbf{A}t}\mathbf{C} = \mathbf{P}e^{\mathbf{D}t}\mathbf{P}^{-1}\mathbf{C} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{t} & 0 \\ 0 & e^{3t} \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2}e^{t} + \frac{1}{2}e^{3t} & -\frac{1}{2}e^{t} + \frac{1}{2}e^{3t} \\ -\frac{1}{2}e^{t} + \frac{1}{2}e^{3t} & \frac{1}{2}e^{t} + \frac{1}{2}e^{3t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

**25.** If $\det(s\mathbf{I} - \mathbf{A}) = 0$, then $s$ is an eigenvalue of $\mathbf{A}$. Thus $s\mathbf{I} - \mathbf{A}$ has an inverse if $s$ is not an eigenvalue of $\mathbf{A}$. For the purposes of the discussion in this section, we take $s$ to be larger than the largest eigenvalue of $\mathbf{A}$. Under this condition $s\mathbf{I} - \mathbf{A}$ has an inverse.

**26.** Since $\mathbf{A}^3 = \mathbf{0}$, $\mathbf{A}$ is nilpotent. Since

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \mathbf{A}^2 \frac{t^2}{2!} + \cdots + \mathbf{A}^k \frac{t^k}{k!} + \cdots ,$$

if $\mathbf{A}$ is nilpotent and $\mathbf{A}^m = \mathbf{0}$, then $\mathbf{A}^k = \mathbf{0}$ for $k \geq m$ and

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \mathbf{A}^2 \frac{t^2}{2!} + \cdots + \mathbf{A}^{m-1} \frac{t^{m-1}}{(m-1)!} .$$

In this problem $\mathbf{A}^3 = \mathbf{0}$, so

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \mathbf{A}^2 \frac{t^2}{2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & 1 & 1 \end{pmatrix} t + \begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix} \frac{t^2}{2}$$

$$= \begin{pmatrix} 1 - t - t^2/2 & t & t + t^2/2 \\ -t & 1 & t \\ -t - t^2/2 & t & 1 + t + t^2/2 \end{pmatrix}$$

and the solution of $\mathbf{X}' = \mathbf{A}\mathbf{X}$ is

$$\mathbf{X}(t) = e^{\mathbf{A}t}\mathbf{C} = e^{\mathbf{A}t} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} c_1(1 - t - t^2/2) + c_2 t + c_3(t + t^2/2) \\ -c_1 t + c_2 + c_3 t \\ c_1(-t - t^2/2) + c_2 t + c_3(1 + t + t^2/2) \end{pmatrix}.$$

**27. (a)** The following commands can be used in *Mathematica*:

```
A = {{4, 2},{3, 3}};
c = {c1, c2};
m = MatrixExp[A t];
sol = Expand[m.c]
Collect[sol, {c1, c2}]//MatrixForm
```

The output gives

$$x(t) = c_1 \left( \frac{2}{5} e^t + \frac{3}{5} e^{6t} \right) + c_2 \left( -\frac{2}{5} e^t + \frac{2}{5} e^{6t} \right)$$

$$y(t) = c_1 \left( -\frac{3}{5} e^t + \frac{3}{5} e^{6t} \right) + c_2 \left( \frac{3}{5} e^t + \frac{2}{5} e^{6t} \right).$$

The eigenvalues are 1 and 6 with corresponding eigenvectors

$$\begin{pmatrix} -2 \\ 3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

so the solution of the system is

$$\mathbf{X}(t) = b_1 \begin{pmatrix} -2 \\ 3 \end{pmatrix} e^t + b_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{6t}$$

or

$$\begin{aligned} x(t) &= -2b_1e^t + b_2e^{6t} \\ y(t) &= 3b_1e^t + b_2e^{6t}. \end{aligned}$$

If we replace $b_1$ with $-\frac{1}{5}c_1 + \frac{1}{5}c_2$ and $b_2$ with $\frac{3}{5}c_1 + \frac{2}{5}c_2$, we obtain the solution found using the matrix exponential.

**(b)** $x(t) = c_1e^{-2t}\cos t - (c_1 + c_2)e^{-2t}\sin t$

$y(t) = c_2e^{-2t}\cos t + (2c_1 + c_2)e^{-2t}\sin t$

**28.** $x(t) = c_1(3e^{-2t} - 2e^{-t}) + c_3(-6e^{-2t} + 6e^{-t})$

$y(t) = c_2(4e^{-2t} - 3e^{-t}) + c_4(4e^{-2t} - 4e^{-t})$

$z(t) = c_1(e^{-2t} - e^{-t}) + c_3(-2e^{-2t} + 3e^{-t})$

$w(t) = c_2(-3e^{-2t} + 3e^{-t}) + c_4(-3e^{-2t} + 4e^{-t})$

## Chapter 8 in Review

**1.** If $\mathbf{X} = k\begin{pmatrix} 4 \\ 5 \end{pmatrix}$, then $\mathbf{X}' = \mathbf{0}$ and

$$k\begin{pmatrix} 1 & 4 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 4 \\ 5 \end{pmatrix} - \begin{pmatrix} 8 \\ 1 \end{pmatrix} = k\begin{pmatrix} 24 \\ 3 \end{pmatrix} - \begin{pmatrix} 8 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

We see that $k = \frac{1}{3}$.

**2.** Solving for $c_1$ and $c_2$ we find $c_1 = -\frac{3}{4}$ and $c_2 = \frac{1}{4}$.

**3.** Since

$$\begin{pmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 12 \\ 4 \\ -4 \end{pmatrix} = 4\begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix},$$

we see that $\lambda = 4$ is an eigenvalue with eigenvector $\mathbf{K}_3$. The corresponding solution is $\mathbf{X}_3 = \mathbf{K}_3e^{4t}$.

**4.** The other eigenvalue is $\lambda_2 = 1 - 2i$ with corresponding eigenvector $\mathbf{K}_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$. The general solution is

$$\mathbf{X}(t) = c_1\begin{pmatrix} \cos 2t \\ -\sin 2t \end{pmatrix}e^t + c_2\begin{pmatrix} \sin 2t \\ \cos 2t \end{pmatrix}e^t.$$

**5.** We have $\det(\mathbf{A}-\lambda\mathbf{I}) = (\lambda-1)^2 = 0$ and $\mathbf{K} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. A solution to $(\mathbf{A}-\lambda\mathbf{I})\mathbf{P} = \mathbf{K}$ is $\mathbf{P} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ so that

$$\mathbf{X} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t + c_2 \left[ \begin{pmatrix} 1 \\ -1 \end{pmatrix} te^t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^t \right].$$

**6.** We have $\det(\mathbf{A}-\lambda\mathbf{I}) = (\lambda+6)(\lambda+2) = 0$ so that

$$\mathbf{X} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-6t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t}.$$

**7.** We have $\det(\mathbf{A}-\lambda\mathbf{I}) = \lambda^2 - 2\lambda + 5 = 0$. For $\lambda = 1+2i$ we obtain $\mathbf{K}_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}$ and

$$\mathbf{X}_1 = \begin{pmatrix} 1 \\ i \end{pmatrix} e^{(1+2i)t} = \begin{pmatrix} \cos 2t \\ -\sin 2t \end{pmatrix} e^t + i \begin{pmatrix} \sin 2t \\ \cos 2t \end{pmatrix} e^t.$$

Then

$$\mathbf{X} = c_1 \begin{pmatrix} \cos 2t \\ -\sin 2t \end{pmatrix} e^t + c_2 \begin{pmatrix} \sin 2t \\ \cos 2t \end{pmatrix} e^t.$$

**8.** We have $\det(\mathbf{A}-\lambda\mathbf{I}) = \lambda^2 - 2\lambda + 2 = 0$. For $\lambda = 1+i$ we obtain $\mathbf{K}_1 = \begin{pmatrix} 3-i \\ 2 \end{pmatrix}$ and

$$\mathbf{X}_1 = \begin{pmatrix} 3-i \\ 2 \end{pmatrix} e^{(1+i)t} = \begin{pmatrix} 3\cos t + \sin t \\ 2\cos t \end{pmatrix} e^t + i \begin{pmatrix} -\cos t + 3\sin t \\ 2\sin t \end{pmatrix} e^t.$$

Then

$$\mathbf{X} = c_1 \begin{pmatrix} 3\cos t + \sin t \\ 2\cos t \end{pmatrix} e^t + c_2 \begin{pmatrix} -\cos t + 3\sin t \\ 2\sin t \end{pmatrix} e^t.$$

**9.** We have $\det(\mathbf{A}-\lambda\mathbf{I}) = -(\lambda-2)(\lambda-4)(\lambda+3) = 0$ so that

$$\mathbf{X} = c_1 \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} e^{4t} + c_3 \begin{pmatrix} 7 \\ 12 \\ -16 \end{pmatrix} e^{-3t}.$$

**10.** We have $\det(\mathbf{A}-\lambda\mathbf{I}) = -(\lambda+2)(\lambda^2-2\lambda+3) = 0$. The eigenvalues are $\lambda_1 = -2$, $\lambda_2 = 1+\sqrt{2}i$, and $\lambda_2 = 1-\sqrt{2}i$, with eigenvectors

$$\mathbf{K}_1 = \begin{pmatrix} -7 \\ 5 \\ 4 \end{pmatrix}, \quad \mathbf{K}_2 = \begin{pmatrix} 1 \\ \frac{1}{2}\sqrt{2}\,i \\ 1 \end{pmatrix}, \quad \text{and} \quad \mathbf{K}_3 = \begin{pmatrix} 1 \\ -\frac{1}{2}\sqrt{2}\,i \\ 1 \end{pmatrix}.$$

Thus

$$\begin{aligned}\mathbf{X} &= c_1\begin{pmatrix}-7\\5\\4\end{pmatrix}e^{-2t} + c_2\left[\begin{pmatrix}1\\0\\1\end{pmatrix}\cos\sqrt{2}\,t - \begin{pmatrix}0\\\frac{1}{2}\sqrt{2}\\0\end{pmatrix}\sin\sqrt{2}\,t\right]e^{t}\\ &\qquad + c_3\left[\begin{pmatrix}0\\\frac{1}{2}\sqrt{2}\\0\end{pmatrix}\cos\sqrt{2}t + \begin{pmatrix}1\\0\\1\end{pmatrix}\sin\sqrt{2}\,t\right]e^{t}\\ &= c_1\begin{pmatrix}-7\\5\\4\end{pmatrix}e^{-2t} + c_2\begin{pmatrix}\cos\sqrt{2}t\\-\frac{1}{2}\sqrt{2}\sin\sqrt{2}t\\\cos\sqrt{2}\,t\end{pmatrix}e^{t} + c_3\begin{pmatrix}\sin\sqrt{2}\,t\\\frac{1}{2}\sqrt{2}\cos\sqrt{2}\,t\\\sin\sqrt{2}\,t\end{pmatrix}e^{t}.\end{aligned}$$

**11.** We have

$$\mathbf{X}_c = c_1\begin{pmatrix}1\\0\end{pmatrix}e^{2t} + c_2\begin{pmatrix}4\\1\end{pmatrix}e^{4t}.$$

Then

$$\mathbf{\Phi} = \begin{pmatrix}e^{2t} & 4e^{4t}\\0 & e^{4t}\end{pmatrix}, \quad \mathbf{\Phi}^{-1} = \begin{pmatrix}e^{-2t} & -4e^{-2t}\\0 & e^{-4t}\end{pmatrix},$$

and

$$\mathbf{U} = \int \mathbf{\Phi}^{-1}\mathbf{F}\,dt = \int\begin{pmatrix}2e^{-2t} - 64te^{-2t}\\16te^{-4t}\end{pmatrix}dt = \begin{pmatrix}15e^{-2t} + 32te^{-2t}\\-e^{-4t} - 4te^{-4t}\end{pmatrix},$$

so that

$$\mathbf{X}_p = \mathbf{\Phi}\mathbf{U} = \begin{pmatrix}11 + 16t\\-1 - 4t\end{pmatrix}.$$

The solution is

$$\mathbf{X} = \mathbf{X}_c + \mathbf{X}_p = c_1\begin{pmatrix}1\\0\end{pmatrix}e^{2t} + c_2\begin{pmatrix}4\\1\end{pmatrix}e^{4t} + \begin{pmatrix}11 + 16t\\-1 - 4t\end{pmatrix}.$$

**12.** We have

$$\mathbf{X}_c = c_1\begin{pmatrix}2\cos t\\-\sin t\end{pmatrix}e^{t} + c_2\begin{pmatrix}2\sin t\\\cos t\end{pmatrix}e^{t}.$$

Then

$$\mathbf{\Phi} = \begin{pmatrix}2\cos t & 2\sin t\\-\sin t & \cos t\end{pmatrix}e^{t}, \quad \mathbf{\Phi}^{-1} = \begin{pmatrix}\frac{1}{2}\cos t & -\sin t\\\frac{1}{2}\sin t & \cos t\end{pmatrix}e^{-t},$$

and

$$\mathbf{U} = \int \mathbf{\Phi}^{-1}\mathbf{F}\,dt = \int\begin{pmatrix}\cos t - \sec t\\\sin t\end{pmatrix}dt = \begin{pmatrix}\sin t - \ln|\sec t + \tan t|\\-\cos t\end{pmatrix},$$

so that

$$\mathbf{X}_p = \mathbf{\Phi U} = \begin{pmatrix} -2\cos t \ln|\sec t + \tan t| \\ -1 + \sin t \ln|\sec t + \tan t| \end{pmatrix} e^t.$$

The solution is

$$\mathbf{X} = \mathbf{X}_c + \mathbf{X}_p = c_1 \begin{pmatrix} 2\cos t \\ -\sin t \end{pmatrix} e^t + c_2 \begin{pmatrix} 2\sin t \\ \cos t \end{pmatrix} e^t + \begin{pmatrix} -2\cos t \ln|\sec t + \tan t| \\ -1 + \sin t \ln|\sec t + \tan t| \end{pmatrix} e^t.$$

**13.** We have

$$\mathbf{X}_c = c_1 \begin{pmatrix} \cos t + \sin t \\ 2\cos t \end{pmatrix} + c_2 \begin{pmatrix} \sin t - \cos t \\ 2\sin t \end{pmatrix}.$$

Then

$$\mathbf{\Phi} = \begin{pmatrix} \cos t + \sin t & \sin t - \cos t \\ 2\cos t & 2\sin t \end{pmatrix}, \quad \mathbf{\Phi}^{-1} = \begin{pmatrix} \sin t & \frac{1}{2}\cos t - \frac{1}{2}\sin t \\ -\cos t & \frac{1}{2}\cos t + \frac{1}{2}\sin t \end{pmatrix},$$

and

$$\mathbf{U} = \int \mathbf{\Phi}^{-1}\mathbf{F}\,dt = \int \begin{pmatrix} \frac{1}{2}\sin t - \frac{1}{2}\cos t + \frac{1}{2}\csc t \\ -\frac{1}{2}\sin t - \frac{1}{2}\cos t + \frac{1}{2}\csc t \end{pmatrix} dt$$

$$= \begin{pmatrix} -\frac{1}{2}\cos t - \frac{1}{2}\sin t + \frac{1}{2}\ln|\csc t - \cot t| \\ \frac{1}{2}\cos t - \frac{1}{2}\sin t + \frac{1}{2}\ln|\csc t - \cot t| \end{pmatrix},$$

so that

$$\mathbf{X}_p = \mathbf{\Phi U} = \begin{pmatrix} -1 \\ -1 \end{pmatrix} + \begin{pmatrix} \sin t \\ \sin t + \cos t \end{pmatrix} \ln|\csc t - \cot t|.$$

The solution is

$$\begin{aligned} \mathbf{X} &= \mathbf{X}_c + \mathbf{X}_p \\ &= c_1 \begin{pmatrix} \cos t + \sin t \\ 2\cos t \end{pmatrix} + c_2 \begin{pmatrix} \sin t - \cos t \\ 2\sin t \end{pmatrix} + \begin{pmatrix} -1 \\ -1 \end{pmatrix} + \begin{pmatrix} \sin t \\ \sin t + \cos t \end{pmatrix} \ln|\csc t - \cot t|. \end{aligned}$$

**14.** We have

$$\mathbf{X}_c = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t} + c_2 \left[ \begin{pmatrix} 1 \\ -1 \end{pmatrix} te^{2t} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t} \right].$$

Then

$$\mathbf{\Phi} = \begin{pmatrix} e^{2t} & te^{2t} + e^{2t} \\ -e^{2t} & -te^{2t} \end{pmatrix}, \quad \mathbf{\Phi}^{-1} = \begin{pmatrix} -te^{-2t} & -te^{-2t} - e^{-2t} \\ e^{-2t} & e^{-2t} \end{pmatrix},$$

and

$$\mathbf{U} = \int \mathbf{\Phi}^{-1}\mathbf{F}\,dt = \int \begin{pmatrix} t - 1 \\ -1 \end{pmatrix} dt = \begin{pmatrix} \frac{1}{2}t^2 - t \\ -t \end{pmatrix},$$

so that

$$\mathbf{X}_p = \mathbf{\Phi U} = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix} t^2 e^{2t} + \begin{pmatrix} -2 \\ 1 \end{pmatrix} te^{2t}.$$

The solution is

$$\mathbf{X} = \mathbf{X}_c + \mathbf{X}_p = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1+t \\ -t \end{pmatrix} e^{2t} + \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix} t^2 e^{2t} + \begin{pmatrix} -2 \\ 1 \end{pmatrix} te^{2t}.$$

**15. (a)** Letting

$$\mathbf{K} = \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix}$$

we note that $(\mathbf{A} - 2\mathbf{I})\mathbf{K} = \mathbf{0}$ implies that $3k_1 + 3k_2 + 3k_3 = 0$, so $k_1 = -(k_2 + k_3)$. Choosing $k_2 = 0$, $k_3 = 1$ and then $k_2 = 1$, $k_3 = 0$ we get

$$\mathbf{K}_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{K}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix},$$

respectively. Thus,

$$\mathbf{X}_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} e^{2t} \quad \text{and} \quad \mathbf{X}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} e^{2t}$$

are two solutions.

**(b)** From $\det(\mathbf{A} - \lambda\mathbf{I}) = \lambda^2(3 - \lambda) = 0$ we see that $\lambda_1 = 3$, and $\lambda_2 = 0$ is an eigenvalue of multiplicity two. Letting

$$\mathbf{K} = \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix},$$

as in part (**A**), we note that $(\mathbf{A} - 0\mathbf{I})\mathbf{K} = \mathbf{AK} = \mathbf{0}$ implies that $k_1 + k_2 + k_3 = 0$, so $k_1 = -(k_2 + k_3)$. Choosing $k_2 = 0$, $k_3 = 1$, and then $k_2 = 1$, $k_3 = 0$ we get

$$\mathbf{K}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{K}_3 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix},$$

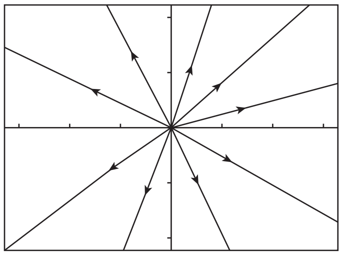
respectively. Since an eigenvector corresponding to $\lambda_1 = 3$ is

$$\mathbf{K}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

the general solution of the system is

$$\mathbf{X} = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.$$

**16.** For $\mathbf{X} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} e^t$ we have $\mathbf{X}' = \mathbf{X} = \mathbf{I}\mathbf{X}$.



# Chapter 9

# Numerical Solutions of Ordinary Differential Equations

## 9.1 Euler Methods and Error Analysis

**1.**

$h = 0.1$

| $x_n$ | $y_n$ |
|---|---|
| 1.00 | 5.0000 |
| 1.10 | 3.9900 |
| 1.20 | 3.2546 |
| 1.30 | 2.7236 |
| 1.40 | 2.3451 |
| 1.50 | 2.0801 |

$h = 0.05$

| $x_n$ | $y_n$ |
|---|---|
| 1.00 | 5.0000 |
| 1.05 | 4.4475 |
| 1.10 | 3.9763 |
| 1.15 | 3.5751 |
| 1.20 | 3.2342 |
| 1.25 | 2.9452 |
| 1.30 | 2.7009 |
| 1.35 | 2.4952 |
| 1.40 | 2.3226 |
| 1.45 | 2.1786 |
| 1.50 | 2.0592 |

**2.**

$h = 0.1$

| $x_n$ | $y_n$ |
|---|---|
| 0.00 | 2.0000 |
| 0.10 | 1.6600 |
| 0.20 | 1.4172 |
| 0.30 | 1.2541 |
| 0.40 | 1.1564 |
| 0.50 | 1.1122 |

$h = 0.05$

| $x_n$ | $y_n$ |
|---|---|
| 0.00 | 2.0000 |
| 0.05 | 1.8150 |
| 0.10 | 1.6571 |
| 0.15 | 1.5237 |
| 0.20 | 1.4124 |
| 0.25 | 1.3212 |
| 0.30 | 1.2482 |
| 0.35 | 1.1916 |
| 0.40 | 1.1499 |
| 0.45 | 1.1217 |
| 0.50 | 1.1056 |

**3.**

$h = 0.1$

| $x_n$ | $y_n$ |
|---|---|
| 0.00 | 0.0000 |
| 0.10 | 0.1005 |
| 0.20 | 0.2030 |
| 0.30 | 0.3098 |
| 0.40 | 0.4234 |
| 0.50 | 0.5470 |

$h = 0.05$

| $x_n$ | $y_n$ |
|---|---|
| 0.00 | 0.0000 |
| 0.05 | 0.0501 |
| 0.10 | 0.1004 |
| 0.15 | 0.1512 |
| 0.20 | 0.2028 |
| 0.25 | 0.2554 |
| 0.30 | 0.3095 |
| 0.35 | 0.3652 |
| 0.40 | 0.4230 |
| 0.45 | 0.4832 |
| 0.50 | 0.5465 |

**4.**

$h = 0.1$

| $x_n$ | $y_n$ |
|---|---|
| 0.00 | 1.0000 |
| 0.10 | 1.1110 |
| 0.20 | 1.2515 |
| 0.30 | 1.4361 |
| 0.40 | 1.6880 |
| 0.50 | 2.0488 |

$h = 0.05$

| $x_n$ | $y_n$ |
|---|---|
| 0.00 | 1.0000 |
| 0.05 | 1.0526 |
| 0.10 | 1.1113 |
| 0.15 | 1.1775 |
| 0.20 | 1.2526 |
| 0.25 | 1.3388 |
| 0.30 | 1.4387 |
| 0.35 | 1.5556 |
| 0.40 | 1.6939 |
| 0.45 | 1.8598 |
| 0.50 | 2.0619 |

**5.**

$h = 0.1$

| $x_n$ | $y_n$ |
|---|---|
| 0.00 | 0.0000 |
| 0.10 | 0.0952 |
| 0.20 | 0.1822 |
| 0.30 | 0.2622 |
| 0.40 | 0.3363 |
| 0.50 | 0.4053 |

$h = 0.05$

| $x_n$ | $y_n$ |
|---|---|
| 0.00 | 0.0000 |
| 0.05 | 0.0488 |
| 0.10 | 0.0953 |
| 0.15 | 0.1397 |
| 0.20 | 0.1823 |
| 0.25 | 0.2231 |
| 0.30 | 0.2623 |
| 0.35 | 0.3001 |
| 0.40 | 0.3364 |
| 0.45 | 0.3715 |
| 0.50 | 0.4054 |

**6.**

$h = 0.1$

| $x_n$ | $y_n$ |
|---|---|
| 0.00 | 0.0000 |
| 0.10 | 0.0050 |
| 0.20 | 0.0200 |
| 0.30 | 0.0451 |
| 0.40 | 0.0805 |
| 0.50 | 0.1266 |

$h = 0.05$

| $x_n$ | $y_n$ |
|---|---|
| 0.00 | 0.0000 |
| 0.05 | 0.0013 |
| 0.10 | 0.0050 |
| 0.15 | 0.0113 |
| 0.20 | 0.0200 |
| 0.25 | 0.0313 |
| 0.30 | 0.0451 |
| 0.35 | 0.0615 |
| 0.40 | 0.0805 |
| 0.45 | 0.1022 |
| 0.50 | 0.1266 |

**7.**

$h = 0.1$

| $x_n$ | $y_n$ |
|---|---|
| 0.00 | 0.5000 |
| 0.10 | 0.5215 |
| 0.20 | 0.5362 |
| 0.30 | 0.5449 |
| 0.40 | 0.5490 |
| 0.50 | 0.5503 |

$h = 0.05$

| $x_n$ | $y_n$ |
|---|---|
| 0.00 | 0.5000 |
| 0.05 | 0.5116 |
| 0.10 | 0.5214 |
| 0.15 | 0.5294 |
| 0.20 | 0.5359 |
| 0.25 | 0.5408 |
| 0.30 | 0.5444 |
| 0.35 | 0.5469 |
| 0.40 | 0.5484 |
| 0.45 | 0.5492 |
| 0.50 | 0.5495 |

**8.**

$h = 0.1$

| $x_n$ | $y_n$ |
|---|---|
| 0.00 | 1.0000 |
| 0.10 | 1.1079 |
| 0.20 | 1.2337 |
| 0.30 | 1.3806 |
| 0.40 | 1.5529 |
| 0.50 | 1.7557 |

$h = 0.05$

| $x_n$ | $y_n$ |
|---|---|
| 0.00 | 1.0000 |
| 0.05 | 1.0519 |
| 0.10 | 1.1079 |
| 0.15 | 1.1684 |
| 0.20 | 1.2337 |
| 0.25 | 1.3043 |
| 0.30 | 1.3807 |
| 0.35 | 1.4634 |
| 0.40 | 1.5530 |
| 0.45 | 1.6503 |
| 0.50 | 1.7560 |

**9.**

$h = 0.1$

| $x_n$ | $y_n$ |
|---|---|
| 1.00 | 1.0000 |
| 1.10 | 1.0095 |
| 1.20 | 1.0404 |
| 1.30 | 1.0967 |
| 1.40 | 1.1866 |
| 1.50 | 1.3260 |

$h = 0.05$

| $x_n$ | $y_n$ |
|---|---|
| 1.00 | 1.0000 |
| 1.05 | 1.0024 |
| 1.10 | 1.0100 |
| 1.15 | 1.0228 |
| 1.20 | 1.0414 |
| 1.25 | 1.0663 |
| 1.30 | 1.0984 |
| 1.35 | 1.1389 |
| 1.40 | 1.1895 |
| 1.45 | 1.2526 |
| 1.50 | 1.3315 |

**10.**

$h = 0.1$

| $x_n$ | $y_n$ |
|---|---|
| 0.00 | 0.5000 |
| 0.10 | 0.5250 |
| 0.20 | 0.5498 |
| 0.30 | 0.5744 |
| 0.40 | 0.5986 |
| 0.50 | 0.6224 |

$h = 0.05$

| $x_n$ | $y_n$ |
|---|---|
| 0.00 | 0.5000 |
| 0.05 | 0.5125 |
| 0.10 | 0.5250 |
| 0.15 | 0.5374 |
| 0.20 | 0.5498 |
| 0.25 | 0.5622 |
| 0.30 | 0.5744 |
| 0.35 | 0.5866 |
| 0.40 | 0.5987 |
| 0.45 | 0.6106 |
| 0.50 | 0.6224 |

**11.** To obtain the analytic solution use the substitution $u = x + y - 1$. The resulting differential equation in $u(x)$ will be separable.

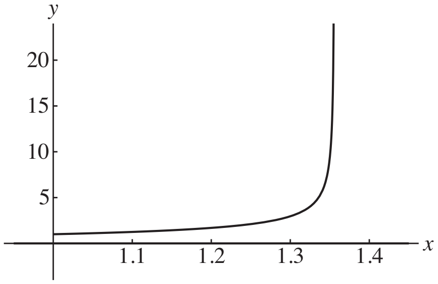
$h = 0.1$

| $x_n$ | $y_n$ | Actual Value |
|---|---|---|
| 0.00 | 2.0000 | 2.0000 |
| 1.10 | 2.1220 | 2.1230 |
| 0.20 | 2.3049 | 2.3085 |
| 0.30 | 2.5858 | 2.5958 |
| 0.40 | 3.0378 | 3.0650 |
| 0.50 | 3.8254 | 3.9082 |

$h = 0.05$

| $x_n$ | $y_n$ | Actual Value |
|---|---|---|
| 0.00 | 2.0000 | 2.0000 |
| 0.05 | 2.0553 | 2.0554 |
| 0.10 | 2.1228 | 2.1230 |
| 0.15 | 2.2056 | 2.2061 |
| 0.20 | 2.3075 | 2.3085 |
| 0.25 | 2.4342 | 2.4358 |
| 0.30 | 2.5931 | 2.5958 |
| 0.35 | 2.7953 | 2.7997 |
| 0.40 | 3.0574 | 3.0650 |
| 0.45 | 3.4057 | 3.4189 |
| 0.50 | 3.8840 | 3.9082 |

**12. (a)**



**(b)**

| $x_n$ | *Euler* | *Imp. Euler* |
|---|---|---|
| 1.00 | 1.0000 | 1.0000 |
| 1.10 | 1.2000 | 1.2000 |
| 1.20 | 1.4938 | 1.4938 |
| 1.30 | 1.9711 | 1.9711 |
| 1.40 | 2.9060 | 2.9060 |

**13. (a)** Using Euler's method we obtain $y(0.1) \approx y_1 = 1.2$.

**(b)** Using $y'' = 4e^{2x}$ we see that the local truncation error is

$$y''(c)\,\frac{h^2}{2} = 4e^{2c}\,\frac{(0.1)^2}{2} = 0.02e^{2c}.$$

Since $e^{2x}$ is an increasing function, $e^{2c} \leq e^{2(0.1)} = e^{0.2}$ for $0 \leq c \leq 0.1$. Thus an upper bound for the local truncation error is $0.02e^{0.2} = 0.0244$.

**(c)** Since $y(0.1) = e^{0.2} = 1.2214$, the actual error is $y(0.1) - y_1 = 0.0214$, which is less than 0.0244.

**(d)** Using Euler's method with $h = 0.05$ we obtain $y(0.1) \approx y_2 = 1.21$.

**(e)** The error in (d) is $1.2214 - 1.21 = 0.0114$. With global truncation error $O(h)$, when the step size is halved we expect the error for $h = 0.05$ to be one-half the error when $h = 0.1$. Comparing 0.0114 with 0.0214 we see that this is the case.

**14.** **(a)** Using the improved Euler's method we obtain $y(0.1) \approx y_1 = 1.22$.

**(b)** Using $y''' = 8e^{2x}$ we see that the local truncation error is

$$y'''(c)\,\frac{h^3}{6} = 8e^{2c}\,\frac{(0.1)^3}{6} = 0.001333e^{2c}.$$

Since $e^{2x}$ is an increasing function, $e^{2c} \le e^{2(0.1)} = e^{0.2}$ for $0 \le c \le 0.1$. Thus an upper bound for the local truncation error is $0.001333e^{0.2} = 0.001628$.

**(c)** Since $y(0.1) = e^{0.2} = 1.221403$, the actual error is $y(0.1) - y_1 = 0.001403$ which is less than 0.001628.

**(d)** Using the improved Euler's method with $h = 0.05$ we obtain $y(0.1) \approx y_2 = 1.221025$.

**(e)** The error in (d) is $1.221403 - 1.221025 = 0.000378$. With global truncation error $O(h^2)$, when the step size is halved we expect the error for $h = 0.05$ to be one-fourth the error for $h = 0.1$. Comparing 0.000378 with 0.001403 we see that this is the case.

**15.** **(a)** Using Euler's method we obtain $y(0.1) \approx y_1 = 0.8$.

**(b)** Using $y'' = 5e^{-2x}$ we see that the local truncation error is

$$5e^{-2c}\,\frac{(0.1)^2}{2} = 0.025e^{-2c}.$$

Since $e^{-2x}$ is a decreasing function, $e^{-2c} \le e^0 = 1$ for $0 \le c \le 0.1$. Thus an upper bound for the local truncation error is $0.025(1) = 0.025$.

**(c)** Since $y(0.1) = 0.8234$, the actual error is $y(0.1) - y_1 = 0.0234$, which is less than 0.025.

**(d)** Using Euler's method with $h = 0.05$ we obtain $y(0.1) \approx y_2 = 0.8125$.

**(e)** The error in (d) is $0.8234 - 0.8125 = 0.0109$. With global truncation error $O(h)$, when the step size is halved we expect the error for $h = 0.05$ to be one-half the error when $h = 0.1$. Comparing 0.0109 with 0.0234 we see that this is the case.

**16.** **(a)** Using the improved Euler's method we obtain $y(0.1) \approx y_1 = 0.825$.

**(b)** Using $y''' = -10e^{-2x}$ we see that the local truncation error is

$$10e^{-2c}\,\frac{(0.1)^3}{6} = 0.001667e^{-2c}.$$

Since $e^{-2x}$ is a decreasing function, $e^{-2c} \le e^0 = 1$ for $0 \le c \le 0.1$. Thus an upper bound for the local truncation error is $0.001667(1) = 0.001667$.

**(c)** Since $y(0.1) = 0.823413$, the actual error is $y(0.1) - y_1 = 0.001587$, which is less than 0.001667.

**(d)** Using the improved Euler's method with $h = 0.05$ we obtain $y(0.1) \approx y_2 = 0.823781$.

**(e)** The error in (d) is $|0.823413 - 0.8237181| = 0.000305$. With global truncation error $O(h^2)$, when the step size is halved we expect the error for $h = 0.05$ to be one-fourth the error when $h = 0.1$. Comparing 0.000305 with 0.001587 we see that this is the case.

**17.** **(a)** Using $y'' = 38e^{-3(x-1)}$ we see that the local truncation error is

$$y''(c)\,\frac{h^2}{2} = 38e^{-3(c-1)}\,\frac{h^2}{2} = 19h^2e^{-3(c-1)}.$$

**(b)** Since $e^{-3(x-1)}$ is a decreasing function for $1 \le x \le 1.5$, $e^{-3(c-1)} \le e^{-3(1-1)} = 1$ for $1 \le c \le 1.5$ and

$$y''(c)\,\frac{h^2}{2} \le 19(0.1)^2(1) = 0.19.$$

**(c)** Using Euler's method with $h = 0.1$ we obtain $y(1.5) \approx 1.8207$. With $h = 0.05$ we obtain $y(1.5) \approx 1.9424$.

**(d)** Since $y(1.5) = 2.0532$, the error for $h = 0.1$ is $E_{0.1} = 0.2325$, while the error for $h = 0.05$ is $E_{0.05} = 0.1109$. With global truncation error $O(h)$ we expect $E_{0.1}/E_{0.05} \approx 2$. We actually have $E_{0.1}/E_{0.05} = 2.10$.

**18.** **(a)** Using $y''' = -114e^{-3(x-1)}$ we see that the local truncation error is

$$\left| y'''(c)\,\frac{h^3}{6} \right| = 114e^{-3(x-1)}\,\frac{h^3}{6} = 19h^3e^{-3(c-1)}.$$

**(b)** Since $e^{-3(x-1)}$ is a decreasing function for $1 \le x \le 1.5$, $e^{-3(c-1)} \le e^{-3(1-1)} = 1$ for $1 \le c \le 1.5$ and

$$\left| y'''(c)\,\frac{h^3}{6} \right| \le 19(0.1)^3(1) = 0.019.$$

**(c)** Using the improved Euler's method with $h = 0.1$ we obtain $y(1.5) \approx 2.080108$. With $h = 0.05$ we obtain $y(1.5) \approx 2.059166$.

**(d)** Since $y(1.5) = 2.053216$, the error for $h = 0.1$ is $E_{0.1} = 0.026892$, while the error for $h = 0.05$ is $E_{0.05} = 0.005950$. With global truncation error $O(h^2)$ we expect $E_{0.1}/E_{0.05} \approx 4$. We actually have $E_{0.1}/E_{0.05} = 4.52$.

**19.** **(a)** Using $y'' = -1/(x+1)^2$ we see that the local truncation error is

$$\left| y''(c)\,\frac{h^2}{2} \right| = \frac{1}{(c+1)^2}\,\frac{h^2}{2}.$$

**(b)** Since $1/(x+1)^2$ is a decreasing function for $0 \le x \le 0.5$, $1/(c+1)^2 \le 1/(0+1)^2 = 1$ for $0 \le c \le 0.5$ and

$$\left| y''(c)\,\frac{h^2}{2} \right| \le (1)\,\frac{(0.1)^2}{2} = 0.005.$$

**(c)** Using Euler's method with $h = 0.1$ we obtain $y(0.5) \approx 0.4198$. With $h = 0.05$ we obtain $y(0.5) \approx 0.4124$.

**(d)** Since $y(0.5) = 0.4055$, the error for $h = 0.1$ is $E_{0.1} = 0.0143$, while the error for $h = 0.05$ is $E_{0.05} = 0.0069$. With global truncation error $O(h)$ we expect $E_{0.1}/E_{0.05} \approx 2$. We actually have $E_{0.1}/E_{0.05} = 2.06$.

**20.** **(a)** Using $y''' = 2/(x+1)^3$ we see that the local truncation error is

$$y'''(c)\,\frac{h^3}{6} = \frac{1}{(c+1)^3}\,\frac{h^3}{3}.$$

**(b)** Since $1/(x+1)^3$ is a decreasing function for $0 \le x \le 0.5$, $1/(c+1)^3 \le 1/(0+1)^3 = 1$ for $0 \le c \le 0.5$ and

$$y'''(c)\,\frac{h^3}{6} \le (1)\,\frac{(0.1)^3}{3} = 0.000333.$$

**(c)** Using the improved Euler's method with $h = 0.1$ we obtain $y(0.5) \approx 0.405281$. With $h = 0.05$ we obtain $y(0.5) \approx 0.405419$.

**(d)** Since $y(0.5) = 0.405465$, the error for $h = 0.1$ is $E_{0.1} = 0.000184$, while the error for $h = 0.05$ is $E_{0.05} = 0.000046$. With global truncation error $O(h^2)$ we expect $E_{0.1}/E_{0.05} \approx 4$. We actually have $E_{0.1}/E_{0.05} = 3.98$.

**21.** Because $y^*_{n+1}$ depends on $y_n$ and is used to determine $y_{n+1}$, all of the $y^*_n$ cannot be computed at one time independently of the corresponding $y_n$ values. For example, the computation of $y^*_4$ involves the value of $y_3$.

## 9.2 Runge–Kutta Methods

**1.**

| $x_n$ | $y_n$ | Actual Value |
|---|---|---|
| 0.00 | 2.0000 | 2.0000 |
| 1.10 | 2.1230 | 2.1230 |
| 0.20 | 2.3085 | 2.3085 |
| 0.30 | 2.5958 | 2.5958 |
| 0.40 | 3.0649 | 3.0650 |
| 0.50 | 3.9078 | 3.9082 |

**2.** In this problem we use $h = 0.1$. Substituting $w_2 = \frac{3}{4}$ into the equations in (4) in the text, we obtain

$$w_1 = 1 - w_2 = \frac{1}{4} \quad \alpha = \frac{1}{2w_2} = \frac{2}{3} \quad \text{and} \quad \beta = \frac{1}{2w_2} = \frac{2}{3}.$$

The resulting second-order Runge-Kutta method is

$$y_{n+1} = y_n + h\left(\frac{1}{4}k_1 + \frac{3}{4}k_2\right) = y_n + \frac{h}{4}(k_1 + 3k_2)$$

where

$$k_1 = f(x_n, y_n) \quad \text{and} \quad k_2 = f\left(x_n + \frac{2}{3}h, y_n + \frac{2}{3}hk_1\right).$$

| $x_n$ | Second–Order Runge–Kutta | Improved Euler |
|---|---|---|
| 0.00 | 2.0000 | 2.0000 |
| 1.10 | 2.1213 | 2.1220 |
| 0.20 | 2.3030 | 2.3049 |
| 0.30 | 2.5814 | 2.5858 |
| 0.40 | 3.0277 | 3.0378 |
| 0.50 | 3.8002 | 3.8254 |

The table compares the values obtained using this second-order Runge-Kutta method with the values obtained using the improved Euler's method.

**3.**

| $x_n$ | $y_n$ |
|---|---|
| 1.00 | 5.0000 |
| 1.10 | 3.9724 |
| 1.20 | 3.2284 |
| 1.30 | 2.6945 |
| 1.40 | 2.3163 |
| 1.50 | 2.0533 |

**4.**

| $x_n$ | $y_n$ |
|---|---|
| 0.00 | 2.0000 |
| 0.10 | 1.6562 |
| 0.20 | 1.4110 |
| 0.30 | 1.2465 |
| 0.40 | 1.1480 |
| 0.50 | 1.1037 |

**5.**

| $x_n$ | $y_n$ |
|---|---|
| 0.00 | 0.0000 |
| 0.10 | 0.1003 |
| 0.20 | 0.2027 |
| 0.30 | 0.3093 |
| 0.40 | 0.4228 |
| 0.50 | 0.5463 |

**6.**

| $x_n$ | $y_n$ |
|---|---|
| 0.00 | 1.0000 |
| 0.10 | 1.1115 |
| 0.20 | 1.2530 |
| 0.30 | 1.4397 |
| 0.40 | 1.6961 |
| 0.50 | 2.0670 |

**7.**

| $x_n$ | $y_n$ |
|---|---|
| 0.00 | 0.0000 |
| 0.10 | 0.0953 |
| 0.20 | 0.1823 |
| 0.30 | 0.2624 |
| 0.40 | 0.3365 |
| 0.50 | 0.4055 |

**8.**

| $x_n$ | $y_n$ |
|---|---|
| 0.00 | 0.0000 |
| 0.10 | 0.0050 |
| 0.20 | 0.0200 |
| 0.30 | 0.0451 |
| 0.40 | 0.0805 |
| 0.50 | 0.1266 |

**9.**

| $x_n$ | $y_n$ |
|---|---|
| 0.00 | 0.5000 |
| 0.10 | 0.5213 |
| 0.20 | 0.5358 |
| 0.30 | 0.5443 |
| 0.40 | 0.5482 |
| 0.50 | 0.5493 |

**10.**

| $x_n$ | $y_n$ |
|---|---|
| 0.00 | 1.0000 |
| 0.10 | 1.1079 |
| 0.20 | 1.2337 |
| 0.30 | 1.3807 |
| 0.40 | 1.5531 |
| 0.50 | 1.7561 |

**11.**

| $x_n$ | $y_n$ |
|---|---|
| 1.00 | 1.0000 |
| 1.10 | 1.0101 |
| 1.20 | 1.0417 |
| 1.30 | 1.0989 |
| 1.40 | 1.1905 |
| 1.50 | 1.3333 |

**12.**

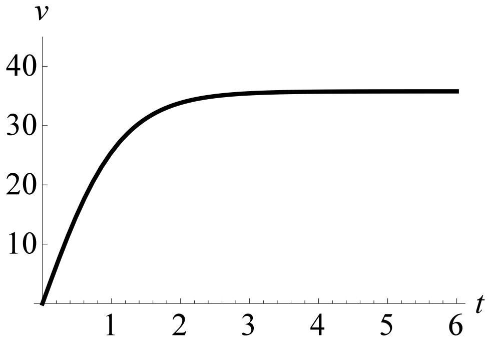
| $x_n$ | $y_n$ |
|---|---|
| 0.00 | 0.5000 |
| 0.10 | 0.5250 |
| 0.20 | 0.5498 |
| 0.30 | 0.5744 |
| 0.40 | 0.5987 |
| 0.50 | 0.6225 |

**13. (a)** Write the equation in the form

$$\frac{dv}{dt} = 32 - 0.025v^2 = f(t, v).$$

| $t_n$ | $y_n$ |
|---|---|
| 0.0 | 0.0000 |
| 1.0 | 25.2570 |
| 2.0 | 32.9390 |
| 3.0 | 34.9772 |
| 4.0 | 35.5503 |
| 5.0 | 35.7128 |

**(b)**



**(c)** Separating variables and using partial fractions we have

$$\frac{1}{2\sqrt{32}}\left(\frac{1}{\sqrt{32}-\sqrt{0.025}\,v} + \frac{1}{\sqrt{32}+\sqrt{0.025}\,v}\right) dv = dt$$

and

$$\frac{1}{2\sqrt{32}\sqrt{0.025}}\left(\ln\left|\sqrt{32}+\sqrt{0.025}\,v\right| - \ln\left|\sqrt{32}-\sqrt{0.025}\,v\right|\right) = t + c.$$

Since $v(0) = 0$ we find $c = 0$. Solving for $v$ we obtain

$$v(t) = \frac{16\sqrt{5}(e^{\sqrt{3.2}\,t} - 1)}{e^{\sqrt{3.2}\,t} + 1}$$

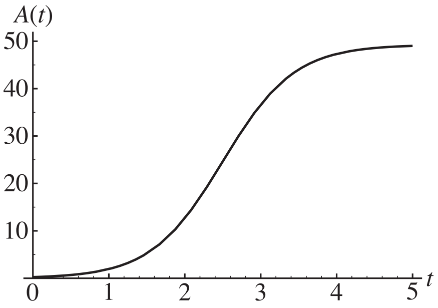
and $v(5) \approx 35.7678$. Alternatively, the solution can be expressed as

$$v(t) = \sqrt{\frac{mg}{k}}\,\tanh\sqrt{\frac{kg}{m}}\,t.$$

**14. (a)**

| *t (days)* | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| ***A (observed)*** | 2.78 | 13.53 | 36.30 | 47.50 | 49.40 |
| ***A (approximated)*** | 1.93 | 12.50 | 36.46 | 47.23 | 49.00 |

**(b)** From the graph we estimate $A(1) \approx 1.68$, $A(2) \approx 13.2$, $A(3) \approx 36.8$, $A(4) \approx 46.9$, and $A(5) \approx 48.9$.



**(c)** Let $\alpha = 2.128$ and $\beta = 0.0432$. Separating variables we obtain

$$\frac{dA}{A(\alpha - \beta A)} = dt$$

$$\frac{1}{\alpha}\left(\frac{1}{A} + \frac{\beta}{\alpha - \beta A}\right) dA = dt$$

$$\frac{1}{\alpha}[\ln A - \ln(\alpha - \beta A)] = t + c$$

$$\ln\frac{A}{\alpha - \beta A} = \alpha(t + c)$$

$$\frac{A}{\alpha - \beta A} = e^{\alpha(t+c)}$$

$$A = \alpha e^{\alpha(t+c)} - \beta A e^{\alpha(t+c)}$$

$$\left[1 + \beta e^{\alpha(t+c)}\right] A = \alpha e^{\alpha(t+c)}.$$

Thus

$$A(t) = \frac{\alpha e^{\alpha(t+c)}}{1 + \beta e^{\alpha(t+c)}} = \frac{\alpha}{\beta + e^{-\alpha(t+c)}} = \frac{\alpha}{\beta + e^{-\alpha c}e^{-\alpha t}}\,.$$

From $A(0) = 0.24$ we obtain

$$0.24 = \frac{\alpha}{\beta + e^{-\alpha c}}$$

so that $e^{-\alpha c} = \alpha/0.24 - \beta \approx 8.8235$ and

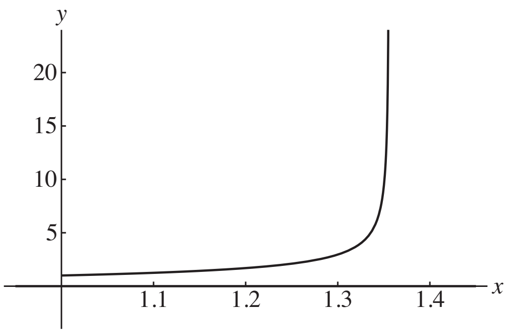
$$A(t) \approx \frac{2.128}{+}0.0432 + 8.8235e^{-2.128t}\,.$$

| *t (days)* | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| ***A (observed)*** | 2.78 | 13.53 | 36.30 | 47.50 | 49.40 |
| ***A (actual)*** | 1.95 | 12.64 | 36.63 | 49.02 | 49.02 |

**15. (a)**

| $x_n$ | $h = 0.05$ | $h = 0.1$ |
|---|---|---|
| 1.00 | 1.0000 | 1.0000 |
| 1.05 | 1.1112 | |
| 1.10 | 1.2511 | 1.2511 |
| 1.15 | 1.4348 | |
| 1.20 | 1.6934 | 1.6934 |
| 1.25 | 2.1047 | |
| 1.30 | 2.9560 | 2.9425 |
| 1.35 | 7.8981 | |
| 1.40 | $1.0608 \times 10^{15}$ | 903.0282 |

**(b)**



**16. (a)** Using the RK4 method we obtain $y(0.1) \approx y_1 = 1.2214$.

**(b)** Using $y^{(5)}(x) = 32e^{2x}$ we see that the local truncation error is

$$y^{(5)}(c)\,\frac{h^5}{120} = 32e^{2c}\,\frac{(0.1)^5}{120} = 0.000002667e^{2c}.$$

Since $e^{2x}$ is an increasing function, $e^{2c} \le e^{2(0.1)} = e^{0.2}$ for $0 \le c \le 0.1$. Thus an upper bound for the local truncation error is $0.000002667e^{0.2} = 0.000003257$.

**(c)** Since $y(0.1) = e^{0.2} = 1.221402758$, the actual error is $y(0.1) - y_1 = 0.000002758$ which is less than 0.000003257.

**(d)** Using the RK4 formula with $h = 0.05$ we obtain $y(0.1) \approx y_2 = 1.221402571$.

**(e)** The error in **(d)** is $1.221402758 - 1.221402571 = 0.000000187$. With global truncation error $O(h^4)$, when the step size is halved we expect the error for $h = 0.05$ to be one-sixteenth the error for $h = 0.1$. Comparing 0.000000187 with 0.000002758 we see that this is the case.

**17. (a)** Using the RK4 method we obtain $y(0.1) \approx y_1 = 0.823416667$.

**(b)** Using $y^{(5)}(x) = -40e^{-2x}$ we see that the local truncation error is

$$40e^{-2c}\,\frac{(0.1)^5}{120} = 0.000003333.$$

Since $e^{-2x}$ is a decreasing function, $e^{-2c} \le e^0 = 1$ for $0 \le c \le 0.1$. Thus an upper bound for the local truncation error is $0.000003333(1) = 0.000003333$.

**(c)** Since $y(0.1) = 0.823413441$, the actual error is $|y(0.1) - y_1| = 0.000003225$, which is less than 0.000003333.

**(d)** Using the RK4 method with $h = 0.05$ we obtain $y(0.1) \approx y_2 = 0.823413627$.

**(e)** The error in **(d)** is $|0.823413441 - 0.823413627| = 0.000000185$. With global truncation error $O(h^4)$, when the step size is halved we expect the error for $h = 0.05$ to be one-sixteenth the error when $h = 0.1$. Comparing 0.000000185 with 0.000003225 we see that this is the case.

**18.** **(a)** Using $y^{(5)} = -1026e^{-3(x-1)}$ we see that the local truncation error is

$$\left| y^{(5)}(c)\,\frac{h^5}{120} \right| = 8.55h^5 e^{-3(c-1)}.$$

**(b)** Since $e^{-3(x-1)}$ is a decreasing function for $1 \le x \le 1.5$, $e^{-3(c-1)} \le e^{-3(1-1)} = 1$ for $1 \le c \le 1.5$ and

$$y^{(5)}(c)\,\frac{h^5}{120} \le 8.55(0.1)^5(1) = 0.0000855.$$

**(c)** Using the RK4 method with $h = 0.1$ we obtain $y(1.5) \approx 2.053338827$. With $h = 0.05$ we obtain $y(1.5) \approx 2.053222989$.

**19.** **(a)** Using $y^{(5)} = 24/(x+1)^5$ we see that the local truncation error is

$$y^{(5)}(c)\,\frac{h^5}{120} = \frac{1}{(c+1)^5}\,\frac{h^5}{5}.$$

**(b)** Since $1/(x+1)^5$ is a decreasing function for $0 \le x \le 0.5$, $1/(c+1)^5 \le 1/(0+1)^5 = 1$ for $0 \le c \le 0.5$ and

$$y^{(5)}(c)\,\frac{h^5}{5} \le (1)\,\frac{(0.1)^5}{5} = 0.000002.$$

**(c)** Using the RK4 method with $h = 0.1$ we obtain $y(0.5) \approx 0.405465168$. With $h = 0.05$ we obtain $y(0.5) \approx 0.405465111$.

**20.** Each step of Euler's method requires only 1 function evaluation, while each step of the improved Euler's method requires 2 function evaluations – once at $(x_n, y_n)$ and again at $(x_{n+1}, y^*_{n+1})$. The second-order Runge-Kutta methods require 2 function evaluations per step, while the RK4 method requires 4 function evaluations per step. To compare the methods we approximate the solution of $y' = (x + y - 1)^2$, $y(0) = 2$, at $x = 0.2$ using $h = 0.1$ for the Runge-Kutta method, $h = 0.05$ for the improved Euler's method, and $h = 0.025$ for Euler's method. For each method a total of 8 function evaluations is required. By comparing with the exact solution we see that the RK4 method appears to still give the most accurate result.

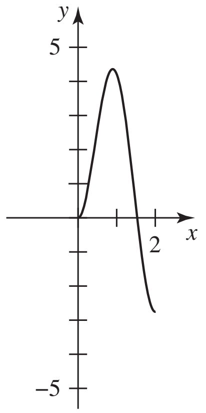
| $x_n$ | Euler $h = 0.025$ | Imp. Euler $h = 0.05$ | RK4 $h = 0.1$ | Actual |
|---|---|---|---|---|
| 0.000 | 2.0000 | 2.0000 | 2.0000 | 2.0000 |
| 0.025 | 2.0250 | | | 2.0263 |
| 0.050 | 2.0526 | 2.0553 | | 2.0554 |
| 0.075 | 2.0830 | | | 2.0875 |
| 0.100 | 2.1165 | 2.1228 | 2.1230 | 2.1230 |
| 0.125 | 2.1535 | | | 2.1624 |
| 0.150 | 2.1943 | 2.2056 | | 2.2061 |
| 0.175 | 2.2395 | | | 2.2546 |
| 0.200 | 2.2895 | 2.3075 | 2.3085 | 2.3085 |

**21. (a)** For $y' + y = 10\sin 3x$ an integrating factor is $e^x$ so that

$$\frac{d}{dx}\left[e^x y\right] = 10e^x \sin 3x$$

$$e^x y = e^x \sin 3x - 3e^x \cos 3x + c$$

$$y = \sin 3x - 3\cos 3x + ce^{-x}.$$



When $x = 0$, $y = 0$, so $0 = -3 + c$ and $c = 3$. The solution is

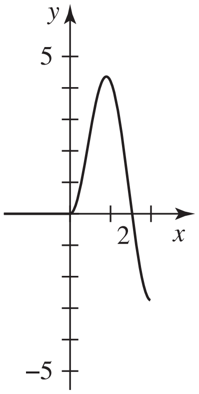
$$y = \sin 3x - 3\cos 3x + 3e^{-x}.$$

Using Newton's method we find that $x = 1.53235$ is the only positive root in $[0, 2]$.

**(b)** Using the RK4 method with $h = 0.1$ we obtain the table of values shown. These values are used to obtain an interpolating function in *Mathematica*. The graph of the interpolating function is shown. Using *Mathematica*'s root finding capability we see that the only positive root in $[0, 2]$ is $x = 1.53236$.

| $x_n$ | $y_n$ |
|---|---|
| 0.0 | 0.0000 |
| 0.1 | 0.1440 |
| 0.2 | 0.5448 |
| 0.3 | 1.1409 |
| 0.4 | 1.8559 |
| 0.5 | 2.6049 |
| 0.6 | 3.3019 |
| 0.7 | 3.8675 |
| 0.8 | 4.2356 |
| 0.9 | 4.3593 |
| 1.0 | 4.2147 |

| $x_n$ | $y_n$ |
|---|---|
| 1.0 | 4.2147 |
| 1.1 | 3.8033 |
| 1.2 | 3.1513 |
| 1.3 | 2.3076 |
| 1.4 | 1.3390 |
| 1.5 | 0.3243 |
| 1.6 | −0.6530 |
| 1.7 | −1.5117 |
| 1.8 | −2.1809 |
| 1.9 | −2.6061 |
| 2.0 | −2.7539 |



## 9.3 Multistep Methods

*In the tables in this section "ABM" stands for Adams-Bashforth-Moulton.*

**1.** Writing the differential equation in the form $y' - y = x - 1$ we see that an integrating factor is $e^{-\int dx} = e^{-x}$, so that

$$\frac{d}{dx}\left[e^{-x}y\right] = (x-1)e^{-x}$$

and

$$y = e^{x}(-xe^{-x} + c) = -x + ce^{x}.$$

From $y(0) = 1$ we find $c = 1$, so the solution of the initial-value problem is $y = -x + e^{x}$. Actual values of the analytic solution above are compared with the approximated values in the table.

| $x_n$ | $y_n$ | Actual | |
|---|---|---|---|
| 0.0 | 1.00000000 | 1.00000000 | init. cond. |
| 0.2 | 1.02140000 | 1.02140276 | RK4 |
| 0.4 | 1.09181796 | 1.09182470 | RK4 |
| 0.6 | 1.22210646 | 1.22211880 | RK4 |
| 0.8 | 1.42552788 | 1.42554093 | ABM |

**2.** The following program is written in *Mathematica*. It uses the Adams-Bashforth-Moulton method to approximate the solution of the initial-value problem $y' = x + y - 1$, $y(0) = 1$, on the interval $[0, 1]$.

```
Clear[f, x, y, h, a, b, y0];
f[x_, y_]:= x + y - 1;          (* define the differential equation *)
h = 0.2;                        (* set the step size *)
a = 0; y0 = 1; b = 1;           (* set the initial condition and the interval *)
f[x, y]                         (* display the DE *)
Clear[k1, k2, k3, k4, x, y, u, v]
x = u[0] = a;
y = v[0] = y0;
n = 0;
While[x < a + 3h,               (* use RK4 to compute the first 3 values after y(0) *)
```

```
        n = n + 1;
        k1 = f[x, y];
        k2 = f[x + h/2, y + h k1/2];
        k3 = f[x + h/2, y + h k2/2];
        k4 = f[x + h, y + h k3];
        x = x + h;
        y = y + (h/6)(k1 + 2k2 + 2k3 + k4);
        u[n] = x;
        v[n] = y];
    While[x ≤ b,                           (* use Adams-Bashforth-Moulton *)
        p3 = f[u[n - 3], v[n - 3]];
        p2 = f[u[n - 2], v[n - 2]];
        p1 = f[u[n - 1], v[n - 1]];
        p0 = f[u[n], v[n]];
        pred = y + (h/24)(55p0 - 59p1 + 37p2 - 9p3);     (* predictor *)
        x = x + h;
        p4 = f[x, pred];
        y = y + (h/24)(9p4 + 19p0 - 5p1 + p2);           (* corrector *)
        n = n + 1;
        u[n] = x;
        v[n] = y]
    TableForm[Prepend[Table[{u[n], v[n]}, {n, 0, (b-a)/h}], {"x(n)", "y(n)"}]];
                                                         (* display the table *)
```

**3.** The first predictor is $y_4^* = 0.73318477$.

| $x_n$ | $y_n$ | |
|---|---|---|
| 0.0 | 1.00000000 | init. cond. |
| 0.2 | 0.73280000 | RK4 |
| 0.4 | 0.64608032 | RK4 |
| 0.6 | 0.65851653 | RK4 |
| 0.8 | 0.72319464 | ABM |

**4.** The first predictor is $y_4^* = 1.21092217$.

| $x_n$ | $y_n$ | |
|---|---|---|
| 0.0 | 2.00000000 | init. cond. |
| 0.2 | 1.41120000 | RK4 |
| 0.4 | 1.14830848 | RK4 |
| 0.6 | 1.10390600 | RK4 |
| 0.8 | 1.20486982 | ABM |

**5.** The first predictor for $h = 0.2$ is $y_4^* = 1.02343488$.

| $x_n$ | $h = 0.2$ | | $h = 0.1$ | |
|---|---|---|---|---|
| 0.0 | 0.00000000 | init. cond. | 0.00000000 | init. cond. |
| 0.1 | | | 0.10033459 | RK4 |
| 0.2 | 0.20270741 | RK4 | 0.20270988 | RK4 |
| 0.3 | | | 0.30933604 | RK4 |
| 0.4 | 0.42278899 | RK4 | 0.42279808 | ABM |
| 0.5 | | | 0.54631491 | ABM |
| 0.6 | 0.68413340 | RK4 | 0.68416105 | ABM |
| 0.7 | | | 0.84233188 | ABM |
| 0.8 | 1.02969040 | ABM | 1.02971420 | ABM |
| 0.9 | | | 1.26028800 | ABM |
| 1.0 | 1.55685960 | ABM | 1.55762558 | ABM |

**6.** The first predictor for $h = 0.2$ is $y_4^* = 3.34828434$.

| $x_n$ | $h = 0.2$ | | $h = 0.1$ | |
|---|---|---|---|---|
| 0.0 | 1.00000000 | init. cond. | 1.00000000 | init. cond. |
| 0.1 | | | 1.21017082 | RK4 |
| 0.2 | 1.44139950 | RK4 | 1.44140511 | RK4 |
| 0.3 | | | 1.69487942 | RK4 |
| 0.4 | 1.97190167 | RK4 | 1.97191536 | ABM |
| 0.5 | | | 2.27400341 | ABM |
| 0.6 | 2.60280694 | RK4 | 2.60283209 | ABM |
| 0.7 | | | 2.96031780 | ABM |
| 0.8 | 3.34860927 | ABM | 3.34863769 | ABM |
| 0.9 | | | 3.77026548 | ABM |
| 1.0 | 4.22797875 | ABM | 4.22801028 | ABM |

**7.** The first predictor for $h = 0.2$ is $y_4^* = 0.13618654$.

| $x_n$ | $h = 0.2$ | | $h = 0.1$ | |
|---|---|---|---|---|
| 0.0 | 0.00000000 | init. cond. | 0.00000000 | init. cond. |
| 0.1 | | | 0.00033209 | RK4 |
| 0.2 | 0.00262739 | RK4 | 0.00262486 | RK4 |
| 0.3 | | | 0.00868768 | RK4 |
| 0.4 | 0.02005764 | RK4 | 0.02004821 | ABM |
| 0.5 | | | 0.03787884 | ABM |
| 0.6 | 0.06296284 | RK4 | 0.06294717 | ABM |
| 0.7 | | | 0.09563116 | ABM |
| 0.8 | 0.13598600 | ABM | 0.13596515 | ABM |
| 0.9 | | | 0.18370712 | ABM |
| 1.0 | 0.23854783 | ABM | 0.23841344 | ABM |

**8.** The first predictor for $h = 0.2$ is $y_4^* = 2.61796154$.

| $x_n$ | $h = 0.2$ | | $h = 0.1$ | |
|---|---|---|---|---|
| 0.0 | 1.00000000 | init. cond. | 1.00000000 | init. cond. |
| 0.1 | | | 1.10793839 | RK4 |
| 0.2 | 1.23369623 | RK4 | 1.23369772 | RK4 |
| 0.3 | | | 1.38068454 | RK4 |
| 0.4 | 1.55308554 | RK4 | 1.55309381 | ABM |
| 0.5 | | | 1.75610064 | ABM |
| 0.6 | 1.99610329 | RK4 | 1.99612995 | ABM |
| 0.7 | | | 2.28119129 | ABM |
| 0.8 | 2.62136177 | ABM | 2.62131818 | ABM |
| 0.9 | | | 3.02914333 | ABM |
| 1.0 | 3.52079042 | ABM | 3.52065536 | ABM |

## 9.4 Higher-Order Equations and Systems

**1.** The substitution $y' = u$ leads to the iteration formulas

$$y_{n+1} = y_n + hu_n, \qquad u_{n+1} = u_n + h(4u_n - 4y_n).$$

The initial conditions are $y_0 = -2$ and $u_0 = 1$. Then

$$\begin{aligned} y_1 &= y_0 + 0.1u_0 = -2 + 0.1(1) = -1.9 \\ u_1 &= u_0 + 0.1(4u_0 - 4y_0) = 1 + 0.1(4 + 8) = 2.2 \\ y_2 &= y_1 + 0.1u_1 = -1.9 + 0.1(2.2) = -1.68. \end{aligned}$$

The general solution of the differential equation is $y = c_1e^{2x} + c_2xe^{2x}$. From the initial conditions we find $c_1 = -2$ and $c_2 = 5$. Thus $y = -2e^{2x} + 5xe^{2x}$ and $y(0.2) \approx 1.4918$.

**2.** The substitution $y' = u$ leads to the iteration formulas

$$y_{n+1} = y_n + hu_n, \qquad u_{n+1} = u_n + h\left(\frac{2}{x}u_n - \frac{2}{x^2}y_n\right).$$

The initial conditions are $y_0 = 4$ and $u_0 = 9$. Then

$$\begin{aligned} y_1 &= y_0 + 0.1u_0 = 4 + 0.1(9) = 4.9 \\ u_1 &= u_0 + 0.1\left(\frac{2}{1}u_0 - \frac{2}{1}y_0\right) = 9 + 0.1[2(9) - 2(4)] = 10 \\ y_2 &= y_1 + 0.1u_1 = 4.9 + 0.1(10) = 5.9. \end{aligned}$$

The general solution of the Cauchy-Euler differential equation is $y = c_1x + c_2x^2$. From the initial conditions we find $c_1 = -1$ and $c_2 = 5$. Thus $y = -x + 5x^2$ and $y(1.2) = 6$.

**3.** The substitution $y' = u$ leads to the system

$$y' = u, \qquad u' = 4u - 4y.$$

Using formula (4), we obtain the table shown.

| $x_n$ | $h = 0.2$ $y_n$ | $h = 0.2$ $u_n$ | $h = 0.1$ $y_n$ | $h = 0.1$ $u_n$ |
|---|---|---|---|---|
| 0.0 | –2.0000 | 1.0000 | –2.0000 | 1.0000 |
| 0.1 | | | –1.8321 | 2.4427 |
| 0.2 | –1.4928 | 4.4731 | –1.4919 | 4.4753 |

**4.** The substitution $y' = u$ leads to the system

$$y' = u, \qquad u' = \frac{2}{x}u - \frac{2}{x^2}y.$$

Using formula (4), we obtain the table shown.

| $x_n$ | $h = 0.2$ $y_n$ | $h = 0.2$ $u_n$ | $h = 0.1$ $y_n$ | $h = 0.1$ $u_n$ |
|---|---|---|---|---|
| 1.0 | 4.0000 | 9.0000 | 4.0000 | 9.0000 |
| 1.1 | | | 4.9500 | 10.0000 |
| 1.2 | 6.0001 | 11.0002 | 6.0000 | 11.0000 |

**5.** The substitution $y' = u$ leads to the system

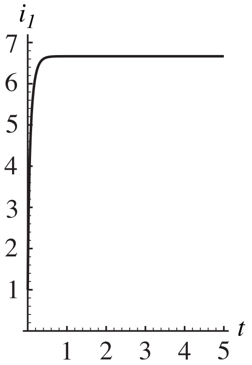
$$y' = u, \qquad u' = 2u - 2y + e^t\cos t.$$

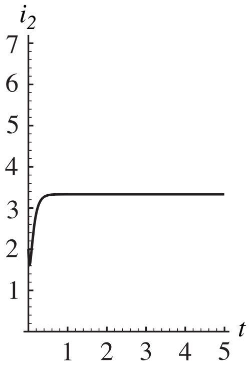
Using formula (4), we obtain the table shown.

| $x_n$ | $h = 0.2$ $y_n$ | $h = 0.2$ $u_n$ | $h = 0.1$ $y_n$ | $h = 0.1$ $u_n$ |
|---|---|---|---|---|
| 0.0 | 1.0000 | 2.0000 | 1.0000 | 2.0000 |
| 0.1 | | | 1.2155 | 2.3150 |
| 0.2 | 1.4640 | 2.6594 | 1.4640 | 2.6594 |

**6.** Using $h = 0.1$, the RK4 method for a system, and a numerical solver, we obtain

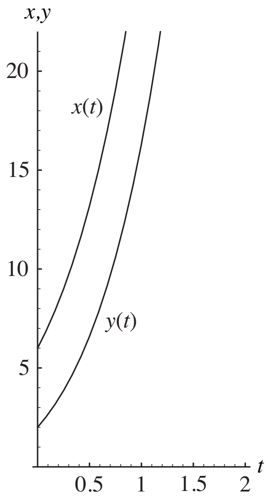
| $t_n$ | $h = 0.2$<br>$i_{1n}$ | $h = 0.2$<br>$i_{3n}$ |
|---|---|---|
| 0.0 | 0.0000 | 0.0000 |
| 0.1 | 2.5000 | 3.7500 |
| 0.2 | 2.8125 | 5.7813 |
| 0.3 | 2.0703 | 7.4023 |
| 0.4 | 0.6104 | 9.1919 |
| 0.5 | –1.5619 | 11.4877 |





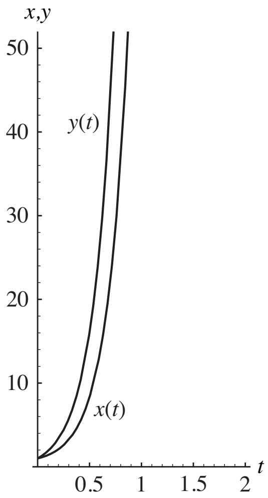
**7.**

| $t_n$ | $h = 0.2$<br>$x_n$ | $h = 0.2$<br>$y_n$ | $h = 0.1$<br>$x_n$ | $h = 0.1$<br>$y_n$ |
|---|---|---|---|---|
| 0.0 | 6.0000 | 2.0000 | 6.0000 | 2.0000 |
| 0.1 | | | 7.0731 | 2.6524 |
| 0.2 | 8.3055 | 3.4199 | 8.3055 | 3.4199 |



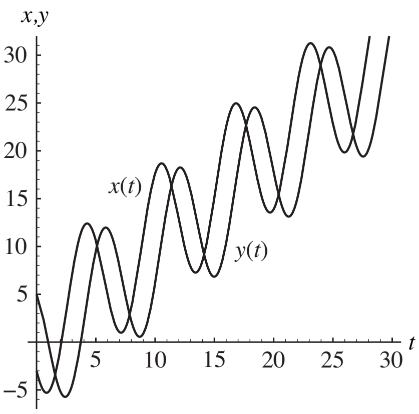
**8.**

| $t_n$ | $h = 0.2$<br>$x_n$ | $h = 0.2$<br>$y_n$ | $h = 0.1$<br>$x_n$ | $h = 0.1$<br>$y_n$ |
|---|---|---|---|---|
| 0.0 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |
| 0.1 | | | 1.4006 | 1.8963 |
| 0.2 | 2.0785 | 3.3382 | 2.0845 | 3.3502 |



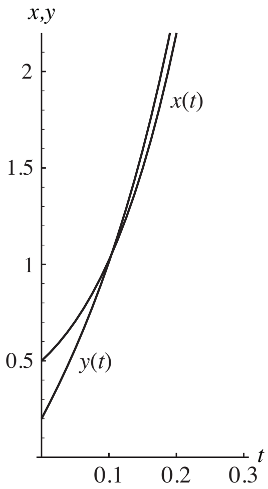
**9.**

| $t_n$ | $h = 0.2$ $x_n$ | $h = 0.2$ $y_n$ | $h = 0.1$ $x_n$ | $h = 0.1$ $y_n$ |
|---|---|---|---|---|
| 0.0 | –3.0000 | 5.0000 | –3.0000 | 5.0000 |
| 0.1 | | | –3.4790 | 4.6707 |
| 0.2 | –3.9123 | 4.2857 | –3.9123 | 4.2857 |



**10.**

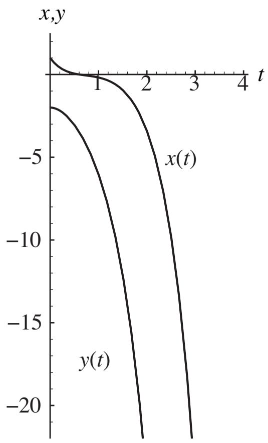
| $t_n$ | $h = 0.2$ $x_n$ | $h = 0.2$ $y_n$ | $h = 0.1$ $x_n$ | $h = 0.1$ $y_n$ |
|---|---|---|---|---|
| 0.0 | 0.5000 | 0.2000 | 0.5000 | 0.2000 |
| 0.1 | | | 1.0207 | 1.0115 |
| 0.2 | 2.1589 | 2.3279 | 2.1904 | 2.3592 |



**11.** Solving for $x'$ and $y'$ we obtain the system

$$x' = -2x + y + 5t$$
$$y' = 2x + y - 2t$$

| $t_n$ | $h = 0.2$ $x_n$ | $h = 0.2$ $y_n$ | $h = 0.1$ $x_n$ | $h = 0.1$ $y_n$ |
|---|---|---|---|---|
| 0.0 | 1.0000 | –2.0000 | 1.0000 | –2.0000 |
| 0.1 | | | 0.6594 | –2.0476 |
| 0.2 | 0.4179 | –2.1824 | 0.4173 | –2.1821 |

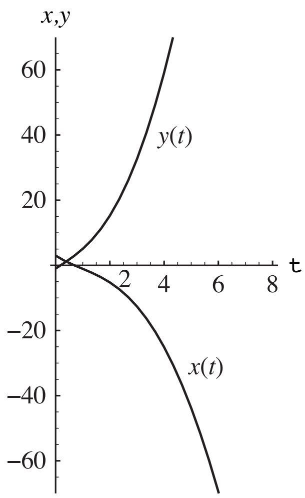


**12.** Solving for $x'$ and $y'$ we obtain the system

$$x' = \frac{1}{2}y - 3t^2 + 2t - 5$$

$$y' = -\frac{1}{2}y + 3t^2 + 2t + 5.$$

| $t_n$ | $h = 0.2$ | $h = 0.2$ | $h = 0.1$ | $h = 0.1$ |
|---|---|---|---|---|
| | $x_n$ | $y_n$ | $x_n$ | $y_n$ |
| 0.0 | 3.0000 | –1.0000 | 3.0000 | –1.0000 |
| 0.1 | | | 2.4727 | –0.4527 |
| 0.2 | 1.9867 | 0.0933 | 1.9867 | 0.0933 |



## 9.5 Second-Order Boundary-Value Problems

**1.** We identify $P(x) = 0$, $Q(x) = 9$, $f(x) = 0$, and $h = (2-0)/4 = 0.5$. Then the finite difference equation is

$$y_{i+1} + 0.25y_i + y_{i-1} = 0.$$

The solution of the corresponding linear system gives

| $x$ | 0.0 | 0.5 | 1.0 | 1.5 | 2.0 |
|---|---|---|---|---|---|
| $y$ | 4.0000 | –5.6774 | –2.5807 | –6.3226 | 1.0000 |

**2.** We identify $P(x) = 0$, $Q(x) = -1$, $f(x) = x^2$, and $h = (1-0)/4 = 0.25$. Then the finite difference equation is

$$y_{i+1} - 2.0625y_i + y_{i-1} = 0.0625x_i^2.$$

The solution of the corresponding linear system gives

| $x$ | 0.00 | 0.25 | 0.50 | 0.75 | 1.00 |
|---|---|---|---|---|---|
| $y$ | 0.0000 | –0.0172 | –0.0316 | –0.0324 | 0.0000 |

**3.** We identify $P(x) = 2$, $Q(x) = 1$, $f(x) = 5x$, and $h = (1-0)/5 = 0.2$. Then the finite difference equation is

$$1.2y_{i+1} - 1.96y_i + 0.8y_{i-1} = 0.04(5x_i).$$

The solution of the corresponding linear system gives

| $x$ | 0.0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|---|---|---|---|---|---|---|
| $y$ | 0.0000 | –0.2259 | –0.3356 | –0.3308 | –0.2167 | 0.0000 |

**4.** We identify $P(x) = -10$, $Q(x) = 25$, $f(x) = 1$, and $h = (1-0)/5 = 0.2$. Then the finite difference equation is

$$-y_i + 2y_{i-1} = 0.04.$$

The solution of the corresponding linear system gives

| $x$ | 0.00 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|---|---|---|---|---|---|---|
| $y$ | 1.0000 | 1.9600 | 3.8800 | 7.7200 | 15.4000 | 0.0000 |

**5.** We identify $P(x) = -4$, $Q(x) = 4$, $f(x) = (1+x)e^{2x}$, and $h = (1-0)/6 = 0.1667$. Then the finite difference equation is

$$0.6667y_{i+1} - 1.8889y_i + 1.3333y_{i-1} = 0.2778(1+x_i)e^{2x_i}.$$

The solution of the corresponding linear system gives

| $x$ | 0.0000 | 0.1667 | 0.3333 | 0.5000 | 0.6667 | 0.8333 | 1.0000 |
|---|---|---|---|---|---|---|---|
| $y$ | 3.0000 | 3.3751 | 3.6306 | 3.6448 | 3.2355 | 2.1411 | 0.0000 |

**6.** We identify $P(x) = 5$, $Q(x) = 0$, $f(x) = 4\sqrt{x}$, and $h = (2-1)/6 = 0.1667$. Then the finite difference equation is

$$1.4167y_{i+1} - 2y_i + 0.5833y_{i-1} = 0.2778(4\sqrt{x_i}\,).$$

The solution of the corresponding linear system gives

| $x$ | 1.0000 | 1.1667 | 1.3333 | 1.5000 | 1.6667 | 1.8333 | 2.0000 |
|---|---|---|---|---|---|---|---|
| $y$ | 1.0000 | −0.5918 | −1.1626 | −1.3070 | −1.2704 | −1.1541 | −1.0000 |

**7.** We identify $P(x) = 3/x$, $Q(x) = 3/x^2$, $f(x) = 0$, and $h = (2-1)/8 = 0.125$. Then the finite difference equation is

$$\left(1 + \frac{0.1875}{x_i}\right)y_{i+1} + \left(-2 + \frac{0.0469}{x_i^2}\right)y_i + \left(1 - \frac{0.1875}{x_i}\right)y_{i-1} = 0.$$

The solution of the corresponding linear system gives

| $x$ | 1.000 | 1.125 | 1.250 | 1.375 | 1.500 | 1.625 | 1.750 | 1.875 | 2.000 |
|---|---|---|---|---|---|---|---|---|---|
| $y$ | 5.0000 | 3.8842 | 2.9640 | 2.2064 | 1.5826 | 1.0681 | 0.6430 | 0.2913 | 0.0000 |

**8.** We identify $P(x) = -1/x$, $Q(x) = x^{-2}$, $f(x) = \ln x/x^2$, and $h = (2-1)/8 = 0.125$. Then the finite difference equation is

$$\left(1 - \frac{0.0625}{x_i}\right)y_{i+1} + \left(-2 + \frac{0.0156}{x_i^2}\right)y_i + \left(1 + \frac{0.0625}{x_i}\right)y_{i-1} = 0.0156\ln x_i.$$

The solution of the corresponding linear system gives

| $x$ | 1.000 | 1.125 | 1.250 | 1.375 | 1.500 | 1.625 | 1.750 | 1.875 | 2.000 |
|---|---|---|---|---|---|---|---|---|---|
| $y$ | 0.0000 | −0.1988 | −0.4168 | −0.6510 | −0.8992 | −1.1594 | −1.4304 | −1.7109 | −2.0000 |

**9.** We identify $P(x) = 1 - x$, $Q(x) = x$, $f(x) = x$, and $h = (1 - 0)/10 = 0.1$. Then the finite difference equation is

$$[1 + 0.05(1 - x_i)]y_{i+1} + [-2 + 0.01x_i]y_i + [1 - 0.05(1 - x_i)]y_{i-1} = 0.01x_i.$$

The solution of the corresponding linear system gives

| $x$ | 0.0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $y$ | 0.0000 | 0.2660 | 0.5097 | 0.7357 | 0.9471 | 1.1465 | 1.3353 | 1.5149 | 1.6855 | 1.8474 | 2.0000 |

**10.** We identify $P(x) = x$, $Q(x) = 1$, $f(x) = x$, and $h = (1 - 0)/10 = 0.1$. Then the finite difference equation is

$$(1 + 0.05x_i)y_{i+1} - 1.99y_i + (1 - 0.05x_i)y_{i-1} = 0.01x_i.$$

The solution of the corresponding linear system gives

| $x$ | 0.0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $y$ | 1.0000 | 0.8929 | 0.7789 | 0.6615 | 0.5440 | 0.4296 | 0.3216 | 0.2225 | 0.1347 | 0.0601 | 0.0000 |

**11.** We identify $P(x) = 0$, $Q(x) = -4$, $f(x) = 0$, and $h = (1 - 0)/8 = 0.125$. Then the finite difference equation is

$$y_{i+1} - 2.0625y_i + y_{i-1} = 0.$$

The solution of the corresponding linear system gives

| $x$ | 0.000 | 0.125 | 0.250 | 0.375 | 0.500 | 0.625 | 0.750 | 0.875 | 1.000 |
|---|---|---|---|---|---|---|---|---|---|
| $y$ | 0.0000 | 0.3492 | 0.7202 | 1.1363 | 1.6233 | 2.2118 | 2.9386 | 3.8490 | 5.0000 |

**12.** We identify $P(r) = 2/r$, $Q(r) = 0$, $f(r) = 0$, and $h = (4 - 1)/6 = 0.5$. Then the finite difference equation is

$$\left(1 + \frac{0.5}{r_i}\right)u_{i+1} - 2u_i + \left(1 - \frac{0.5}{r_i}\right)u_{i-1} = 0.$$

The solution of the corresponding linear system gives

| $r$ | 1.0 | 1.5 | 2.0 | 2.5 | 3.0 | 3.5 | 4.0 |
|---|---|---|---|---|---|---|---|
| $u$ | 50.0000 | 72.2222 | 83.3333 | 90.0000 | 94.4444 | 97.6190 | 100.0000 |

**13.** **(a)** The difference equation

$$\left(1+\frac{h}{2}P_i\right)y_{i+1}+(-2+h^2Q_i)y_i+\left(1-\frac{h}{2}P_i\right)y_{i-1}=h^2f_i$$

is the same as equation (8) in the text. The equations are the same because the derivation was based only on the differential equation, not the boundary conditions. If we allow $i$ to range from 0 to $n-1$ we obtain $n$ equations in the $n+1$ unknowns $y_{-1}$, $y_0$, $y_1$, $\ldots$, $y_{n-1}$. Since $y_n$ is one of the given boundary conditions, it is not an unknown.

**(b)** Identifying $y_0=y(0)$, $y_{-1}=y(0-h)$, and $y_1=y(0+h)$ we have from equation (5) in the text

$$\frac{1}{2h}[y_1-y_{-1}]=y'(0)=1 \quad \text{or} \quad y_1-y_{-1}=2h.$$

The difference equation corresponding to $i=0$,

$$\left(1+\frac{h}{2}P_0\right)y_1+(-2+h^2Q_0)y_0+\left(1-\frac{h}{2}P_0\right)y_{-1}=h^2f_0$$

becomes, with $y_{-1}=y_1-2h$,

$$\left(1+\frac{h}{2}P_0\right)y_1+(-2+h^2Q_0)y_0+\left(1-\frac{h}{2}P_0\right)(y_1-2h)=h^2f_0$$

or

$$2y_1+(-2+h^2Q_0)y_0=h^2f_0+2h-P_0.$$

Alternatively, we may simply add the equation $y_1-y_{-1}=2h$ to the list of $n$ difference equations obtaining $n+1$ equations in the $n+1$ unknowns $y_{-1}$, $y_0$, $y_1$, $\ldots$, $y_{n-1}$.

**(c)** Using $n=5$ we obtain

| $x$ | 0.0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|---|---|---|---|---|---|---|
| $y$ | –2.2755 | –2.0755 | –1.8589 | –1.6126 | –1.3275 | –1.0000 |

**14.** Using $h=0.1$ and, after shooting a few times, $y'(0)=0.43535$ we obtain the following table with the RK4 method.

| $x$ | 0.0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $y$ | 1.00000 | 1.04561 | 1.09492 | 1.14714 | 1.20131 | 1.25633 | 1.31096 | 1.36392 | 1.41388 | 1.45962 | 1.50003 |

## Chapter 9 in Review

**1.**

| $x_n$ | Euler $h = 0.1$ | Euler $h = 0.05$ | Imp. Euler $h = 0.1$ | Imp. Euler $h = 0.05$ | RK4 $h = 0.1$ | RK4 $h = 0.05$ |
|---|---|---|---|---|---|---|
| 1.00 | 2.0000 | 2.0000 | 2.0000 | 2.0000 | 2.0000 | 2.0000 |
| 1.05 | | 2.0693 | | 2.0735 | | 2.0736 |
| 1.10 | 2.1386 | 2.1469 | 2.1549 | 2.1554 | 2.1556 | 2.1556 |
| 1.15 | | 2.2328 | | 2.2459 | | 2.2462 |
| 1.20 | 2.3097 | 2.3272 | 2.3439 | 2.3450 | 2.3454 | 2.3454 |
| 1.25 | | 2.4299 | | 2.4527 | | 2.4532 |
| 1.30 | 2.5136 | 2.5409 | 2.5672 | 2.5689 | 2.5695 | 2.5695 |
| 1.35 | | 2.6604 | | 2.6937 | | 2.6944 |
| 1.40 | 2.7504 | 2.7883 | 2.8246 | 2.8269 | 2.8278 | 2.8278 |
| 1.45 | | 2.9245 | | 2.9686 | | 2.9696 |
| 1.50 | 3.0201 | 3.0690 | 3.1157 | 3.1187 | 3.1197 | 3.1197 |

**2.**

| $x_n$ | Euler $h = 0.1$ | Euler $h = 0.05$ | Imp. Euler $h = 0.1$ | Imp. Euler $h = 0.05$ | RK4 $h = 0.1$ | RK4 $h = 0.05$ |
|---|---|---|---|---|---|---|
| 0.00 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.05 | | 0.0500 | | 0.0501 | | 0.0500 |
| 0.10 | 0.1000 | 0.1001 | 0.1005 | 0.1004 | 0.1003 | 0.1003 |
| 0.15 | | 0.1506 | | 0.1512 | | 0.1511 |
| 0.20 | 0.2010 | 0.2017 | 0.2030 | 0.2027 | 0.2026 | 0.2026 |
| 0.25 | | 0.2537 | | 0.2552 | | 0.2551 |
| 0.30 | 0.3049 | 0.3067 | 0.3092 | 0.3088 | 0.3087 | 0.3087 |
| 0.35 | | 0.3610 | | 0.3638 | | 0.3637 |
| 0.40 | 0.4135 | 0.4167 | 0.4207 | 0.4202 | 0.4201 | 0.4201 |
| 0.45 | | 0.4739 | | 0.4782 | | 0.4781 |
| 0.50 | 0.5279 | 0.5327 | 0.5382 | 0.5378 | 0.5376 | 0.5376 |

**3.**

| $x_n$ | Euler $h = 0.1$ | Euler $h = 0.05$ | Imp. Euler $h = 0.1$ | Imp. Euler $h = 0.05$ | RK4 $h = 0.1$ | RK4 $h = 0.05$ |
|---|---|---|---|---|---|---|
| 0.50 | 0.5000 | 0.5000 | 0.5000 | 0.5000 | 0.5000 | 0.5000 |
| 0.55 | | 0.5500 | | 0.5512 | | 0.5512 |
| 0.60 | 0.6000 | 0.6024 | 0.6048 | 0.6049 | 0.6049 | 0.6049 |
| 0.65 | | 0.6573 | | 0.6609 | | 0.6610 |
| 0.70 | 0.7090 | 0.7144 | 0.7191 | 0.7193 | 0.7194 | 0.7194 |
| 0.75 | | 0.7739 | | 0.7800 | | 0.7801 |
| 0.80 | 0.8283 | 0.8356 | 0.8427 | 0.8430 | 0.8431 | 0.8431 |
| 0.85 | | 0.8996 | | 0.9082 | | 0.9083 |
| 0.90 | 0.9559 | 0.9657 | 0.9752 | 0.9755 | 0.9757 | 0.9757 |
| 0.95 | | 1.0340 | | 1.0451 | | 1.0452 |
| 1.00 | 1.0921 | 1.1044 | 1.1163 | 1.1168 | 1.1169 | 1.1169 |

**4.**

| $x_n$ | Euler $h = 0.1$ | Euler $h = 0.05$ | Imp. Euler $h = 0.1$ | Imp. Euler $h = 0.05$ | RK4 $h = 0.1$ | RK4 $h = 0.05$ |
|---|---|---|---|---|---|---|
| 1.00 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |
| 1.05 | | 1.1000 | | 1.1091 | | 1.1095 |
| 1.10 | 1.2000 | 1.2183 | 1.2380 | 1.2405 | 1.2415 | 1.2415 |
| 1.15 | | 1.3595 | | 1.4010 | | 1.4029 |
| 1.20 | 1.4760 | 1.5300 | 1.5910 | 1.6001 | 1.6036 | 1.6036 |
| 1.25 | | 1.7389 | | 1.8523 | | 1.8586 |
| 1.30 | 1.8710 | 1.9988 | 2.1524 | 2.1799 | 2.1909 | 2.1911 |
| 1.35 | | 2.3284 | | 2.6197 | | 2.6401 |
| 1.40 | 2.4643 | 2.7567 | 3.1458 | 3.2360 | 3.2745 | 3.2755 |
| 1.45 | | 3.3296 | | 4.1528 | | 4.2363 |
| 1.50 | 3.4165 | 4.1253 | 5.2510 | 5.6404 | 5.8338 | 5.8446 |

**5.** Using

$$y_{n+1} = y_n + hu_n, \qquad y_0 = 3$$
$$u_{n+1} = u_n + h(2x_n + 1)y_n, \qquad u_0 = 1$$

we obtain (when $h = 0.2$) $y_1 = y(0.2) = y_0 + hu_0 = 3 + (0.2)1 = 3.2$. When $h = 0.1$ we have

$$y_1 = y_0 + 0.1u_0 = 3 + (0.1)1 = 3.1$$
$$u_1 = u_0 + 0.1(2x_0 + 1)y_0 = 1 + 0.1(1)3 = 1.3$$
$$y_2 = y_1 + 0.1u_1 = 3.1 + 0.1(1.3) = 3.23.$$

**6.** The first predictor is $y_3^* = 1.14822731$.

| $x_n$ | $y_n$ | |
|---|---|---|
| 0.0 | 2.00000000 | init. cond. |
| 0.1 | 1.65620000 | RK4 |
| 0.2 | 1.41097281 | RK4 |
| 0.3 | 1.24645047 | RK4 |
| 0.4 | 1.14796764 | ABM |

**7.** Using $x_0 = 1$, $y_0 = 2$, and $h = 0.1$ we have

$$x_1 = x_0 + h(x_0 + y_0) = 1 + 0.1(1 + 2) = 1.3$$
$$y_1 = y_0 + h(x_0 - y_0) = 2 + 0.1(1 - 2) = 1.9$$

and

$$x_2 = x_1 + h(x_1 + y_1) = 1.3 + 0.1(1.3 + 1.9) = 1.62$$
$$y_2 = y_1 + h(x_1 - y_1) = 1.9 + 0.1(1.3 - 1.9) = 1.84.$$

Thus, $x(0.2) \approx 1.62$ and $y(0.2) \approx 1.84$.

**8.** We identify $P(x) = 0$, $Q(x) = 6.55(1 + x)$, $f(x) = 1$, and $h = (1 - 0)/10 = 0.1$. Then the finite difference equation is

$$y_{i+1} + [-2 + 0.0655(1 + x_i)]y_i + y_{i-1} = 0.001$$

or

$$y_{i+1} + (0.0655x_i - 1.9345)y_i + y_{i-1} = 0.001.$$

The solution of the corresponding linear system gives

| $x$ | 0.0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $y$ | 0.0000 | 4.1987 | 8.1049 | 11.3840 | 13.7038 | 14.7770 | 14.4083 | 12.5396 | 9.2847 | 4.9450 | 0.0000 |

# Chapter 10

## Systems of Nonlinear First-Order Differential Equations

### 10.1 Autonomous Systems

**1.** The corresponding plane autonomous system is

$$x' = y, \quad y' = -9\sin x.$$

If $(x, y)$ is a critical point, $y = 0$ and $-9\sin x = 0$. Therefore $x = \pm n\pi$ and so the critical points are $(\pm n\pi, 0)$ for $n = 0,\ 1,\ 2,\ \ldots$ .

**2.** The corresponding plane autonomous system is

$$x' = y, \quad y' = -2x - y^2.$$

If $(x, y)$ is a critical point, then $y = 0$ and so $-2x - y^2 = -2x = 0$. Therefore $(0, 0)$ is the sole critical point.

**3.** The corresponding plane autonomous system is

$$x' = y, \quad y' = x^2 - y(1 - x^3).$$

If $(x, y)$ is a critical point, $y = 0$ and so $x^2 - y(1 - x^3) = x^2 = 0$. Therefore $(0, 0)$ is the sole critical point.

**4.** The corresponding plane autonomous system is

$$x' = y, \quad y' = -4\frac{x}{1 + x^2} - 2y.$$

If $(x, y)$ is a critical point, $y = 0$ and so $-4x/(1 + x^2) - 2(0) = 0$. Therefore $x = 0$ and so $(0, 0)$ is the sole critical point.

**5.** The corresponding plane autonomous system is

$$x' = y, \quad y' = -x + \epsilon x^3.$$

If $(x, y)$ is a critical point, $y = 0$ and $-x + \epsilon x^3 = 0$. Hence $x(-1 + \epsilon x^2) = 0$ and so $x = 0$, $\sqrt{1/\epsilon}\,$, $-\sqrt{1/\epsilon}\,$. The critical points are $(0, 0)$, $(\sqrt{1/\epsilon}\,, 0)$ and $(-\sqrt{1/\epsilon}\,, 0)$.

**6.** The corresponding plane autonomous system is

$$x' = y, \quad y' = -x + \epsilon x|x|.$$

If $(x, y)$ is a critical point, $y = 0$ and $-x + \epsilon x|x| = x(-1 + \epsilon|x|) = 0$. Hence $x = 0$, $1/\epsilon$, $-1/\epsilon$. The critical points are $(0, 0)$, $(1/\epsilon, 0)$ and $(-1/\epsilon, 0)$.

**7.** From $x+xy = 0$ we have $x(1+y) = 0$. Therefore $x = 0$ or $y = -1$. If $x = 0$, then, substituting into $-y - xy = 0$, we obtain $y = 0$. Likewise, if $y = -1$, $1 + x = 0$ or $x = -1$. We can conclude that $(0, 0)$ and $(-1, -1)$ are critical points of the system.

**8.** From $y^2 - x = 0$ we have $x = y^2$. Substituting into $x^2 - y = 0$, we obtain $y^4 - y = 0$ or $y(y^3 - 1) = 0$. It follows that $y = 0$, $1$ and so $(0, 0)$ and $(1, 1)$ are the critical points of the system.

**9.** From $x - y = 0$ we have $y = x$. Substituting into $3x^2 - 4y = 0$ we obtain $3x^2 - 4x = x(3x - 4) = 0$. It follows that $(0, 0)$ and $(4/3, 4/3)$ are the critical points of the system.

**10.** From $x^3 - y = 0$ we have $y = x^3$. Substituting into $x - y^3 = 0$ we obtain $x - x^9 = 0$ or $x(1 - x^8)$. Therefore $x = 0$, $1$, $-1$ and so the critical points of the system are $(0, 0)$, $(1, 1)$, and $(-1, -1)$.

**11.** From $x(10 - x - \frac{1}{2}y) = 0$ we obtain $x = 0$ or $x + \frac{1}{2}y = 10$. Likewise $y(16 - y - x) = 0$ implies that $y = 0$ or $x + y = 16$. We therefore have four cases. If $x = 0$, $y = 0$ or $y = 16$. If $x + \frac{1}{2}y = 10$, we can conclude that $y(-\frac{1}{2}y + 6) = 0$ and so $y = 0$, $12$. Therefore the critical points of the system are $(0, 0)$, $(0, 16)$, $(10, 0)$, and $(4, 12)$.

**12.** Adding the two equations we obtain $10 - 15y/(y + 5) = 0$. It follows that $y = 10$, and from $-2x + y + 10 = 0$ we can conclude that $x = 10$. Therefore $(10, 10)$ is the sole critical point of the system.

**13.** From $x^2e^y = 0$ we have $x = 0$. Since $e^x - 1 = e^0 - 1 = 0$, the second equation is satisfied for an arbitrary value of $y$. Therefore any point of the form $(0, y)$ is a critical point.

**14.** From $\sin y = 0$ we have $y = \pm n\pi$. From $e^{x-y} = 1$, we can conclude that $x - y = 0$ or $x = y$. The critical points of the system are therefore $(\pm n\pi, \pm n\pi)$ for $n = 0, 1, 2, \ldots$ .

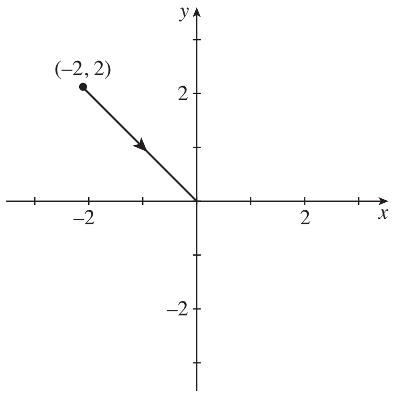
**15.** From $x(1 - x^2 - 3y^2) = 0$ we have $x = 0$ or $x^2 + 3y^2 = 1$. If $x = 0$, then substituting into $y(3 - x^2 - 3y^2)$ gives $y(3 - 3y^2) = 0$. Therefore $y = 0,\ 1,\ -1$. Likewise $x^2 = 1 - 3y^2$ yields $2y = 0$ so that $y = 0$ and $x^2 = 1 - 3(0)^2 = 1$. The critical points of the system are therefore $(0,0)$, $(0,1)$, $(0,-1)$, $(1,0)$, and $(-1,0)$.

**16.** From $-x(4 - y^2) = 0$ we obtain $x = 0$, $y = 2$, or $y = -2$. If $x = 0$, then substituting into $4y(1 - x^2)$ yields $y = 0$. Likewise $y = 2$ gives $8(1 - x^2) = 0$ or $x = 1,\ -1$. Finally $y = -2$ yields $-8(1 - x^2) = 0$ or $x = 1,\ -1$. The critical points of the system are therefore $(0,0)$, $(1,2)$, $(-1,2)$, $(1,-2)$, and $(-1,-2)$.

**17.** **(a)** From Exercises 8.2, Problem 1, $x = c_1e^{5t} - c_2e^{-t}$ and $y = 2c_1e^{5t} + c_2e^{-t}$.

**(b)** From $\mathbf{X}(0) = (-2,2)$ it follows that $c_1 = 0$ and $c_2 = 2$. Therefore $x = -2e^{-t}$ and $y = 2e^{-t}$.

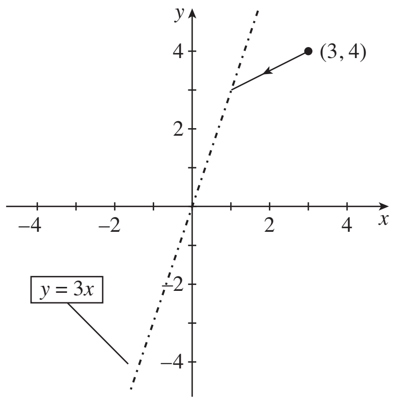
**(c)**



**18.** **(a)** From Exercises 8.2, Problem 6, $x = c_1 + 2c_2e^{-5t}$ and $y = 3c_1 + c_2e^{-5t}$, which is not periodic.

**(b)** From $\mathbf{X}(0) = (3,4)$ it follows that $c_1 = c_2 = 1$. Therefore $x = 1 + 2e^{-5t}$ and $y = 3 + e^{-5t}$ gives $y = \frac{1}{2}(x - 1) + 3$.

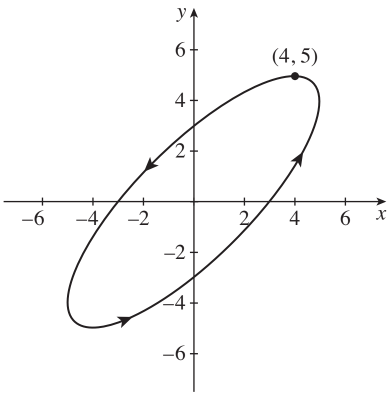
**(c)**



**19.** **(a)** From Exercises 8.2, Problem 34, $x = c_1(4\cos 3t - 3\sin 3t) + c_2(4\sin 3t + 3\cos 3t)$ and $y = c_1(5\cos 3t) + c_2(5\sin 3t)$. All solutions are periodic with $p = 2\pi/3$.

**(b)** From $\mathbf{X}(0) = (4, 5)$ it follows that $c_1 = 1$ and $c_2 = 0$. Therefore $x = 4\cos 3t - 3\sin 3t$ and $y = 5\cos 3t$.

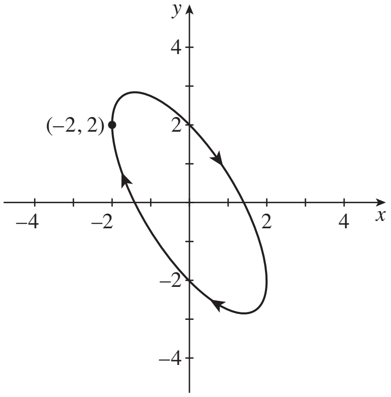
**(c)**



**20.** **(a)** From Exercises 8.2, Problem 35, $x = c_1(\sin t - \cos t) + c_2(-\cos t - \sin t)$ and $y = 2c_1\cos t + 2c_2\sin t$. All solutions are periodic with $p = 2\pi$.

**(b)** From $\mathbf{X}(0) = (-2, 2)$ it follows that $c_1 = c_2 = 1$. Therefore $x = -2\cos t$ and $y = 2\cos t + 2\sin t$.

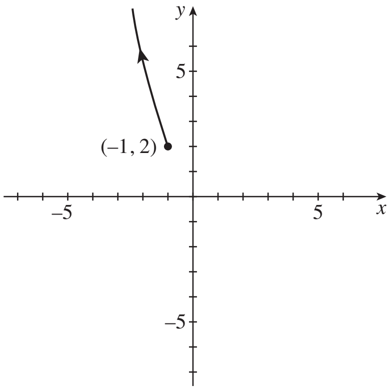
**(c)**



**21.** **(a)** From Exercises 8.2, Problem 38, $x = c_1(\sin t - \cos t)e^{4t} + c_2(-\sin t - \cos t)e^{4t}$ and $y = 2c_1(\cos t)\,e^{4t} + 2c_2(\sin t)\,e^{4t}$. Because of the presence of $e^{4t}$, there are no periodic solutions.

**(b)** From $\mathbf{X}(0) = (-1, 2)$ it follows that $c_1 = 1$ and $c_2 = 0$. Therefore $x = (\sin t - \cos t)e^{4t}$ and $y = 2(\cos t)\,e^{4t}$.

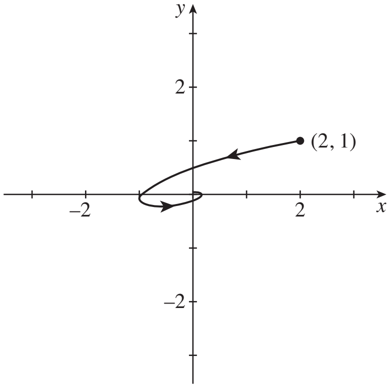
**(c)**



**22.** **(a)** From Exercises 10.2, Problem 40, $x = c_1e^{-t}(2\cos 2t - 2\sin 2t) + c_2e^{-t}(2\cos 2t + 2\sin 2t)$ and $y = c_1e^{-t}\cos 2t + c_2e^{-t}\sin 2t$. Because of the presence of $e^{-t}$, there are no periodic solutions.

**(b)** From $\mathbf{X}(0) = (2, 1)$ it follows that $c_1 = 1$ and $c_2 = 0$. Therefore $x = e^{-t}(2\cos 2t - 2\sin 2t)$ and $y = e^{-t}\cos 2t$.

**(c)**



**23.** Switching to polar coordinates,

$$\frac{dr}{dt} = \frac{1}{r}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right) = \frac{1}{r}(-xy - x^2r^4 + xy - y^2r^4) = -r^5$$

$$\frac{d\theta}{dt} = \frac{1}{r^2}\left(-y\frac{dx}{dt} + x\frac{dy}{dt}\right) = \frac{1}{r^2}(y^2 + xyr^4 + x^2 - xyr^4) = 1\,.$$

If we use separation of variables on $\dfrac{dr}{dt} = -r^5$ we obtain

$$r = \left(\frac{1}{4t + c_1}\right)^{1/4} \quad \text{and} \quad \theta = t + c_2.$$

Since $\mathbf{X}(0) = (4, 0)$, $r = 4$ and $\theta = 0$ when $t = 0$. It follows that $c_2 = 0$ and $c_1 = \frac{1}{256}$. The final solution can be written as

$$r = \frac{4}{\sqrt[4]{1024t + 1}}\,, \qquad \theta = t$$

and so the solution spirals toward the origin as $t$ increases.

**24.** Switching to polar coordinates,

$$\frac{dr}{dt} = \frac{1}{r}\left(x\,\frac{dx}{dt} + y\,\frac{dy}{dt}\right) = \frac{1}{r}(xy + x^2r^2 - xy + y^2r^2) = r^3$$

$$\frac{d\theta}{dt} = \frac{1}{r^2}\left(-y\,\frac{dx}{dt} + x\,\frac{dy}{dt}\right) = \frac{1}{r^2}(-y^2 - xyr^2 - x^2 + xyr^2) = -1\,.$$

If we use separation of variables, it follows that

$$r = \frac{1}{\sqrt{-2t + c_1}} \quad \text{and} \quad \theta = -t + c_2.$$

Since $\mathbf{X}(0) = (4, 0)$, $r = 4$ and $\theta = 0$ when $t = 0$. It follows that $c_2 = 0$ and $c_1 = \frac{1}{16}$. The final solution can be written as

$$r = \frac{4}{\sqrt{1 - 32t}}\,, \qquad \theta = -t.$$

Note that $r \to \infty$ as $t \to \left(\frac{1}{32}\right)$. Because $0 \le t \le \frac{1}{32}$, the curve is not a spiral.

**25.** Switching to polar coordinates,

$$\frac{dr}{dt} = \frac{1}{r}\left(x\,\frac{dx}{dt} + y\,\frac{dy}{dt}\right) = \frac{1}{r}[-xy + x^2(1 - r^2) + xy + y^2(1 - r^2)] = r(1 - r^2)$$

$$\frac{d\theta}{dt} = \frac{1}{r^2}\left(-y\,\frac{dx}{dt} + x\,\frac{dy}{dt}\right) = \frac{1}{r^2}[y^2 - xy(1 - r^2) + x^2 + xy(1 - r^2)] = 1.$$

Now $dr/dt = r - r^3$ or $(dr/dt) - r = -r^3$ is a Bernoulli differential equation. Following the procedure in Section 2.5 of the text, we let $w = r^{-2}$ so that $w' = -2r^{-3}\,(dr/dt)$. Therefore $w' + 2w = 2$, a linear first order differential equation. It follows that $w = 1 + c_1e^{-2t}$ and so $r^2 = 1/(1 + c_1e^{-2t})$. The general solution can be written as

$$r = \frac{1}{\sqrt{1 + c_1e^{-2t}}}\,, \qquad \theta = t + c_2.$$

If $\mathbf{X}(0) = (1, 0)$, $r = 1$ and $\theta = 0$ when $t = 0$. Therefore $c_1 = 0 = c_2$ and so $x = r\cos t = \cos t$ and $y = r\sin t = \sin t$. This solution generates the circle $r = 1$. If $\mathbf{X}(0) = (2, 0)$, $r = 2$ and $\theta = 0$ when $t = 0$. Therefore $c_1 = -3/4$, $c_2 = 0$ and so

$$r = \frac{1}{\sqrt{1 - \frac{3}{4}\,e^{-2t}}}\,, \qquad \theta = t.$$

This solution spirals toward the circle $r = 1$ as $t$ increases.

**26.** Switching to polar coordinates,

$$\frac{dr}{dt} = \frac{1}{r}\left(x\,\frac{dx}{dt} + y\,\frac{dy}{dt}\right) = \frac{1}{r}\left[xy - \frac{x^2}{r}(4 - r^2) - xy - \frac{y^2}{r}(4 - r^2)\right] = r^2 - 4$$

$$\frac{d\theta}{dt} = \frac{1}{r^2}\left(-y\,\frac{dx}{dt} + x\,\frac{dy}{dt}\right) = \frac{1}{r^2}\left[-y^2 + \frac{xy}{r}(4 - r^2) - x^2 - \frac{xy}{r}(4 - r^2)\right] = -1.$$

From Example 3, Section 2.2,

$$r = 2\,\frac{1 + c_1 e^{4t}}{1 - c_1 e^{4t}} \quad \text{and} \quad \theta = -t + c_2.$$

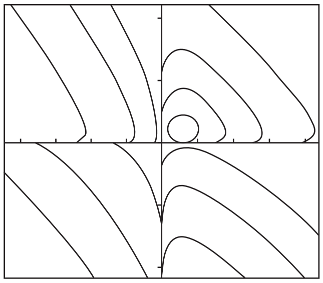
If $\mathbf{X}(0) = (1, 0)$, $r = 1$ and $\theta = 0$ when $t = 0$. It follows that $c_2 = 0$ and $c_1 = -\frac{1}{3}$. Therefore

$$r = 2\,\frac{1 - \frac{1}{3}e^{4t}}{1 + \frac{1}{3}e^{4t}} \quad \text{and} \quad \theta = -t.$$

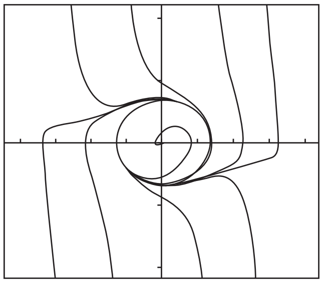
Note that $r = 0$ when $e^{4t} = 3$ or $t = (\ln 3)/4$ and $r \to -2$ as $t \to \infty$. The solution therefore approaches the circle $r = 2$. If $\mathbf{X}(0) = (2, 0)$, it follows that $c_1 = c_2 = 0$. Therefore $r = 2$ and $\theta = -t$ so that the solution generates the circle $r = 2$ traversed in the clockwise direction. Note also that the original system is not defined at $(0, 0)$ but the corresponding polar system is defined for $r = 0$. If the Runge-Kutta method is applied to the original system, the solution corresponding to $\mathbf{X}(0) = (1, 0)$ will stall at the origin.

**27.** The system has no critical points, so there are no periodic solutions.

**28.** From $x(6y - 1) = 0$ and $y(2 - 8x) = 0$ we see that $(0, 0)$ and $(1/4, 1/6)$ are critical points. From the graph we see that there are periodic solutions around $(1/4, 1/6)$.



**29.** The only critical point is $(0, 0)$. There appears to be a single periodic solution around $(0, 0)$.

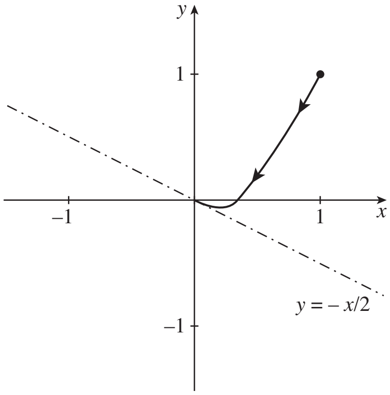


**30.** The system has no critical points, so there are no periodic solutions.

## 10.2 Stability of Linear Systems

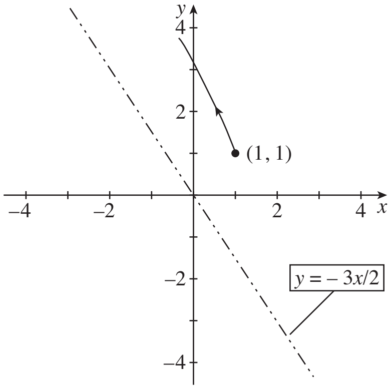
**1. (a)** If $\mathbf{X}(0) = \mathbf{X}_0$ lies on the line $y = 2x$, then $\mathbf{X}(t)$ approaches $(0, 0)$ along this line. For all other initial conditions, $\mathbf{X}(t)$ approaches $(0, 0)$ from the direction determined by the line $y = -x/2$.

**(b)**



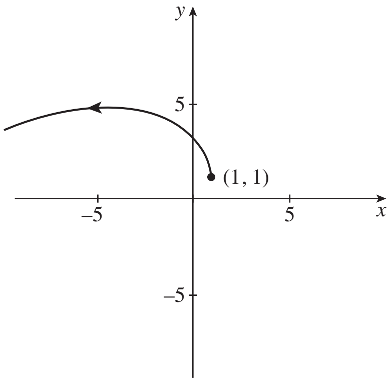
**2. (a)** If $\mathbf{X}(0) = \mathbf{X}_0$ lies on the line $y = -x$, then $\mathbf{X}(t)$ becomes unbounded along this line. For all other initial conditions, $\mathbf{X}(t)$ becomes unbounded and $y = -3x/2$ serves as an asymptote.

**(b)**



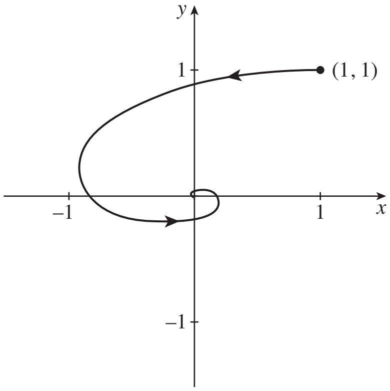
**3. (a)** All solutions are unstable spirals which become unbounded as $t$ increases.

**(b)**



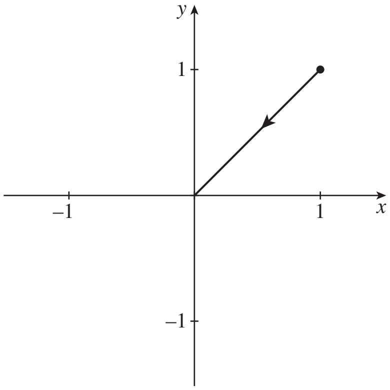
**4.** **(a)** All solutions are spirals which approach the origin.

**(b)**



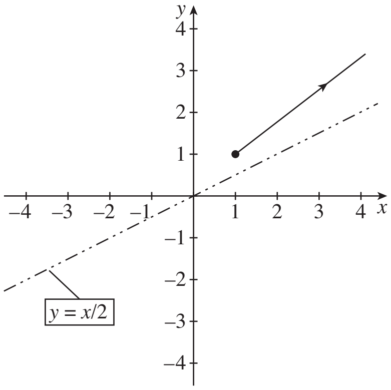
**5.** **(a)** All solutions approach $(0, 0)$ from the direction specified by the line $y = x$.

**(b)**



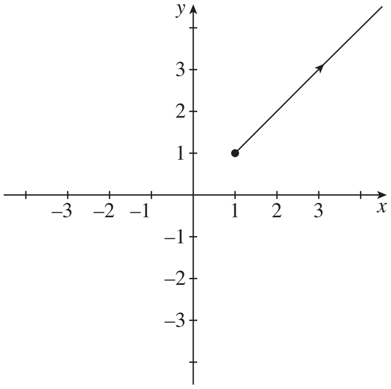
**6.** **(a)** All solutions become unbounded and $y = x/2$ serves as the asymptote.

**(b)**



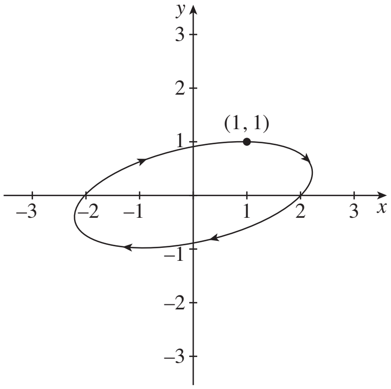
**7. (a)** If $\mathbf{X}(0) = \mathbf{X}_0$ lies on the line $y = 3x$, then $\mathbf{X}(t)$ approaches $(0, 0)$ along this line. For all other initial conditions, $\mathbf{X}(t)$ becomes unbounded and $y = x$ serves as the asymptote.

**(b)**



**8. (a)** The solutions are ellipses which encircle the origin.

**(b)**



**9.** Since $\Delta = -41 < 0$, we can conclude from Figure 10.2.12 that $(0, 0)$ is a saddle point.

**10.** Since $\Delta = 29$ and $\tau = -12$, $\tau^2 - 4\Delta > 0$ and so from Figure 10.2.12, $(0, 0)$ is a stable node.

**11.** Since $\Delta = -19 < 0$, we can conclude from Figure 10.2.12 that $(0, 0)$ is a saddle point.

**12.** Since $\Delta = 1$ and $\tau = -1$, $\tau^2 - 4\Delta = -3$ and so from Figure 10.2.12, $(0, 0)$ is a stable spiral point.

**13.** Since $\Delta = 1$ and $\tau = -2$, $\tau^2 - 4\Delta = 0$ and so from Figure 10.2.12, $(0, 0)$ is a degenerate stable node.

**14.** Since $\Delta = 1$ and $\tau = 2$, $\tau^2 - 4\Delta = 0$ and so from Figure 10.2.12, $(0, 0)$ is a degenerate unstable node.

**15.** Since $\Delta = 0.01$ and $\tau = -0.03$, $\tau^2 - 4\Delta < 0$ and so from Figure 10.2.12, $(0, 0)$ is a stable spiral point.

**16.** Since $\Delta = 0.0016$ and $\tau = 0.08$, $\tau^2 - 4\Delta = 0$ and so from Figure 10.2.12, $(0, 0)$ is a degenerate unstable node.

**17.** $\Delta = 1-\mu^2$, $\tau = 0$, and so we need $\Delta = 1-\mu^2 > 0$ for $(0,0)$ to be a center. Therefore $|\mu| < 1$.

**18.** Note that $\Delta = 1$ and $\tau = \mu$. Therefore we need both $\tau = \mu < 0$ and $\tau^2 - 4\Delta = \mu^2 - 4 < 0$ for $(0,0)$ to be a stable spiral point. These two conditions can be written as $-2 < \mu < 0$.

**19.** Note that $\Delta = \mu + 1$ and $\tau = \mu + 1$ and so $\tau^2 - 4\Delta = (\mu+1)^2 - 4(\mu+1) = (\mu+1)(\mu-3)$. It follows that $\tau^2 - 4\Delta < 0$ if and only if $-1 < \mu < 3$. We can conclude that $(0,0)$ will be a saddle point when $\mu < -1$. Likewise $(0,0)$ will be an unstable spiral point when $\tau = \mu + 1 > 0$ and $\tau^2 - 4\Delta < 0$. This condition reduces to $-1 < \mu < 3$.

**20.** $\tau = 2\alpha$, $\Delta = \alpha^2 + \beta^2 > 0$, and $\tau^2 - 4\Delta = -4\beta < 0$. If $\alpha < 0$, $(0,0)$ is a stable spiral point. If $\alpha > 0$, $(0,0)$ is an unstable spiral point. Therefore $(0,0)$ cannot be a node or saddle point.

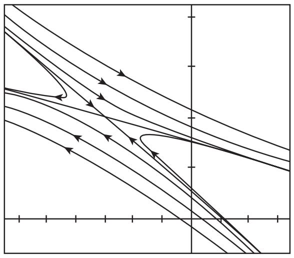
**21.** $\mathbf{AX}_1 + \mathbf{F} = \mathbf{0}$ implies that $\mathbf{AX}_1 = -\mathbf{F}$ or $\mathbf{X}_1 = -\mathbf{A}^{-1}\mathbf{F}$. Since $\mathbf{X}_p(t) = -\mathbf{A}^{-1}\mathbf{F}$ is a particular solution, it follows from Theorem 10.1.6 that $\mathbf{X}(t) = \mathbf{X}_c(t) + \mathbf{X}_1$ is the general solution to $\mathbf{X}' = \mathbf{AX} + \mathbf{F}$. If $\tau < 0$ and $\Delta > 0$ then $\mathbf{X}_c(t)$ approaches $(0,0)$ by Theorem 10.2.1(a). It follows that $\mathbf{X}(t)$ approaches $\mathbf{X}_1$ as $t \to \infty$.

**22.** If $bc < 1$, $\Delta = ad\hat{x}\hat{y}(1-bc) > 0$ and $\tau^2 - 4\Delta = (a\hat{x} - d\hat{y})^2 + 4abcd\hat{x}\hat{y} > 0$. Therefore $(0,0)$ is a stable node.

**23.** **(a)** The critical point is $\mathbf{X}_1 = (-3, 4)$.

**(b)** From the graph, $\mathbf{X}_1$ appears to be an unstable node or a saddle point.

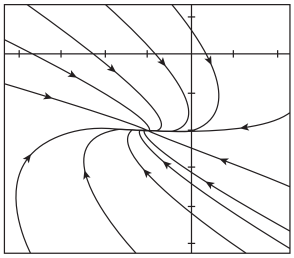
**(c)** Since $\Delta = -1$, $(0,0)$ is a saddle point.



**24.** **(a)** The critical point is $\mathbf{X}_1 = (-1, -2)$.

**(b)** From the graph, $\mathbf{X}_1$ appears to be a stable node or a degenerate stable node.

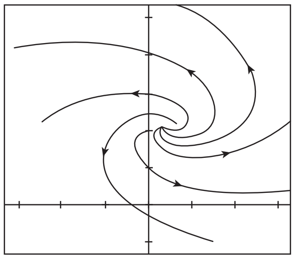
**(c)** Since $\tau = -16$, $\Delta = 64$, and $\tau^2 - 4\Delta = 0$, $(0,0)$ is a degenerate stable node.



**25.** **(a)** The critical point is $\mathbf{X}_1 = (0.5, 2)$.

**(b)** From the graph, $\mathbf{X}_1$ appears to be an unstable spiral point.

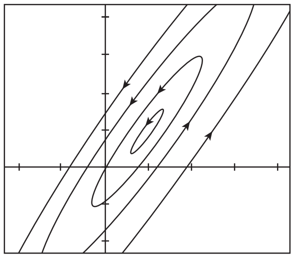
**(c)** Since $\tau = 0.2$, $\Delta = 0.03$, and $\tau 2 - 4\Delta = -0.08$, $(0,0)$ is an unstable spiral point.



**26.** **(a)** The critical point is $\mathbf{X}_1 = (1, 1)$.

**(b)** From the graph, $\mathbf{X}_1$ appears to be a center.

**(c)** Since $\tau = 0$ and $\Delta = 1$, $(0, 0)$ is a center.



## 10.3 Linearization and Local Stability

**1.** Switching to polar coordinates,

$$\frac{dr}{dt} = \frac{1}{r}\left(x\,\frac{dx}{dt} + y\,\frac{dy}{dt}\right) = \frac{1}{r}(\alpha x^2 - \beta xy + xy^2 + \beta xy + \alpha y^2 - xy^2) = \frac{1}{r}\,\alpha r^2 = \alpha r.$$

Therefore $r = ce^{\alpha t}$ and so $r \to 0$ if and only if $\alpha < 0$.

**2.** The differential equation $dr/dt = \alpha r(5 - r)$ is a logistic differential equation. [See Section 3.2, (4) and (5).] It follows that

$$r = \frac{5}{1 + c_1 e^{-5\alpha t}} \quad \text{and} \quad \theta = -t + c_2.$$

If $\alpha > 0$, $r \to 5$ as $t \to +\infty$ and so the critical point $(0, 0)$ is unstable. If $\alpha < 0$, $r \to 0$ as $t \to +\infty$ and so $(0, 0)$ is asymptotically stable.

**3.** The critical points are $x = 0$ and $x = n+1$. Since $g'(x) = k(n+1) - 2kx$, $g'(0) = k(n+1) > 0$ and $g'(n+1) = -k(n+1) < 0$. Therefore $x = 0$ is unstable while $x = n+1$ is asymptotically stable. See Theorem 10.2.

**4.** Note that $x = k$ is the only critical point since $\ln(x/k)$ is not defined at $x = 0$. Since $g'(x) = -k - k\ln(x/k)$, $g'(k) = -k < 0$. Therefore $x = k$ is an asymptotically stable critical point by Theorem 10.2.

**5.** The only critical point is $T = T_0$. Since $g'(T) = k$, $g'(T_0) = k > 0$. Therefore $T = T_0$ is unstable by Theorem 10.2.

**6.** The only critical point is $v = mg/k$. Now $g(v) = g - (k/m)v$ and so $g'(v) = -k/m < 0$. Therefore $v = mg/k$ is an asymptotically stable critical point by Theorem 10.2.

**7.** Critical points occur at $x = \alpha$, $\beta$. Since $g'(x) = k(-\alpha - \beta + 2x)$, $g'(\alpha) = k(\alpha - \beta)$ and $g'(\beta) = k(\beta - \alpha)$. Since $\alpha > \beta$, $g'(\alpha) > 0$ and so $x = \alpha$ is unstable. Likewise $x = \beta$ is asymptotically stable.

**8.** Critical points occur at $x = \alpha$, $\beta$, $\gamma$. Since

$$g'(x) = k(\alpha - x)(-\beta - \gamma - 2x) + k(\beta - x)(\gamma - x)(-1),$$

$g'(\alpha) = -k(\beta - \alpha)(\gamma - \alpha) < 0$ since $\alpha > \beta > \gamma$. Therefore $x = \alpha$ is asymptotically stable. Similarly $g'(\beta) > 0$ and $g'(\gamma) < 0$. Therefore $x = \beta$ is unstable while $x = \gamma$ is asymptotically stable.

**9.** Critical points occur at $P = a/b$, $c$ but not at $P = 0$. Since $g'(P) = (a - bP) + (P - c)(-b)$,

$$g'(a/b) = (a/b - c)(-b) = -a + bc \quad \text{and} \quad g'(c) = a - bc.$$

Since $a < bc$, $-a + bc > 0$ and $a - bc < 0$. Therefore $P = a/b$ is unstable while $P = c$ is asymptotically stable.

**10.** Since $A > 0$, the only critical point is $A = K^2$. Since $g'(A) = \frac{1}{2}kKA^{-1/2} - k$, $g'(K^2) = -k/2 < 0$. Therefore $A = K^2$ is asymptotically stable.

**11.** The sole critical point is $(1/2, 1)$ and

$$\mathbf{g}'(\mathbf{X}) = \begin{pmatrix} -2y & -2x \\ 2y & 2x - 1 \end{pmatrix}.$$

Computing $\mathbf{g}'((1/2, 1))$ we find that $\tau = -2$ and $\Delta = 2$ so that $\tau^2 - 4\Delta = -4 < 0$. Therefore $(1/2, 1)$ is a stable spiral point.

**12.** Critical points are $(1, 0)$ and $(-1, 0)$, and

$$\mathbf{g}'(\mathbf{X}) = \begin{pmatrix} 2x & -2y \\ 0 & 2 \end{pmatrix}.$$

At $\mathbf{X} = (1, 0)$, $\tau = 4$, $\Delta = 4$, and so $\tau^2 - 4\Delta = 0$. We can conclude that $(1, 0)$ is unstable but we are unable to classify this critical point any further. At $\mathbf{X} = (-1, 0)$, $\Delta = -4 < 0$ and so $(-1, 0)$ is a saddle point.

**13.** $y' = 2xy - y = y(2x - 1)$. Therefore if $(x, y)$ is a critical point, either $x = 1/2$ or $y = 0$. The case $x = 1/2$ and $y - x^2 + 2 = 0$ implies that $(x, y) = (1/2, -7/4)$. The case $y = 0$ leads to the critical points $(\sqrt{2}, 0)$ and $(-\sqrt{2}, 0)$. We next use the Jacobian matrix

$$\mathbf{g}'(\mathbf{X}) = \begin{pmatrix} -2x & 1 \\ 2y & 2x - 1 \end{pmatrix}$$

to classify these three critical points. For $\mathbf{X} = (\sqrt{2}, 0)$ or $(-\sqrt{2}, 0)$, $\tau = -1$ and $\Delta < 0$. Therefore both critical points are saddle points. For $\mathbf{X} = (1/2, -7/4)$, $\tau = -1$, $\Delta = 7/2$ and so $\tau^2 - 4\Delta = -13 < 0$. Therefore $(1/2, -7/4)$ is a stable spiral point.

**14.** $y' = -y + xy = y(-1 + x)$. Therefore if $(x, y)$ is a critical point, either $y = 0$ or $x = 1$. The case $y = 0$ and $2x - y^2 = 0$ implies that $(x, y) = (0, 0)$. The case $x = 1$ leads to the critical points $(1, \sqrt{2})$ and $(1, -\sqrt{2})$. We next use the Jacobian matrix

$$\mathbf{g}'(\mathbf{X}) = \begin{pmatrix} 2 & -2y \\ y & x - 1 \end{pmatrix}$$

to classify these critical points. For $\mathbf{X} = (0,0)$, $\Delta = -2 < 0$ and so $(0,0)$ is a saddle point. For either $(1, \sqrt{2}\,)$ or $(1, -\sqrt{2}\,)$, $\tau = 2$, $\Delta = 4$, and so $\tau^2 - 4\Delta = -12$. Therefore $(1, \sqrt{2}\,)$ and $(1, -\sqrt{2}\,)$ are unstable spiral points.

**15.** Since $x^2 - y^2 = 0$, $y^2 = x^2$ and so $x^2 - 3x + 2 = (x-1)(x-2) = 0$. It follows that the critical points are $(1,1)$, $(1,-1)$, $(2,2)$, and $(2,-2)$. We next use the Jacobian

$$\mathbf{g}'(\mathbf{X}) = \begin{pmatrix} -3 & 2y \\ 2x & -2y \end{pmatrix}$$

to classify these four critical points. For $\mathbf{X} = (1,1)$, $\tau = -5$, $\Delta = 2$, and so $\tau^2 - 4\Delta = 17 > 0$. Therefore $(1,1)$ is a stable node. For $\mathbf{X} = (1,-1)$, $\Delta = -2 < 0$ and so $(1,-1)$ is a saddle point. For $\mathbf{X} = (2,2)$, $\Delta = -4 < 0$ and so we have another saddle point. Finally, if $\mathbf{X} = (2,-2)$, $\tau = 1$, $\Delta = 4$, and so $\tau^2 - 4\Delta = -15 < 0$. Therefore $(2,-2)$ is an unstable spiral point.

**16.** From $y^2 - x^2 = 0$, $y = x$ or $y = -x$. The case $y = x$ leads to $(4,4)$ and $(-1,1)$ but the case $y = -x$ leads to $x^2 - 3x + 4 = 0$ which has no real solutions. Therefore $(4,4)$ and $(-1,1)$ are the only critical points. We next use the Jacobian matrix

$$\mathbf{g}'(\mathbf{X}) = \begin{pmatrix} y & x-3 \\ -2x & 2y \end{pmatrix}$$

to classify these two critical points. For $\mathbf{X} = (4,4)$, $\tau = 12$, $\Delta = 40$, and so $\tau^2 - 4\Delta < 0$. Therefore $(4,4)$ is an unstable spiral point. For $\mathbf{X} = (-1,1)$, $\tau = -3$, $\Delta = 10$, and so $x^2 - 4\Delta < 0$. It follows that $(-1,-1)$ is a stable spiral point.

**17.** Since $x' = -2xy = 0$, either $x = 0$ or $y = 0$. If $x = 0$, $y(1 - y^2) = 0$ and so $(0,0)$, $(0,1)$, and $(0,-1)$ are critical points. The case $y = 0$ leads to $x = 0$. We next use the Jacobian matrix

$$\mathbf{g}'(\mathbf{X}) = \begin{pmatrix} -2y & -2x \\ -1+y & 1+x-3y^2 \end{pmatrix}$$

to classify these three critical points. For $\mathbf{X} = (0,0)$, $\tau = 1$ and $\Delta = 0$ and so the test is inconclusive. For $\mathbf{X} = (0,1)$, $\tau = -4$, $\Delta = 4$ and so $\tau^2 - 4\Delta = 0$. We can conclude that $(0,1)$ is a stable critical point but we are unable to classify this critical point further in this borderline case. For $\mathbf{X} = (0,-1)$, $\Delta = -4 < 0$ and so $(0,-1)$ is a saddle point.

**18.** We found that $(0,0)$, $(0,1)$, $(0,-1)$, $(1,0)$ and $(-1,0)$ were the critical points in Problem 15, Section 10.1. The Jacobian is

$$\mathbf{g}'(\mathbf{X}) = \begin{pmatrix} 1 - 3x^2 - 3y^2 & -6xy \\ -2xy & 3 - x^2 - 9y^2 \end{pmatrix}.$$

For $\mathbf{X} = (0,0)$, $\tau = 4$, $\Delta = 3$ and so $\tau^2 - 4\Delta = 4 > 0$. Therefore $(0,0)$ is an unstable node. Both $(0,1)$ and $(0,-1)$ give $\tau = -8$, $\Delta = 12$, and $\tau^2 - 4\Delta = 16 > 0$. These two critical points are therefore stable nodes. For $\mathbf{X} = (1,0)$ or $(-1,0)$, $\Delta = -4 < 0$ and so saddle points occur.

**19.** We found the critical points $(0,0)$, $(10,0)$, $(0,16)$ and $(4,12)$ in Problem 11, Section 10.1. Since the Jacobian is

$$\mathbf{g}'(\mathbf{X}) = \begin{pmatrix} 10-2x-\frac{1}{2}y & -\frac{1}{2}x \\ -y & 16-2y-x \end{pmatrix}$$

we can classify the critical points as follows:

| $\mathbf{X}$ | $\tau$ | $\Delta$ | $\tau^2-4\Delta$ | Conclusion |
|---|---|---|---|---|
| $(0,0)$ | $26$ | $160$ | $36$ | unstable node |
| $(10,0)$ | $-4$ | $-60$ | – | saddle point |
| $(0,16)$ | $-14$ | $-32$ | – | saddle point |
| $(4,12)$ | $-16$ | $24$ | $160$ | stable node |

**20.** We found the sole critical point $(10,10)$ in Problem 12, Section 10.1. The Jacobian is

$$\mathbf{g}'(\mathbf{X}) = \begin{pmatrix} -2 & 1 \\ 2 & -1-\dfrac{15}{(y+5)^2} \end{pmatrix},$$

$\mathbf{g}'((10,10))$ has trace $\tau=-46/15$, $\Delta=2/15$, and $\tau^2-4\Delta>0$. Therefore $(0,0)$ is a stable node.

**21.** The corresponding plane autonomous system is

$$\theta'=y, \quad y'=(\cos\theta-\frac{1}{2})\sin\theta.$$

Since $|\theta|<\pi$, it follows that critical points are $(0,0)$, $(\pi/3,0)$ and $(-\pi/3,0)$. The Jacobian matrix is

$$\mathbf{g}'(\mathbf{X}) = \begin{pmatrix} 0 & 1 \\ \cos 2\theta-\frac{1}{2}\cos\theta & 0 \end{pmatrix}$$

and so at $(0,0)$, $\tau=0$ and $\Delta=-1/2$. Therefore $(0,0)$ is a saddle point. For $\mathbf{X}=(\pm\pi/3,0)$, $\tau=0$ and $\Delta=3/4$. It is not possible to classify either critical point in this borderline case.

**22.** The corresponding plane autonomous system is

$$x'=y, \quad y'=-x+\left(\frac{1}{2}-3y^2\right)y-x^2.$$

If $(x,y)$ is a critical point, $y=0$ and so $-x-x^2=-x(1+x)=0$. Therefore $(0,0)$ and $(-1,0)$ are the only two critical points. We next use the Jacobian matrix

$$\mathbf{g}'(\mathbf{X}) = \begin{pmatrix} 0 & 1 \\ -1-2x & \frac{1}{2}-9y^2 \end{pmatrix}$$

to classify these critical points. For $\mathbf{X}=(0,0)$, $\tau=1/2$, $\Delta=1$, and $\tau^2-4\Delta<0$. Therefore $(0,0)$ is an unstable spiral point. For $\mathbf{X}=(-1,0)$, $\tau=1/2$, $\Delta=-1$ and so $(-1,0)$ is a saddle point.

**23.** The corresponding plane autonomous system is

$$x' = y, \quad y' = x^2 - y(1 - x^3)$$

and the only critical point is $(0,0)$. Since the Jacobian matrix is

$$\mathbf{g}'(\mathbf{X}) = \begin{pmatrix} 0 & 1 \\ 2x + 3x^2y & x^3 - 1 \end{pmatrix},$$

$\tau = -1$ and $\Delta = 0$, and we are unable to classify the critical point in this borderline case.

**24.** The corresponding plane autonomous system is

$$x' = y, \quad y' = -\frac{4x}{1 + x^2} - 2y$$

and the only critical point is $(0,0)$. Since the Jacobian matrix is

$$\mathbf{g}'(\mathbf{X}) = \begin{pmatrix} 0 & 1 \\ -4\,\dfrac{1 - x^2}{(1 + x^2)^2} & -2 \end{pmatrix},$$

$\tau = -2$, $\Delta = 4$, $\tau^2 - 4\Delta = -12$, and so $(0,0)$ is a stable spiral point.

**25.** In Problem 5, Section 10.1, we showed that $(0,0)$, $(\sqrt{1/\epsilon}\,, 0)$ and $(-\sqrt{1/\epsilon}\,, 0)$ are the critical points. We will use the Jacobian matrix

$$\mathbf{g}'(\mathbf{X}) = \begin{pmatrix} 0 & 1 \\ -1 + 3\epsilon x^2 & 0 \end{pmatrix}$$

to classify these three critical points. For $\mathbf{X} = (0,0)$, $\tau = 0$ and $\Delta = 1$ and we are unable to classify this critical point. For $(\pm\sqrt{1/\epsilon}\,, 0)$, $\tau = 0$ and $\Delta = -2$ and so both of these critical points are saddle points.

**26.** In Problem 6, Section 10.1, we showed that $(0,0)$, $(1/\epsilon, 0)$, and $(-1/\epsilon, 0)$ are the critical points. Since $D_x x|x| = 2|x|$, the Jacobian matrix is

$$\mathbf{g}'(\mathbf{X}) = \begin{pmatrix} 0 & 1 \\ 2\epsilon|x| - 1 & 0 \end{pmatrix}.$$

For $\mathbf{X} = (0,0)$, $\tau = 0$, $\Delta = 1$ and we are unable to classify this critical point. For $(\pm 1/\epsilon, 0)$, $\tau = 0$, $\Delta = -1$, and so both of these critical points are saddle points.

**27.** The corresponding plane autonomous system is

$$x' = y, \quad y' = -\frac{(\beta + \alpha^2 y^2)x}{1 + \alpha^2 x^2}$$

and the Jacobian matrix is

$$\mathbf{g}'(\mathbf{X}) = \begin{pmatrix} 0 & 1 \\ \dfrac{(\beta + \alpha y^2)(\alpha^2 x^2 - 1)}{(1 + \alpha^2 x^2)^2} & \dfrac{-2\alpha^2 yx}{1 + \alpha^2 x^2} \end{pmatrix}.$$

For $\mathbf{X} = (0,0)$, $\tau = 0$ and $\Delta = \beta$. Since $\beta < 0$, we can conclude that $(0,0)$ is a saddle point.

**28.** From $x' = -\alpha x + xy = x(-\alpha + y) = 0$, either $x = 0$ or $y = \alpha$. If $x = 0$, then $1 - \beta y = 0$ and so $y = 1/\beta$. The case $y = \alpha$ implies that $1 - \beta\alpha - x^2 = 0$ or $x^2 = 1 - \alpha\beta$. Since $\alpha\beta > 1$, this equation has no real solutions. It follows that $(0, 1/\beta)$ is the unique critical point. Since the Jacobian matrix is

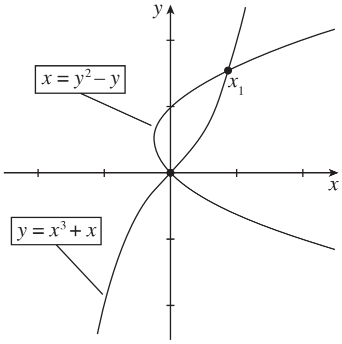
$$\mathbf{g}'(\mathbf{X}) = \begin{pmatrix} -\alpha + y & x \\ -2x & -\beta \end{pmatrix},$$

$\tau = -\alpha - \beta + \dfrac{1}{\beta} = -\beta + \dfrac{1 - \alpha\beta}{\beta} < 0$ and $\Delta = \alpha\beta - 1 > 0$. Therefore $(0, 1/\beta)$ is a stable critical point.

**29.** **(a)** The graphs of $-x + y - x^3 = 0$ and $-x - y + y^2 = 0$ are shown in the figure. The Jacobian matrix is

$$\mathbf{g}'(\mathbf{X}) = \begin{pmatrix} -1 - 3x^2 & 1 \\ -1 & -1 + 2y \end{pmatrix}.$$

For $\mathbf{X} = (0, 0)$, $\tau = -2$, $\Delta = 2$, $\tau^2 - 4\Delta = -4$, and so $(0, 0)$ is a stable spiral point.



**(b)** For $\mathbf{X}_1$, $\Delta = -6.07 < 0$ and so a saddle point occurs at $\mathbf{X}_1$.

**30.** **(a)** The corresponding plane autonomous system is

$$x' = y, \quad y' = \epsilon\left(y - \frac{1}{3}y^3\right) - x$$

and so the only critical point is $(0, 0)$. Since the Jacobian matrix is

$$\mathbf{g}'(\mathbf{X}) = \begin{pmatrix} 0 & 1 \\ -1 & \epsilon(1 - y^2) \end{pmatrix},$$

$\tau = \epsilon$, $\Delta = 1$, and so $\tau^2 - 4\Delta = \epsilon^2 - 4$ at the critical point $(0, 0)$.

**(b)** When $\tau = \epsilon > 0$, $(0, 0)$ is an unstable critical point.

**(c)** When $\epsilon < 0$ and $\tau^2 - 4\Delta = \epsilon^2 - 4 < 0$, $(0, 0)$ is a stable spiral point. These two requirements can be written as $-2 < \epsilon < 0$.

**(d)** When $\epsilon = 0$, $x'' + x = 0$ and so $x = c_1 \cos t + c_2 \sin t$. Therefore all solutions are periodic (with period $2\pi$) and so $(0, 0)$ is a center.

**31.** The differential equation $dy/dx = y'/x' = -2x^3/y$ can be solved by separating variables. It follows that $y^2 + x^4 = c$. If $\mathbf{X}(0) = (x_0, 0)$ where $x_0 > 0$, then $c = x_0^4$ so that $y^2 = x_0^4 - x^4$. Therefore if $-x_0 < x < x_0$, $y^2 > 0$ and so there are two values of $y$ corresponding to each value of $x$. Therefore the solution $\mathbf{X}(t)$ with $\mathbf{X}(0) = (x_0, 0)$ is periodic and so $(0, 0)$ is a center.

**32.** The differential equation $dy/dx = y'/x' = (x^2 - 2x)/y$ can be solved by separating variables. It follows that $y^2/2 = (x^3/3) - x^2 + c$ and since $\mathbf{X}(0) = (x(0), x'(0)) = (1, 0)$, $c = \frac{2}{3}$. Therefore

$$\frac{y^2}{2} = \frac{x^3 - 3x^2 + 2}{3} = \frac{(x-1)(x^2 - 2x - 2)}{3}.$$

But $(x-1)(x^2 - 2x - 2) > 0$ for $1 - \sqrt{3} < x < 1$ and so each $x$ in this interval has 2 corresponding values of $y$. therefore $\mathbf{X}(t)$ is a periodic solution.

**33.** **(a)** $x' = 2xy = 0$ implies that either $x = 0$ or $y = 0$. If $x = 0$, then from $1 - x^2 + y^2 = 0$, $y^2 = -1$ and there are no real solutions. If $y = 0$, $1 - x^2 = 0$ and so $(1, 0)$ and $(-1, 0)$ are critical points. The Jacobian matrix is

$$\mathbf{g}'(\mathbf{X}) = \begin{pmatrix} 2y & 2x \\ -2x & 2y \end{pmatrix}$$

and so $\tau = 0$ and $\Delta = 4$ at either $\mathbf{X} = (1, 0)$ or $(-1, 0)$. We obtain no information about these critical points in this borderline case.

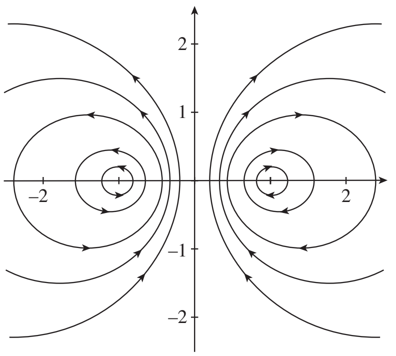
**(b)** The differential equation is

$$\frac{dy}{dx} = \frac{y'}{x'} = \frac{1 - x^2 + y^2}{2xy}$$

or

$$2xy\,\frac{dy}{dx} = 1 - x^2 + y^2.$$

Letting $\mu = y^2/x$, it follows that $d\mu/dx = (1/x^2) - 1$ and so $\mu = -(1/x) - x + 2c$. Therefore $y^2/x = -(1/x) - x + 2c$ which can be put in the form $(x - c)^2 + y^2 = c^2 - 1$. The solution curves are shown and so both $(1, 0)$ and $(-1, 0)$ are centers.



**34.** **(a)** The differential equation is $dy/dx = y'/x' = (-x - y^2)/y = -(x/y) - y$ and so $dy/dx + y = -xy^{-1}$.

**(b)** Let $w = y^{1-n} = y^2$. It follows that $dw/dx + 2w = -2x$, a linear first order differential equation whose solution is $y^2 = w = ce^{-2x} + \left(\frac{1}{2} - x\right)$. Since $x(0) = \frac{1}{2}$ and $y(0) = x'(0) = 0$, $0 = c$ and so $y^2 = \frac{1}{2} - x$, a parabola with vertex at $(1/2, 0)$. Therefore the solution $\mathbf{X}(t)$ with $\mathbf{X}(0) = (1/2, 0)$ is not periodic.

**35.** The differential equation is $dy/dx = y'/x' = (x^3 - x)/y$ and so $y^2/2 = x^4/4 - x^2/2 + c$ or $y^2 = x^4/2 - x^2 + c_1$. Since $x(0) = 0$ and $y(0) = x'(0) = v_0$, it follows that $c_1 = v_0^2$ and so

$$y^2 = \frac{1}{2}x^4 - x^2 + v_0^2 = \frac{(x^2-1)^2 + 2v_0^2 - 1}{2}.$$

The $x$-intercepts on this graph satisfy

$$x^2 = 1 \pm \sqrt{1 - 2v_0^2}$$

and so we must require that $1 - 2v_0^2 \geq 0$ (or $|v_0| \leq \frac{1}{2}\sqrt{2}$ ) for real solutions to exist. If $x_0^2 = 1 - \sqrt{1 - 2v_0^2}$ and $-x_0 < x < x_0$, then $(x^2 - 1)^2 + 2v_0^2 - 1 > 0$ and so there are two corresponding values of $y$. Therefore $\mathbf{X}(t)$ with $\mathbf{X}(0) = (0, v_0)$ is periodic provided that $|v_0| \leq \frac{1}{2}\sqrt{2}$.

**36.** The corresponding plane autonomous system is

$$x' = y, \quad y' = \epsilon x^2 - x + 1$$

and so the critical points must satisfy $y = 0$ and

$$x = \frac{1 \pm \sqrt{1 - 4\epsilon}}{2\epsilon}.$$

Therefore we must require that $\epsilon \leq \frac{1}{4}$ for real solutions to exist. We will use the Jacobian matrix

$$\mathbf{g}'(\mathbf{X}) = \begin{pmatrix} 0 & 1 \\ 2\epsilon x - 1 & 0 \end{pmatrix}$$

to attempt to classify $((1 \pm \sqrt{1 - 4\epsilon})/2\epsilon, 0)$ when $\epsilon \leq 1/4$. Note that $\tau = 0$ and $\Delta = \mp\sqrt{1 - 4\epsilon}$. For $\mathbf{X} = ((1 + \sqrt{1 - 4\epsilon})/2\epsilon, 0)$ and $\epsilon < 1/4$, $\Delta < 0$ and so a saddle point occurs. For $\mathbf{X} = ((1 - \sqrt{1 - 4\epsilon})/2\epsilon, 0)$, $\Delta \geq 0$ and we are not able to classify this critical point using linearization.

**37.** The corresponding plane autonomous system is

$$x' = y, \quad y' = -\frac{\alpha}{L}\,x - \frac{\beta}{L}\,x^3 - \frac{R}{L}\,y$$

where $x = q$ and $y = q'$. If $\mathbf{X} = (x, y)$ is a critical point, $y = 0$ and $-\alpha x - \beta x^3 = -x(\alpha + \beta x^2) = 0$. If $\beta > 0$, $\alpha + \beta x^2 = 0$ has no real solutions and so $(0, 0)$ is the only critical point. Since

$$\mathbf{g}'(\mathbf{X}) = \begin{pmatrix} 0 & 1 \\ \dfrac{-\alpha - 3\beta x^2}{L} & -\dfrac{R}{L} \end{pmatrix},$$

$\tau = -R/L < 0$ and $\Delta = \alpha/L > 0$. Therefore $(0, 0)$ is a stable critical point. If $\beta < 0$, $(0, 0)$ and $(\pm\hat{x}, 0)$, where $\hat{x}^2 = -\alpha/\beta$ are critical points. At $\mathbf{X}(\pm\hat{x}, 0)$, $\tau = -R/L < 0$ and $\Delta = -2\alpha/L < 0$. Therefore both critical points are saddles.

**38.** If we let $dx/dt = y$, then $dy/dt = -x^3 - x$. From this we obtain the first-order differential equation

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = -\frac{x^3 + x}{y}\,.$$

Separating variables and integrating we obtain

$$\int y\,dy = -\int (x^3 + x)\,dx$$

and

$$\frac{1}{2}y^2 = -\frac{1}{4}x^4 - \frac{1}{2}x^2 + c_1.$$

Completing the square we can write the solution as $y^2 = -\frac{1}{2}(x^2 + 1)^2 + c_2$. If $\mathbf{X}(0) = (x_0, 0)$, then $c_2 = \frac{1}{2}(x_0^2 + 1)^2$ and so

$$\begin{aligned} y^2 &= -\frac{1}{2}(x^2 + 1)^2 + \frac{1}{2}(x_0^2 + 1)^2 = \frac{x_0^4 + 2x_0^2 + 1 - x^4 - 2x^2 - 1}{2} \\ &= \frac{(x_0^2 + x^2)(x_0^2 - x^2) + 2(x_0^2 - x^2)}{2} = \frac{(x_0^2 + x^2 + 2)(x_0^2 - x^2)}{2}\,. \end{aligned}$$

Note that $y = 0$ when $x = -x_0$. In addition, the right-hand side is positive for $-x_0 < x < x_0$, and so there are two corresponding values of $y$ for each $x$ between $-x_0$ and $x_0$. The solution $\mathbf{X} = \mathbf{X}(t)$ that satisfies $\mathbf{X}(0) = (x_0, 0)$ is therefore periodic, and so $(0, 0)$ is a center.

**39.** **(a)** Letting $x = \theta$ and $y = x'$ we obtain the system $x' = y$ and $y' = 1/2 - \sin x$. Since $\sin \pi/6 = \sin 5\pi/6 = 1/2$ we see that $(\pi/6, 0)$ and $(5\pi/6, 0)$ are critical points of the system.

**(b)** The Jacobian matrix is

$$\mathbf{g}'(\mathbf{X}) = \begin{pmatrix} 0 & 1 \\ -\cos x & 0 \end{pmatrix}$$

and so

$$\mathbf{A}_1 = \mathbf{g}' = ((\pi/6, 0)) = \begin{pmatrix} 0 & 1 \\ -\sqrt{3}/2 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{A}_2 = \mathbf{g}' = ((5\pi/6, 0)) = \begin{pmatrix} 0 & 1 \\ \sqrt{3}/2 & 0 \end{pmatrix}.$$

Since $\det \mathbf{A}_1 > 0$ and the trace of $\mathbf{A}_1$ is 0, no conclusion can be drawn regarding the critical point $(\pi/6, 0)$. Since $\det \mathbf{A}_2 < 0$, we see that $(5\pi/6, 0)$ is a saddle point.

**(c)** From the system in part (**A**) we obtain the first-order differential equation

$$\frac{dy}{dx} = \frac{1/2 - \sin x}{y}.$$

Separating variables and integrating we obtain

$$\int y\,dy = \int \left(\frac{1}{2} - \sin x\right) dx$$

and

$$\frac{1}{2}y^2 = \frac{1}{2}x + \cos x + c_1$$

or

$$y^2 = x + 2\cos x + c_2.$$

For $x_0$ near $\pi/6$, if $\mathbf{X}(0) = (x_0, 0)$ then $c_2 = -x_0 - 2\cos x_0$ and $y^2 = x + 2\cos x - x_0 - 2\cos x_0$. Thus, there are two values of $y$ for each $x$ in a sufficiently small interval around $\pi/6$. Therefore $(\pi/6, 0)$ is a center.

**40. (a)** Writing the system as $x' = x(x^3 - 2y^3)$ and $y' = y(2x^3 - y^3)$ we see that $(0, 0)$ is a critical point. Setting $x^3 - 2y^3 = 0$ we have $x^3 = 2y^3$ and $2x^3 - y^3 = 4y^3 - y^3 = 3y^3$. Thus, $(0, 0)$ is the only critical point of the system.

**(b)** From the system we obtain the first-order differential equation

$$\frac{dy}{dx} = \frac{2x^3y - y^4}{x^4 - 2xy^3}$$

or

$$(2x^3y - y^4)\,dx + (2xy^3 - x^4)\,dy = 0$$

which is homogeneous. If we let $y = ux$ it follows that

$$\begin{aligned}(2x^4u - x^4u^4)\,dx + (2x^4u^3 - x^4)(u\,dx + x\,du) &= 0\\ x^4u(1 + u^3)\,dx + x^5(2u^3 - 1)\,du &= 0\\ \frac{1}{x}\,dx + \frac{2u^3 - 1}{u(u^3 + 1)}\,du &= 0\\ \frac{1}{x}\,dx + \left(\frac{1}{u + 1} - \frac{1}{u} + \frac{2u - 1}{u^2 - u + 1}\right) du &= 0.\end{aligned}$$

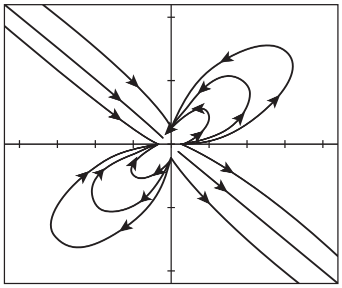
Integrating gives

$$\ln|x| + \ln|u+1| - \ln|u| + \ln|u^2 - u + 1| = c_1$$

or

$$\begin{aligned} x\left(\frac{u+1}{u}\right)(u^2 - u + 1) &= c_2 \\ x\left(\frac{y+x}{y}\right)\left(\frac{y^2}{x^2} - \frac{y}{x} + 1\right) &= c_2 \\ (xy + x^2)(y^2 - xy + x^2) &= c_2 x^2 y \\ xy^3 + x^4 &= c_2 x^2 y \\ x^3 + y^2 &= 3c_3 xy. \end{aligned}$$

**(c)** We see from the graph that $(0,0)$ is unstable. It is not possible to classify the critical point as a node, saddle, center, or spiral point.



## 10.4 Autonomous Systems as Mathematical Models

**1.** We are given that $x(0) = \theta(0) = \pi/3$ and $y(0) = \theta'(0) = w_0$. Since $y^2 = (2g/l)\cos x + c$, $w_0^2 = (2g/l)\cos(\pi/3) + c = g/l + c$ and so $c = w_0^2 - g/l$. Therefore

$$y^2 = \frac{2g}{l}\left(\cos x - \frac{1}{2} + \frac{l}{2g}\,w_0^2\right)$$

and the $x$-intercepts occur where $\cos x = 1/2 - (l/2g)w_0^2$ and so $1/2 - (l/2g)w_0^2$ must be greater than $-1$ for solutions to exist. This condition is equivalent to $|w_0| < \sqrt{3g/l}$.

**2.** **(a)** Since $y^2 = (2g/l)\cos x + c$, $x(0) = \theta(0) = \theta_0$ and $y(0) = \theta'(0) = 0$, $c = -(2g/l)\cos\theta_0$ and so $y^2 = 2g(\cos\theta - \cos\theta_0)/l$. When $\theta = -\theta_0$, $y^2 = 2g[\cos(-\theta_0) - \cos\theta_0]/l = 0$. Therefore $y = d\theta/dt = 0$ when $\theta = -\theta_0$.

**(b)** Since $y = d\theta/dt$ and $\theta$ is decreasing between the time when $\theta = \theta_0$, $t = 0$, and $\theta = -\theta_0$, that is, $t = T$,

$$\frac{d\theta}{dt} = -\sqrt{\frac{2g}{l}}\,\sqrt{\cos\theta - \cos\theta_0}\,.$$

Therefore

$$\frac{dt}{d\theta} = -\sqrt{\frac{l}{2g}}\,\frac{1}{\sqrt{\cos\theta - \cos\theta_0}}$$

and so

$$T = -\sqrt{\frac{l}{2g}} \int_{\theta=\theta_0}^{\theta=-\theta_0} \frac{1}{\sqrt{\cos\theta - \cos\theta_0}}\, d\theta = \sqrt{\frac{l}{2g}} \int_{-\theta_0}^{\theta_0} \frac{1}{\sqrt{\cos\theta - \cos\theta_0}}\, d\theta.$$

**3.** The corresponding plane autonomous system is

$$x' = y, \quad y' = -g\,\frac{f'(x)}{1+[f'(x)]^2} - \frac{\beta}{m}\,y$$

and

$$\frac{\partial}{\partial x}\left(-g\,\frac{f'(x)}{1+[f'(x)]^2} - \frac{\beta}{m}\,y\right) = -g\,\frac{(1+[f'(x)]^2)f''(x) - f'(x)2f'(x)f''(x)}{(1+[f'(x)]^2)^2}\,.$$

If $\mathbf{X}_1 = (x_1, y_1)$ is a critical point, $y_1 = 0$ and $f'(x_1) = 0$. The Jacobian at this critical point is therefore

$$\mathbf{g}'(\mathbf{X}_1) = \begin{pmatrix} 0 & 1 \\ -gf''(x_1) & -\dfrac{\beta}{m} \end{pmatrix}.$$

**4.** When $\beta = 0$ the Jacobian matrix is

$$\begin{pmatrix} 0 & 1 \\ -gf''(x_1) & 0 \end{pmatrix}$$

which has complex eigenvalues $\lambda = \pm\sqrt{gf''(x_1)}\,i$. The approximating linear system with $x'(0) = 0$ has solution

$$x(t) = x(0)\cos\sqrt{gf''(x_1)}\,t$$

and period $2\pi/\sqrt{gf''(x_1)}$. Therefore $p \approx 2\pi/\sqrt{gf''(x_1)}$ for the actual solution.

**5. (a)** If $f(x) = x^2/2$, $f'(x) = x$ and so

$$\frac{dy}{dx} = \frac{y'}{x'} = -g\,\frac{x}{1+x^2}\,\frac{1}{y}\,.$$

We can separate variables to show that $y^2 = -g\ln(1+x^2) + c$. But $x(0) = x_0$ and $y(0) = x'(0) = v_0$. Therefore $c = v_0^2 + g\ln(1+x_0^2)$ and so

$$y^2 = v_0^2 - g\ln\left(\frac{1+x^2}{1+x_0^2}\right).$$

Now

$$v_0^2 - g\ln\left(\frac{1+x^2}{1+x_0^2}\right) \ge 0 \quad \text{if and only if} \quad x^2 \le e^{v_0^2/g}(1+x_0^2) - 1.$$

Therefore, if $|x| \le [e^{v_0^2/g}(1+x_0^2) - 1]^{1/2}$, there are two values of $y$ for a given value of $x$ and so the solution is periodic.

**(b)** Since $z = x^2/2$, the maximum height occurs at the largest value of $x$ on the cycle. From (**A**), $x_{\text{max}} = [e^{v_0^2/g}(1 + x_0^2) - 1]^{1/2}$ and so

$$z_{\text{max}} = \frac{x_{\text{max}}^2}{2} = \frac{1}{2}[e^{v_0^2/g}(1 + x_0^2) - 1].$$

**6.** **(a)** If $f(x) = \cosh x$, $f'(x) = \sinh x$ and $[f'(x)]^2 + 1 = \sinh^2 x + 1 = \cosh^2 x$. Therefore

$$\frac{dy}{dx} = \frac{y'}{x'} = -g\,\frac{\sinh x}{\cosh^2 x}\,\frac{1}{y}\,.$$

We can separate variables to show that $y^2 = 2g/\cosh x + c$. But $x(0) = x_0$ and $y(0) = x'(0) = v_0$. Therefore $c = v_0^2 - (2g/\cosh x_0)$ and so

$$y^2 = \frac{2g}{\cosh x} - \frac{2g}{\cosh x_0} + v_0^2.$$

Now

$$\frac{2g}{\cosh x} - \frac{2g}{\cosh x_0} + v_0^2 \geq 0 \quad \text{if and only if} \quad \cosh x \leq \frac{2g\cosh x_0}{2g - v_0^2\cosh x_0}$$

and the solution to this inequality is an interval $[-a, a]$. Therefore each $x$ in $(-a, a)$ has two corresponding values of $y$ and so the solution is periodic.

**(b)** Since $z = \cosh x$, the maximum height occurs at the largest value of $x$ on the cycle. From (a), $x_{\text{max}} = a$ where $\cosh a = 2g\cosh x_0/(2g - v_0^2\cosh x_0)$. Therefore

$$z_{\text{max}} = \frac{2g\cosh x_0}{2g - v_0^2\cosh x_0}\,.$$

**7.** If $x_m < x_1 < x_n$, then $F(x_1) > F(x_m) = F(x_n)$. Letting $x = x_1$,

$$G(y) = \frac{c_0}{F(x_1)} = \frac{F(x_m)G(a/b)}{F(x_1)} < G(a/b).$$

Therefore from Property $(ii)$ on page 418 in this section of the text, $G(y) = c_0/F(x_1)$ has two solutions $y_1$ and $y_2$ that satisfy $y_1 < a/b < y_2$.

**8.** From Property $(i)$ on page 418 in this section of the text, when $y = a/b$, $x_n$ is taken on at some time $t$. From Property $(iii)$, if $x > x_n$ there is no corresponding value of $y$. Therefore the maximum number of predators is $x_n$ and $x_n$ occurs when $y = a/b$.

**9.** **(a)** In the Lotka–Volterra Model the average number of predators is $d/c$ and the average number of prey is $a/b$. But

$$\begin{aligned} x' &= -ax + bxy - \epsilon_1 x = -(a + \epsilon_1)x + bxy \\ y' &= -cxy + dy - \epsilon_2 y = -cxy + (d - \epsilon_2)y \end{aligned}$$

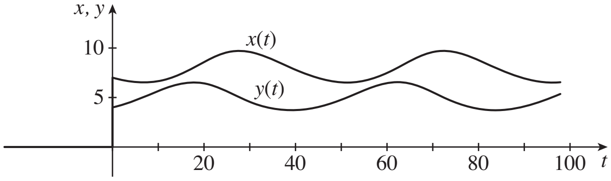
and so the new critical point in the first quadrant is $(d/c - \epsilon_2/c, a/b + \epsilon_1/b)$.

**(b)** The average number of predators $d/c - \epsilon_2/c$ has decreased while the average number of prey $a/b + \epsilon_1/b$ has increased. The fishery science model is consistent with Volterra's principle.

**10. (a)** Solving

$$x(-0.1 + 0.02y) = 0$$
$$y(0.2 - 0.025x) = 0$$

in the first quadrant we obtain the critical point $(8, 5)$. The graphs are plotted using $x(0) = 7$ and $y(0) = 4$.



**(b)** The graph in part (**A**) was obtained using **NDSolve** in *Mathematica*. We see that the period is around 40. Since $x(0) = 7$, we use the **FindRoot** equation solver in *Mathematica* to approximate the solution of $x(t) = 7$ for $t$ near 40. From this we see that the period is more closely approximated by $t = 44.65$.

**11.** Solving

$$x(20 - 0.4x - 0.3y) = 0$$
$$y(10 - 0.1y - 0.3x) = 0$$

we see that critical points are $(0, 0)$, $(0, 100)$, $(50, 0)$, and $(20, 40)$. The Jacobian matrix is

$$\mathbf{g}'(\mathbf{X}) = \begin{pmatrix} 0.08(20 - 0.8x - 0.3y) & -0.024x \\ -0.018y & 0.06(10 - 0.2y - 0.3x) \end{pmatrix}$$

and so

$$\mathbf{A}_1 = \mathbf{g}'((0,0)) = \begin{pmatrix} 1.6 & 0 \\ 0 & 0.6 \end{pmatrix} \qquad \mathbf{A}_2 = \mathbf{g}'((0,100)) = \begin{pmatrix} -0.8 & 0 \\ -1.8 & -0.6 \end{pmatrix}$$

$$\mathbf{A}_3 = \mathbf{g}'((50,0)) = \begin{pmatrix} -1.6 & -1.2 \\ 0 & -0.3 \end{pmatrix} \qquad \mathbf{A}_4 = \mathbf{g}'((20,40)) = \begin{pmatrix} -0.64 & -0.48 \\ -0.72 & -0.24 \end{pmatrix}.$$

Since $\det(\mathbf{A}_1) = \Delta_1 = 0.96 > 0$, $\tau = 2.2 > 0$, and $\tau_1^2 - 4\Delta_1 = 1 > 0$, we see that $(0, 0)$ is an unstable node. Since $\det(\mathbf{A}_2) = \Delta_2 = 0.48 > 0$, $\tau = -1.4 < 0$, and $\tau_2^2 - 4\Delta_2 = 0.04 > 0$, we see that $(0, 100)$ is a stable node. Since $\det(\mathbf{A}_3) = \Delta_3 = 0.48 > 0$, $\tau = -1.9 < 0$, and $\tau_3^2 - 4\Delta_3 = 1.69 > 0$, we see that $(50, 0)$ is a stable node. Since $\det(\mathbf{A}_4) = -0.192 < 0$ we see that $(20, 40)$ is a saddle point.

**12.** $\Delta = r_1 r_2$, $\tau = r_1 + r_2$ and $\tau^2 - 4\Delta = (r_1 + r_2)^2 - 4r_1 r_2 = (r_1 - r_2)^2$. Therefore when $r_1 \neq r_2$, $(0, 0)$ is an unstable node.

**13.** For $\mathbf{X} = (K_1, 0)$, $\tau = -r_1 + r_2[1 - (K_1\alpha_{21}/K_2)]$ and $\Delta = -r_1r_2[1 - (K_1\alpha_{21}/K_2)]$. If we let $c = 1 - K_1\alpha_{21}/K_2$, $\tau^2 - 4\Delta = (cr_2 + r_1)^2 > 0$. Now if $k_1 > K_2/\alpha_{21}$, $c < 0$ and so $\tau < 0$, $\Delta > 0$. Therefore $(K_1, 0)$ is a stable node. If $K_1 < K_2/\alpha_{21}$, $c > 0$ and so $\Delta < 0$. In this case $(K_1, 0)$ is a saddle point.

**14.** $(\hat{x}, \hat{y})$ is a stable node if and only if $K_1/\alpha_{12} > K_2$ and $K_2/\alpha_{21} > K_1$. [See Figure 10.4.11(a) in the text.] From Problem 12, $(0.0)$ is an unstable node and from Problem 13, since $K_1 < K_2/\alpha_{21}$, $(K_1, 0)$ is a saddle point. Finally, when $K_2 < K_1/\alpha_{12}$, $(0, K_2)$ is a saddle point. This is Problem 12 with the roles of 1 and 2 interchanged. Therefore $(0, 0)$, $(K_1, 0)$, and $(0, K_2)$ are unstable.

**15.** $K_1/\alpha_{12} < K_2 < K_1\alpha_{21}$ and so $\alpha_{12}\alpha_{21} > 1$. Therefore $\Delta = (1 - \alpha_{12}\alpha_{21})\hat{x}\hat{y}\, r_1r_2/K_1K_2 < 0$ and so $(\hat{x}, \hat{y})$ is a saddle point.

**16.** **(a)** The corresponding plane autonomous system is

$$x' = y, \quad y' = \frac{-g}{l}\sin x - \frac{\beta}{ml}\,y$$

and so critical points must satisfy both $y = 0$ and $\sin x = 0$. Therefore $(\pm n\pi, 0)$ are critical points.

**(b)** The Jacobian matrix

$$\begin{pmatrix} 0 & 1 \\ -\dfrac{g}{l}\cos x & -\dfrac{\beta}{ml} \end{pmatrix}$$

has trace $\tau = -\beta/ml$ and determinant $\Delta = g/l > 0$ at $(0, 0)$. Therefore

$$\tau^2 - 4\Delta = \frac{\beta^2}{m^2l^2} - 4\,\frac{g}{l} = \frac{\beta^2 - 4glm^2}{m^2l^2}\,.$$

We can conclude that $(0, 0)$ is a stable spiral point provided
$\beta^2 - 4glm^2 < 0$ or $\beta < 2m\sqrt{gl}\,$.

**17.** **(a)** The corresponding plane autonomous system is

$$x = y, \quad y' = -\frac{\beta}{m}\,y|y| - \frac{k}{m}\,x$$

and so a critical point must satisfy both $y = 0$ and $x = 0$. Therefore $(0, 0)$ is the unique critical point.

**(b)** The Jacobian matrix is

$$\begin{pmatrix} 0 & 1 \\ -\dfrac{k}{m} & -\dfrac{\beta}{m}\,2|y| \end{pmatrix}$$

and so $\tau = 0$ and $\Delta = k/m > 0$. Therefore $(0, 0)$ is a center, stable spiral point, or an unstable spiral point. Physical considerations suggest that $(0, 0)$ must be asymptotically stable and so $(0, 0)$ must be a stable spiral point.

**18. (a)** The magnitude of the frictional force between the bead and the wire is $\mu(mg\cos\theta)$ for some $\mu > 0$. The component of this frictional force in the $x$-direction is

$$(\mu mg\cos\theta)\cos\theta = \mu mg\cos^2\theta.$$

But

$$\cos\theta = \frac{1}{\sqrt{1+[f'(x)]^2}} \quad \text{and so} \quad \mu mg\cos^2\theta = \frac{\mu mg}{1+[f'(x)]^2}.$$

It follows from Newton's Second Law that

$$mx'' = -mg\frac{f'(x)}{1+[f'(x)]^2} - \beta x' + mg\frac{\mu}{1+[f'(x)]^2}$$

and so

$$x'' = g\frac{\mu - f'(x)}{1+[f'(x)]^2} - \frac{\beta}{m}x'.$$

**(b)** A critical point $(x,y)$ must satisfy $y=0$ and $f'(x)=\mu$. Therefore critical points occur at $(x_1,0)$ where $f'(x_1)=\mu$. The Jacobian matrix of the plane autonomous system is

$$\mathbf{g}'(\mathbf{X}) = \begin{pmatrix} 0 & 1 \\ g\dfrac{(1+[f'(x)]^2)(-f''(x)) - (\mu - f'(x))2f'(x)f''(x)}{(1+[f'(x)]^2)^2} & -\dfrac{\beta}{m} \end{pmatrix}$$

and so at a critical point $\mathbf{X}_1$,

$$\mathbf{g}'(\mathbf{X}) = \begin{pmatrix} 0 & 1 \\ \dfrac{-gf''(x_1)}{1+\mu^2} & -\dfrac{\beta}{m} \end{pmatrix}.$$

Therefore $\tau = -\beta/m < 0$ and $\Delta = gf''(x_1)/(1+\mu^2)$. When $f''(x_1) < 0$, $\Delta < 0$ and so a saddle point occurs. When $f''(x_1) > 0$ and

$$\tau^2 - 4\Delta = \frac{\beta^2}{m^2} - 4g\frac{f''(x_1)}{1+\mu^2} < 0,$$

$(x_1,0)$ is a stable spiral point. This condition can also be written as

$$\beta^2 < 4gm^2\frac{f''(x_1)}{1+\mu^2}.$$

**19.** We have $dy/dx = y'/x' = -f(x)/y$ and so, using separation of variables,

$$\frac{y^2}{2} = -\int_0^x f(\mu)\,d\mu + c \qquad \text{or} \qquad y^2 + 2F(x) = c.$$

We can conclude that for a given value of $x$ there are at most two corresponding values of $y$. If $(0,0)$ were a stable spiral point there would exist an $x$ with more than two corresponding values of $y$. Note that the condition $f(0)=0$ is required for $(0,0)$ to be a critical point of the corresponding plane autonomous system $x'=y$, $y'=-f(x)$.

**20.** **(a)** $x' = x(-a+by) = 0$ implies that $x = 0$ or $y = a/b$. If $x = 0$, then, from

$$-cxy + \frac{r}{K}\,y(K-y) = 0,$$

$y = 0$ or $K$. Therefore $(0,0)$ and $(0,K)$ are critical points. If $\hat{y} = a/b$, then

$$\hat{y}\left[-cx + \frac{r}{K}(K-\hat{y})\right] = 0.$$

The corresponding value of $x$, $x = \hat{x}$, therefore satisfies the equation $c\hat{x} = r(K-\hat{y})/K$.

**(b)** The Jacobian matrix is

$$\mathbf{g}'(\mathbf{X}) = \begin{pmatrix} -a+by & bx \\ -cy & -cx + \dfrac{r}{K}(K-2y) \end{pmatrix}$$

and so at $\mathbf{X}_1 = (0,0)$, $\Delta = -ar < 0$. For $\mathbf{X}_1 = (0,K)$, $\Delta = n(Kb-a) = -rb(K-a/b)$. Since we are given that $K > a/b$, $\Delta < 0$ in this case. Therefore $(0,0)$ and $(0,K)$ are each saddle points. For $\mathbf{X}_1 = (\hat{x},\hat{y})$ where $\hat{y} = a/b$ and $c\hat{x} = r(K-\hat{y})/K$, we can write the Jacobian matrix as

$$\mathbf{g}'((\hat{x},\hat{y})) = \begin{pmatrix} 0 & b\hat{x} \\ -c\hat{y} & -\dfrac{r}{K}\,\hat{y} \end{pmatrix}$$

and so $\tau = -r\hat{y}/K < 0$ and $\Delta = bc\hat{x}\hat{y} > 0$. Therefore $(\hat{x},\hat{y})$ is a stable critical point and so it is either a stable node (perhaps degenerate) or a stable spiral point.

**(c)** Write

$$\tau^2 - 4\Delta = \frac{r^2}{K^2}\,\hat{y}^2 - 4bc\hat{x}\hat{y} = \hat{y}\left[\frac{r^2}{K^2}\,\hat{y} - 4bc\hat{x}\right] = \hat{y}\left[\frac{r^2}{K^2}\,\hat{y} - 4b\,\frac{r}{K}(K-\hat{y})\right]$$

using

$$c\hat{x} = \frac{r}{K}(K-\hat{y}) = \frac{r}{K}\,\hat{y}\left[\left(\frac{r}{K}+4b\right)\hat{y} - 4bK\right].$$

Therefore $\tau^2 - 4\Delta < 0$ if and only if

$$\hat{y} < \frac{4bK}{\frac{r}{K}+4b} = \frac{4bK^2}{r+4bK}.$$

Note that

$$\frac{4bK^2}{r+4bK} = \frac{4bK}{r+4bK}\cdot K \approx K$$

where $K$ is large, and $\hat{y} = a/b < K$. Therefore $\tau^2 - 4\Delta < 0$ when $K$ is large and a stable spiral point will result.

**21.** The equation

$$x' = \alpha \frac{y}{1+y} x - x = x\left(\frac{\alpha y}{1+y} - 1\right) = 0$$

implies that $x = 0$ or $y = 1/(\alpha - 1)$. When $\alpha > 0$, $\hat{y} = 1/(\alpha - 1) > 0$. If $x = 0$, then from the differential equation for $y'$, $y = \beta$. On the other hand, if $\hat{y} = 1/(\alpha - 1)$, $\hat{y}/(1 + \hat{y}) = 1/\alpha$ and so $\hat{x}/\alpha - 1/(\alpha - 1) + \beta = 0$. It follows that

$$\hat{x} = \alpha\left(\beta - \frac{1}{\alpha - 1}\right) = \frac{\alpha}{\alpha - 1}[(\alpha - 1)\beta - 1]$$

and if $\beta(\alpha - 1) > 1$, $\hat{x} > 0$. Therefore $(\hat{x}, \hat{y})$ is the unique critical point in the first quadrant. The Jacobian matrix is

$$\mathbf{g}'(\mathbf{X}) = \begin{pmatrix} \alpha \dfrac{y}{y+1} - 1 & \dfrac{\alpha x}{(1+y)^2} \\ -\dfrac{y}{1+y} & \dfrac{-x}{(1+y)^2} - 1 \end{pmatrix}$$

and for $\mathbf{X} = (\hat{x}, \hat{y})$, the Jacobian can be written in the form

$$\mathbf{g}'((\hat{x}, \hat{y})) = \begin{pmatrix} 0 & \dfrac{(\alpha - 1)^2}{\alpha} \hat{x} \\ -\dfrac{1}{\alpha} & -\dfrac{(\alpha - 1)^2}{\alpha^2} - 1 \end{pmatrix}.$$

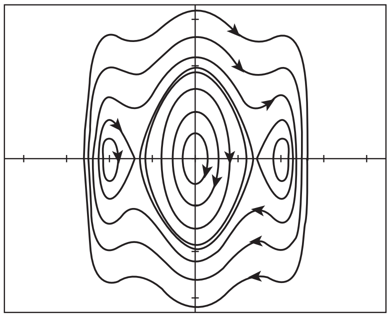
It follows that

$$\tau = -\left[\frac{(\alpha - 1)^2}{\alpha^2} \hat{x} + 1\right] < 0, \qquad \Delta = \frac{(\alpha - 1)^2}{\alpha^2} \hat{x}$$

and so $\tau = -(\Delta + 1)$. Therefore $\tau^2 - 4\Delta = (\Delta + 1)^2 - 4\Delta = (\Delta - 1)^2 > 0$. Therefore $(\hat{x}, \hat{y})$ is a stable node.

**22.** Letting $y = x'$ we obtain the plane autonomous system

$$\begin{aligned} x' &= y \\ y' &= -8x + 6x^3 - x^5. \end{aligned}$$



Solving $x^5 - 6x^3 + 8x = x(x^2 - 4)(x^2 - 2) = 0$ we see that critical points are $(0, 0)$, $(0, -2)$, $(0, 2)$, $(0, -\sqrt{2}\,)$, and $(0, \sqrt{2}\,)$. The Jacobian matrix is

$$\mathbf{g}'(\mathbf{X}) = \begin{pmatrix} 0 & 1 \\ -8 + 18x^2 - 5x^4 & 0 \end{pmatrix}$$

and we see that $\det(\mathbf{g}'(\mathbf{X})) = 5x^4 - 18x^2 + 8$ and the trace of $\mathbf{g}'(\mathbf{X})$ is 0. Since $\det(g'((\pm\sqrt{2}\,, 0))) = -8 < 0$, $(\pm\sqrt{2}\,, 0)$ are saddle points. For the other critical points the determinant is positive and linearization discloses no information. The graph of the phase plane suggests that $(0, 0)$ and $(\pm 2, 0)$ are centers.

## Chapter 10 in Review

**1.** True

**2.** True

**3.** A center or a saddle point

**4.** Complex with negative real parts

**5.** False; there are initial conditions for which $\lim_{t\to\infty} \mathbf{X}(t) = (0,0)$.

**6.** True

**7.** False; this is a borderline case. See Figure 10.3.7 in the text.

**8.** False; see Figure 10.4.2 in the text.

**9.** The system is linear and we identify $\Delta = -\alpha$ and $\tau = \alpha + 1$. Since a critical point will be a center when $\Delta > 0$ and $\tau = 0$ we see that for $\alpha = -1$ critical points will be centered and solutions will be periodic. Not also that when $\alpha = -1$ the system is

$$\begin{aligned} x' &= -x - 2y \\ y' &= x + y \end{aligned}$$

which does have an isolated critical point at $(0,0)$.

**10.** We identify $g(x) = \sin x$ in Theorem 10.3.1. Then $x_1 = n\pi$ is a critical point for $n$ an integer and $g'(n\pi) = \cos n\pi < 0$ when $n$ is an odd integer. Thus, $n\pi$ is an asymptotically stable critical point when $n$ is an odd integer.

**11.** Switching to polar coordinates,

$$\begin{aligned} \frac{dr}{dt} &= \frac{1}{r}\left(x\,\frac{dx}{dt} + y\,\frac{dy}{dt}\right) = \frac{1}{r}(-xy - x^2r^3 + xy - y^2r^3) = -r^4 \\ \frac{d\theta}{dt} &= \frac{1}{r^2}\left(-y\,\frac{dx}{dt} + x\,\frac{dy}{dt}\right) = \frac{1}{r^2}(y^2 + xyr^3 + x^2 - xyr^3) = 1. \end{aligned}$$

Using separation of variables it follows that $r = \dfrac{1}{\sqrt[3]{3t + c_1}}$ and $\theta = t + c_2$. Since $\mathbf{X}(0) = (1,0)$, $r = 1$ and $\theta = 0$. It follows that $c_1 = 1$, $c_2 = 0$, and so

$$r = \frac{1}{\sqrt[3]{3t+1}}, \quad \theta = t.$$

As $t \to \infty$, $r \to 0$ and the solution spirals toward the origin.

**12.** **(a)** If $\mathbf{X}(0) = \mathbf{X}_0$ lies on the line $y = -2x$, then $\mathbf{X}(t)$ approaches $(0, 0)$ along this line. For all other initial conditions, $\mathbf{X}(t)$ approaches $(0, 0)$ from the direction determined by the line $y = x$.

**(b)** If $\mathbf{X}(0) = \mathbf{X}_0$ lies on the line $y = -x$, then $\mathbf{X}(t)$ approaches $(0, 0)$ along this line. For all other initial conditions, $\mathbf{X}(t)$ becomes unbounded and $y = 2x$ serves as an asymptote.

**13.** **(a)** $\tau = 0$, $\Delta = 11 > 0$ and so $(0, 0)$ is a center.

**(b)** $\tau = -2$, $\Delta = 1$, $\tau^2 - 4\Delta = 0$ and so $(0, 0)$ is a degenerate stable node.

**14.** From $x' = x(1 + y - 3x) = 0$, either $x = 0$ or $1 + y - 3x = 0$. If $x = 0$, then, from $y(4 - 2x - y) = 0$ we obtain $y(4 - y) = 0$. It follows that $(0, 0)$ and $(0, 4)$ are critical points. If $1 + y - 3x = 0$, then $y(5 - 5x) = 0$. Therefore $(1/3, 0)$ and $(1, 2)$ are the remaining critical points. We will use the Jacobian matrix

$$\mathbf{g}'(\mathbf{X}) = \begin{pmatrix} 1 + y - 6x & x \\ -2y & 4 - 2x - 2y \end{pmatrix}$$

to classify these four critical points. The results are as follows:

| $\mathbf{X}$ | $\tau$ | $\Delta$ | $\tau^2 - 4\Delta$ | Conclusion |
|---|---|---|---|---|
| $(0, 0)$ | 5 | 4 | 9 | unstable node |
| $(0, 4)$ | – | $-20$ | – | saddle point |
| $\left(\frac{1}{3}, 0\right)$ | – | $-\frac{10}{3}$ | – | saddle point |
| $(1, 2)$ | $-5$ | 10 | $-15$ | stable spiral point |

**15.** From $x = r\cos\theta$, $y = r\sin\theta$ we have

$$\frac{dx}{dt} = -r\sin\theta\,\frac{d\theta}{dt} + \frac{dr}{dt}\cos\theta$$
$$\frac{dy}{dt} = r\cos\theta\,\frac{d\theta}{dt} + \frac{dr}{dt}\sin\theta.$$

Then $r' = \alpha r$, $\theta' = 1$ gives

$$\frac{dx}{dt} = -r\sin\theta + \alpha r\cos\theta$$
$$\frac{dy}{dt} = r\cos\theta + \alpha r\sin\theta.$$

We see that $r = 0$, which corresponds to $\mathbf{X} = (0, 0)$, is a critical point. Solving $r' = \alpha r$ we have $r = c_1 e^{\alpha t}$. Thus, when $\alpha < 0$, $\lim_{t\to\infty} r(t) = 0$ and $(0, 0)$ is a stable critical point. When $\alpha = 0$, $r' = 0$ and $r = c_1$. In this case $(0, 0)$ is a center, which is stable. Therefore, $(0, 0)$ is a stable critical point for the system when $\alpha \le 0$.

**16.** The corresponding plane autonomous system is $x' = y$, $y' = \mu(1-x^2)y - x$ and so the Jacobian at the critical point $(0,0)$ is

$$\mathbf{g}'((0,0)) = \begin{pmatrix} 0 & 1 \\ -1 & \mu \end{pmatrix}.$$

Therefore $\tau = \mu$, $\Delta = 1$ and $\tau^2 - 4\Delta = \mu^2 - 4$. Now $\mu^2 - 4 < 0$ if and only if $-2 < \mu < 2$. We may therefore conclude that $(0,0)$ is a stable node for $\mu < -2$, a stable spiral point for $-2 < \mu < 0$, an unstable spiral point for $0 < \mu < 2$, and an unstable node for $\mu > 2$.

In the case that $\mu = 2$, $(0,0)$ is unstable but is a borderline case that may be an unstable spiral, unstable node, or degenerate unstable node. Similarly, in the case that $\mu = -2$, $(0,0)$ is but is a borderline case that may be a stable spiral, a stable node, or a degenerate stable node.

**17.** Critical points occur at $x = \pm 1$. Since

$$g'(x) = -\frac{1}{2}e^{-x/2}(x^2 - 4x - 1),$$

$g'(1) > 0$ and $g'(-1) < 0$. Therefore $x = 1$ is unstable and $x = -1$ is asymptotically stable.

**18.** $\dfrac{dy}{dx} = \dfrac{y'}{x'} = \dfrac{-2x\sqrt{y^2+1}}{y}$. We may separate variables to show that $\sqrt{y^2+1} = -x^2 + c$. But $x(0) = x_0$ and $y(0) = x'(0) = 0$. It follows that $c = 1 + x_0^2$ so that

$$y^2 = (1 + x_0^2 - x^2)^2 - 1.$$

Note that $1 + x_0^2 - x^2 > 1$ for $-x_0 < x < x_0$ and $y = 0$ for $x = \pm x_0$. Each $x$ with $-x_0 < x < x_0$ has two corresponding values of $y$ and so the solution $\mathbf{X}(t)$ with $\mathbf{X}(0) = (x_0, 0)$ is periodic.

**19.** The corresponding plane autonomous system is

$$x' = y, \quad y' = -\frac{\beta}{m}y - \frac{k}{m}(s+x)^3 + g$$

and so the Jacobian is

$$\mathbf{g}'(\mathbf{X}) = \begin{pmatrix} 0 & 1 \\ -\dfrac{3k}{m}(s+x)^2 & -\dfrac{\beta}{m} \end{pmatrix}.$$

For $\mathbf{X} = (0,0)$, $\tau = -\dfrac{\beta}{m} < 0$, $\Delta = \dfrac{3k}{m}s^2 > 0$.

Therefore

$$\tau^2 - 4\Delta = \frac{\beta^2}{m^2} - \frac{12k}{m}s^2 = \frac{1}{m^2}(\beta^2 - 12kms^2).$$

Therefore $(0,0)$ is a stable node if $\beta^2 > 12kms^2$ and a stable spiral point provided $\beta^2 < 12kms^2$, where $ks^3 = mg$.

**20.** **(a)** If $(x, y)$ is a critical point, $y = 0$ and so $\sin x(\omega^2 \cos x - g/l) = 0$. Either $\sin x = 0$ (in which case $x = 0$) of $\cos x = g/\omega^2 l$. But if $\omega^2 < g/l$, $g/\omega^2 l > 1$ and so the latter equation has no real solutions. Therefore $(0, 0)$ is the only critical point if $\omega^2 < g/l$. The Jacobian matrix is

$$\mathbf{g}'(\mathbf{X}) = \begin{pmatrix} 0 & 1 \\ \omega^2 \cos 2x - \dfrac{g}{l} \cos x & -\dfrac{\beta}{ml} \end{pmatrix}$$

and so $\tau = -\beta/ml < 0$ and $\Delta = g/l - \omega^2 > 0$ for $\mathbf{X} = (0, 0)$. It follows that $(0, 0)$ is asymptotically stable and so after a small displacement, the pendulum will return to $\theta = 0$, $\theta' = 0$.

**(b)** If $\omega^2 > g/l$, $\cos x = g/\omega^2 l$ will have two solutions $x = \pm\hat{x}$ that satisfy $-\pi < x < \pi$. Therefore $(\pm\hat{x}, 0)$ are two additional critical points. If $\mathbf{X}_1 = (0, 0)$, $\Delta = g/l - \omega^2 < 0$ and so $(0, 0)$ is a saddle point. If $\mathbf{X}_1 = (\pm\hat{x}, 0)$, $\tau = -\beta/ml < 0$ and

$$\Delta = \frac{g}{l} \cos \hat{x} - \omega^2 \cos 2\hat{x} = \frac{g^2}{\omega^2 l^2} - \omega^2 \left( 2 \frac{g^2}{\omega^4 l^2} - 1 \right) = \omega^2 - \frac{g^2}{\omega^2 l^2} > 0.$$

Therefore $(\hat{x}, 0)$ and $(-\hat{x}, 0)$ are each stable. When $\theta(0) = \theta_0$, $\theta'(0) = 0$ and $\theta_0$ is small we expect the pendulum to reach one of these two stable equilibrium positions.

# Chapter 11

# Fourier Series

## 11.1 Orthogonal Functions

**1.** $\displaystyle\int_{-2}^{2} xx^2\,dx = \frac{1}{4}x^4\Big|_{-2}^{2} = 0$

**2.** $\displaystyle\int_{-1}^{1} x^3(x^2+1)\,dx = \frac{1}{6}x^6\Big|_{-1}^{1} + \frac{1}{4}x^4\Big|_{-1}^{1} = 0$

**3.** $\displaystyle\int_{0}^{2} e^x(xe^{-x}-e^{-x})\,dx = \int_0^2 (x-1)\,dx = \left(\frac{1}{2}x^2 - x\right)\Big|_0^2 = 0$

**4.** $\displaystyle\int_0^{\pi} \cos x\sin^2 x\,dx = \frac{1}{3}\sin^3 x\Big|_0^{\pi} = 0$

**5.** $\displaystyle\int_{-\pi/2}^{\pi/2} x\cos 2x\,dx = \frac{1}{2}\left(\frac{1}{2}\cos 2x + x\sin 2x\right)\Big|_{-\pi/2}^{\pi/2} = 0$

**6.** $\displaystyle\int_{\pi/4}^{5\pi/4} e^x\sin x\,dx = \left(\frac{1}{2}e^x\sin x - \frac{1}{2}e^x\cos x\right)\Big|_{\pi/4}^{5\pi/4} = 0$

**7.** For $m \neq n$

$$\begin{aligned}\int_0^{\pi/2}\sin(2n+1)x\sin(2m+1)x\,dx &= \frac{1}{2}\int_0^{\pi/2}\left(\cos 2(n-m)x - \cos 2(n+m+1)x\right)\,dx\\ &= \frac{1}{4(n-m)}\sin 2(n-m)x\Big|_0^{\pi/2} - \frac{1}{4(n+m+1)}\sin 2(n+m+1)x\Big|_0^{\pi/2} = 0.\end{aligned}$$

For $m = n$

$$\begin{aligned}\int_0^{\pi/2} \sin^2(2n+1)x\,dx &= \int_0^{\pi/2}\left(\frac{1}{2}-\frac{1}{2}\cos 2(2n+1)x\right)dx\\ &= \frac{1}{2}x\Big|_0^{\pi/2} - \frac{1}{4(2n+1)}\sin 2(2n+1)x\Big|_0^{\pi/2} = \frac{\pi}{4}\end{aligned}$$

so that

$$\|\sin(2n+1)x\| = \frac{1}{2}\sqrt{\pi}\,.$$

**8.** For $m \neq n$

$$\begin{aligned}\int_0^{\pi/2} \cos(2n+1)x\cos(2m+1)x\,dx &= \frac{1}{2}\int_0^{\pi/2}(\cos 2(n-m)x + \cos 2(n+m+1)x)\,dx\\ &= \frac{1}{4(n-m)}\sin 2(n-m)x\Big|_0^{\pi/2} + \frac{1}{4(n+m+1)}\sin 2(n+m+1)x\Big|_0^{\pi/2} = 0.\end{aligned}$$

For $m = n$

$$\begin{aligned}\int_0^{\pi/2} \cos^2(2n+1)x\,dx &= \int_0^{\pi/2}\left(\frac{1}{2}+\frac{1}{2}\cos 2(2n+1)x\right)dx\\ &= \frac{1}{2}x\Big|_0^{\pi/2} + \frac{1}{4(2n+1)}\sin 2(2n+1)x\Big|_0^{\pi/2} = \frac{\pi}{4}\end{aligned}$$

so that

$$\|\cos(2n+1)x\| = \frac{1}{2}\sqrt{\pi}\,.$$

**9.** For $m \neq n$

$$\begin{aligned}\int_0^{\pi} \sin nx\sin mx\,dx &= \frac{1}{2}\int_0^{\pi}(\cos(n-m)x - \cos(n+m)x)\,dx\\ &= \frac{1}{2(n-m)}\sin(n-m)x\Big|_0^{\pi} - \frac{1}{2(n+m)}\sin(n+m)x\Big|_0^{\pi} = 0.\end{aligned}$$

For $m = n$

$$\int_0^{\pi}\sin^2 nx\,dx = \int_0^{\pi}\left(\frac{1}{2}-\frac{1}{2}\cos 2nx\right)dx = \frac{1}{2}x\Big|_0^{\pi} - \frac{1}{4n}\sin 2nx\Big|_0^{\pi} = \frac{\pi}{2}$$

so that

$$\|\sin nx\| = \sqrt{\frac{\pi}{2}}\,.$$

**10.** For $m \neq n$

$$\int_0^p \sin\frac{n\pi}{p}x\,\sin\frac{m\pi}{p}x\,dx = \frac{1}{2}\int_0^p \left(\cos\frac{(n-m)\pi}{p}x - \cos\frac{(n+m)\pi}{p}x\right)dx$$

$$= \frac{p}{2(n-m)\pi}\sin\frac{(n-m)\pi}{p}x\Big|_0^p - \frac{p}{2(n+m)\pi}\sin\frac{(n+m)\pi}{p}x\Big|_0^p = 0.$$

For $m = n$

$$\int_0^p \sin^2\frac{n\pi}{p}x\,dx = \int_0^p\left(\frac{1}{2} - \frac{1}{2}\cos\frac{2n\pi}{p}x\right)dx = \frac{1}{2}x\Big|_0^p - \frac{p}{4n\pi}\sin\frac{2n\pi}{p}x\Big|_0^p = \frac{p}{2}$$

so that

$$\left\|\sin\frac{n\pi}{p}x\right\| = \sqrt{\frac{p}{2}}\,.$$

**11.** For $m \neq n$

$$\int_0^p \cos\frac{n\pi}{p}x\,\cos\frac{m\pi}{p}x\,dx = \frac{1}{2}\int_0^p \left(\cos\frac{(n-m)\pi}{p}x + \cos\frac{(n+m)\pi}{p}x\right)dx$$

$$= \frac{p}{2(n-m)\pi}\sin\frac{(n-m)\pi}{p}x\Big|_0^p + \frac{p}{2(n+m)\pi}\sin\frac{(n+m)\pi}{p}x\Big|_0^p = 0.$$

For $m = n$

$$\int_0^p \cos^2\frac{n\pi}{p}x\,dx = \int_0^p\left(\frac{1}{2} + \frac{1}{2}\cos\frac{2n\pi}{p}x\right)dx = \frac{1}{2}x\Big|_0^p + \frac{p}{4n\pi}\sin\frac{2n\pi}{p}x\Big|_0^p = \frac{p}{2}\,.$$

Also

$$\int_0^p 1\cdot\cos\frac{n\pi}{p}x\,dx = \frac{p}{n\pi}\sin\frac{n\pi}{p}x\Big|_0^p = 0 \quad\text{and}\quad \int_0^p 1^2\,dx = p$$

so that

$$\|1\| = \sqrt{p} \quad\text{and}\quad \left\|\cos\frac{n\pi}{p}x\right\| = \sqrt{\frac{p}{2}}\,.$$

**12.** For $m \neq n$, we use Problems 11 and 10:

$$\int_{-p}^p \cos\frac{n\pi}{p}x\,\cos\frac{m\pi}{p}x\,dx = 2\int_0^p \cos\frac{n\pi}{p}x\,\cos\frac{m\pi}{p}x\,dx = 0$$

$$\int_{-p}^p \sin\frac{n\pi}{p}x\,\sin\frac{m\pi}{p}x\,dx = 2\int_0^p \sin\frac{n\pi}{p}x\,\sin\frac{m\pi}{p}x\,dx = 0.$$

Also

$$\int_{-p}^{p}\sin\frac{n\pi}{p}x\,\cos\frac{m\pi}{p}x\,dx=\frac{1}{2}\int_{-p}^{p}\left(\sin\frac{(n-m)\pi}{p}x+\sin\frac{(n+m)\pi}{p}x\right)\,dx=0,$$

$$\int_{-p}^{p}1\cdot\cos\frac{n\pi}{p}x\,dx=\frac{p}{n\pi}\sin\frac{n\pi}{p}x\bigg|_{-p}^{p}=0,$$

$$\int_{-p}^{p}1\cdot\sin\frac{n\pi}{p}x\,dx=-\frac{p}{n\pi}\cos\frac{n\pi}{p}x\bigg|_{-p}^{p}=0,$$

and

$$\int_{-p}^{p}\sin\frac{n\pi}{p}x\,\cos\frac{n\pi}{p}x\,dx=\int_{-p}^{p}\frac{1}{2}\sin\frac{2n\pi}{p}x\,dx=\frac{-}{p}4n\pi\cos\frac{2n\pi}{p}x\bigg|_{-p}^{p}=0.$$

For $m=n$

$$\int_{-p}^{p}\cos^2\frac{n\pi}{p}x\,dx=\int_{-p}^{p}\left(\frac{1}{2}+\frac{1}{2}\cos\frac{2n\pi}{p}x\right)\,dx=p,$$

$$\int_{-p}^{p}\sin^2\frac{n\pi}{p}x\,dx=\int_{-p}^{p}\left(\frac{1}{2}-\frac{1}{2}\cos\frac{2n\pi}{p}x\right)\,dx=p,$$

and

$$\int_{-p}^{p}1^2\,dx=2p$$

so that

$$\|1\|=\sqrt{2p}\,,\qquad\left\|\cos\frac{n\pi}{p}x\right\|=\sqrt{p}\,,\quad\text{and}\quad\left\|\sin\frac{n\pi}{p}x\right\|=\sqrt{p}\,.$$

**13.** Since

$$\int_{-\infty}^{\infty}e^{-x^2}\cdot 1\cdot 2x\,dx=0,\quad\longleftarrow\ \text{integrand is an odd function}$$

$$\begin{aligned}\int_{-\infty}^{\infty}e^{-x^2}\cdot 1\cdot(4x^2-2)\,dx&=2\int_{0}^{\infty}e^{-x^2}(4x^2-2)\,dx\quad\longleftarrow\ \text{integrand is an even function}\\&=4\int_{0}^{\infty}x\left(2xe^{-x^2}\right)\,dx-4\int_{0}^{\infty}e^{-x^2}\,dx\\&=4\left(-xe^{-x^2}\bigg|_{0}^{\infty}+\int_{0}^{\infty}e^{-x^2}\,dx\right)-4\int_{0}^{\infty}e^{-x^2}\,dx\\&=-4xe^{-x^2}\bigg|_{0}^{\infty}+4\int_{0}^{\infty}e^{-x^2}\,dx-4\int_{0}^{\infty}e^{-x^2}\,dx=0\end{aligned}$$

and

$$\int_{-\infty}^{\infty}e^{-x^2}\cdot 2x\cdot(4x^2-2)\,dx=0,\quad\longleftarrow\ \text{integrand is an odd function}$$

the functions are orthogonal.

**14.** Since

$$\int_0^\infty e^{-x}\cdot 1(1-x)\,dx = (x-1)e^{-x}\Big|_0^\infty - \int_0^\infty e^{-x}\,dx = 0,$$

$$\begin{aligned}\int_0^\infty e^{-x}\cdot 1\cdot\left(\frac{1}{2}x^2-2x+1\right)dx &= \left(2x-1-\frac{1}{2}x^2\right)e^{-x}\Big|_0^\infty + \int_0^\infty e^{-x}(x-2)\,dx\\ &= 1+(2-x)e^{-x}\Big|_0^\infty + \int_0^\infty e^{-x}\,dx = 0,\end{aligned}$$

and

$$\begin{aligned}\int_0^\infty e^{-x}\cdot(1-x)\left(\frac{1}{2}x^2-2x+1\right)dx &= \int_0^\infty e^{-x}\left(-\frac{1}{2}x^3+\frac{5}{2}x^2-3x+1\right)dx\\ &= e^{-x}\left(\frac{1}{2}x^3-\frac{5}{2}x^2+3x-1\right)\Big|_0^\infty + \int_0^\infty e^{-x}\left(-\frac{3}{2}x^2+5x-3\right)dx\\ &= 1+e^{-x}\left(\frac{3}{2}x^2-5x+3\right)\Big|_0^\infty + \int_0^\infty e^{-x}(5-3x)\,dx\\ &= 1-3+e^{-x}(3x-5)\Big|_0^\infty - 3\int_0^\infty e^{-x}\,dx = 0,\end{aligned}$$

the functions are orthogonal.

**15.** By orthogonality $\int_a^b \phi_0(x)\phi_n(x)dx = 0$ for $n = 1, 2, 3, \ldots$; that is, $\int_a^b \phi_n(x)\,dx = 0$ for $n = 1, 2, 3, \ldots$.

**16.** Using the facts that $\phi_0$ and $\phi_1$ are orthogonal to $\phi_n$ for $n > 1$, we have

$$\begin{aligned}\int_a^b (\alpha x+\beta)\phi_n(x)\,dx &= \alpha\int_a^b x\phi_n(x)\,dx + \beta\int_a^b 1\cdot\phi_n(x)\,dx\\ &= \alpha\int_a^b \phi_1(x)\phi_n(x)\,dx + \beta\int_a^b \phi_0(x)\phi_n(x)\,dx\\ &= \alpha\cdot 0+\beta\cdot 0 = 0\end{aligned}$$

for $n = 2, 3, 4, \ldots$.

**17.** Using the fact that $\phi_n$ and $\phi_m$ are orthogonal for $n \neq m$ we have

$$\begin{aligned}\|\phi_m(x)+\phi_n(x)\|^2 &= \int_a^b [\phi_m(x)+\phi_n(x)]^2\,dx = \int_a^b \left[\phi_m^2(x)+2\phi_m(x)\phi_n(x)+\phi_n^2(x)\right]dx\\ &= \int_a^b \phi_m^2(x)\,dx + 2\int_a^b \phi_m(x)\phi_n(x)\,dx + \int_a^b \phi_n^2(x)\,dx\\ &= \|\phi_m(x)\|^2+\|\phi_n(x)\|^2.\end{aligned}$$

**18.** Setting

$$0 = \int_{-2}^{2} f_3(x) f_1(x)\, dx = \int_{-2}^{2} \left(x^2 + c_1 x^3 + c_2 x^4\right)\, dx = \frac{16}{3} + \frac{64}{5} c_2$$

and

$$0 = \int_{-2}^{2} f_3(x) f_2(x)\, dx = \int_{-2}^{2} \left(x^3 + c_1 x^4 + c_2 x^5\right)\, dx = \frac{64}{5} c_1$$

we obtain $c_1 = 0$ and $c_2 = -5/12$.

**19.** The fundamental period is $2\pi/2\pi = 1$.

**20.** The fundamental period is $2\pi/(4/L) = \frac{1}{2}\pi L$.

**21.** The fundamental period of $\sin x + \sin 2x$ is $2\pi$.

**22.** The fundamental period of $\sin 2x + \cos 4x$ is $2\pi/2 = \pi$.

**23.** The fundamental period of $\sin 3x + \cos 2x$ is $2\pi$ since the smallest integer multiples of $2\pi/3$ and $2\pi/2 = \pi$ that are equal are 3 and 2, respectively.

**24.** The fundamental period of $\sin^2 \pi x$ is 1 since

$$\begin{aligned} \sin \pi(x+1) &= \sin(\pi x + \pi) = \sin \pi x \cos \pi + \cos \pi x \sin \pi = -\sin \pi x \\ \sin^2 \pi(x+1) &= (-\sin \pi x)^2 = \sin^2 \pi x. \end{aligned}$$

Alternatively, we can also use the trigonometric identity $\sin^2 \pi x = \frac{1}{2}(1 - \cos 2\pi x)$.

**25.** **(a)** For $m$ and $n$ positive integers, we have orthogonality on the interval $[-\pi, \pi]$ because

$$\begin{aligned} \int_{-\pi}^{\pi} \sin nx \sin mx\, dx &= \frac{1}{2} \int_{-\pi}^{\pi} [\cos(m-n)x - \cos(m+n)x]\, dx \\ &= \frac{1}{2} \left[ \frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n} \right]_{-\pi}^{\pi} = 0, \quad m \neq n \end{aligned}$$

**(b)** The function $f(x) = 1$ is continuous on $[-\pi, \pi]$ but is orthogonal to every function in the orthogonal set:

$$\int_{\pi}^{\pi} 1 \cdot \sin nx\, dx = \frac{\cos \pi - \cos(-\pi)}{n} = 0, \quad n = 1, 2, 3, \ldots$$

Therefore the set $\{\sin nx\}$, $n = 1, 2, 3, \ldots$ is not complete on the interval $[-\pi, \pi]$.

**26.** **(a)** Following the pattern established by $\phi_1(x)$ and $\phi_2(x)$ we have

$$\phi_3(x) = f_3(x) - \frac{(f_3, \phi_0)}{(\phi_0, \phi_0)}\, \phi_0(x) - \frac{(f_3, \phi_1)}{(\phi_1, \phi_1)}\, \phi_1(x) - \frac{(f_3, \phi_2)}{(\phi_2, \phi_2)}\, \phi_2(x).$$

**(b)** To show mutual orthogonality we compute $(\phi_0, \phi_1)$, $(\phi_0, \phi_2)$, and $(\phi_1, \phi_2)$ using properties (*i*), (*ii*), (*iii*), and (*iv*) from this section in the text.

$$(\phi_0, \phi_1) = \left(\phi_0, f_1 - \frac{(f_1, \phi_0)}{(\phi_0, \phi_0)}\,\phi_0\right) = (\phi_0, f_1) - \frac{(f_1, \phi_0)}{(\phi_0, \phi_0)}\,(\phi_0, \phi_0) = (\phi_0, f_1) - (f_1, \phi_0) = 0$$

$$\begin{aligned}(\phi_0, \phi_2) &= \left(\phi_0, f_2 - \frac{(f_2, \phi_0)}{(\phi_0, \phi_0)}\,\phi_0 - \frac{(f_2, \phi_1)}{(\phi_1, \phi_1)}\,\phi_1\right) = (\phi_0, f_2) - \frac{(f_2, \phi_0)}{(\phi_0, \phi_0)}\,(\phi_0, \phi_0) - \frac{(f_2, \phi_1)}{(\phi_1, \phi_1)}\,(\phi_0, \phi_1)\\ &= (\phi_0, f_2) - (f_2, \phi_0) - 0 = 0\end{aligned}$$

$$\begin{aligned}(\phi_1, \phi_2) &= \left(\phi_1, f_2 - \frac{(f_2, \phi_0)}{(\phi_0, \phi_0)}\,\phi_0 - \frac{(f_2, \phi_1)}{(\phi_1, \phi_1)}\,\phi_1\right) = (\phi_1, f_2) - \frac{(f_2, \phi_0)}{(\phi_0, \phi_0)}\,(\phi_1, \phi_0) - \frac{(f_2, \phi_1)}{(\phi_1, \phi_1)}\,(\phi_1, \phi_1)\\ &= (\phi_1, f_2) - 0 - (f_2, \phi_1) = 0.\end{aligned}$$

**27.** First we identify $f_0(x) = 1, f_1(x) = x, f_2(x) = x^2$, and $f_3(x) = x^3$. Then, we use the formulas from Problem 26. First, we have $\phi_0(x) = f_0(x) = 1$. Then

$$(f_1, \phi_0) = (x, 1) = \int_{-1}^{1} x\,dx = 0 \quad \text{and} \quad (\phi_0, \phi_0) = \int_{-1}^{1} 1\,dx = 2,$$

so

$$\phi_1(x) = f_1(x) - \frac{(f_1, \phi_0)}{(\phi_0, \phi_0)}\,\phi_0(x) = x - \frac{0}{2}\,(1) = x.$$

Next

$$(f_2, \phi_0) = (x^2, 1) = \int_{-1}^{1} x^2\,dx = \frac{2}{3}, \;\; (f_2, \phi_1) = (x^2, x) = \int_{-1}^{1} x^3\,dx = 0,$$

$$\text{and } (\phi_1, \phi_1) = \int_{-1}^{1} x^2\,dx = \frac{2}{3},$$

so

$$\phi_2(x) = f_2(x) - \frac{(f_2, \phi_0)}{(\phi_0, \phi_0)}\,\phi_0(x) - \frac{(f_2, \phi_1)}{(\phi_1, \phi_1)}\,\phi_1(x) = x^2 - \frac{2/3}{2}\,(1) - \frac{0}{2}\,(x) = x^2 - \frac{1}{3}\,.$$

Finally,

$$(f_3, \phi_0) = (x^3, 1) = \int_{-1}^{1} x^3\,dx = 0, \quad (f_3, \phi_1) = (x^3, x) = \int_{-1}^{1} x^4\,dx = \frac{2}{5},$$

and

$$(f_3, \phi_2) = \left(x^3, x^2 - \frac{1}{3}\right) = \int_{-1}^{1} \left(x^5 - \frac{1}{3}\,x^3\right)\,dx = 0,$$

so

$$\begin{aligned}\phi_3(x) &= f_3(x) - \frac{(f_3, \phi_0)}{(\phi_0, \phi_0)}\,\phi_0(x) - \frac{(f_3, \phi_1)}{(\phi_1, \phi_1)}\,\phi_1(x) - \frac{(f_3, \phi_2)}{(\phi_2, \phi_2)}\,\phi_2(x)\\ &= x^3 - 0 - \frac{2/5}{2/3}\,(x) - 0 = x^3 - \frac{3}{5}\,x.\end{aligned}$$

**28.** Let $P_n(x)$ denote the Legendre polynomials given in Section 6.4. The entries in the orthogonal set found by the Gram-Schmidt process in Problem 27 are proportional to the Legenre polynomials.

$$\begin{aligned} \phi_0 &= P_0 = 1 \\ \phi_1 &= P_1 = x \\ \frac{3}{2}\phi_2 &= P_2 = \frac{1}{2}\left(3x^2 - 1\right) \\ \frac{5}{2}\phi_3 &= P_3 = \frac{1}{2}\left(5x^3 - 3x\right) \end{aligned}$$

## 11.2 Fourier Series

**1.** $a_0 = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\,dx = \dfrac{1}{\pi}\int_0^{\pi} 1\,dx = 1$

$a_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\cos\frac{n\pi}{pi}x\,dx = \dfrac{1}{\pi}\int_0^{\pi}\cos nx\,dx = 0$

$b_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\sin\frac{n\pi}{\pi}x\,dx = \dfrac{1}{\pi}\int_0^{\pi}\sin nx\,dx = \dfrac{1}{n\pi}(1-\cos n\pi) = \dfrac{1}{n\pi}[1-(-1)^n]$

$f(x) = \dfrac{1}{2} + \dfrac{1}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n}\sin nx$

Converges to $\frac{1}{2}$ at $x = 0$.

**2.** $a_0 = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\,dx = \dfrac{1}{\pi}\int_{-\pi}^{0}(-1)\,dx + \dfrac{1}{\pi}\int_0^{\pi} 2\,dx = 1$

$a_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \dfrac{1}{\pi}\int_{-\pi}^{0}(-\cos nx)\,dx + \dfrac{1}{\pi}\int_0^{\pi} 2\cos nx\,dx = 0$

$b_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \dfrac{1}{\pi}\int_{-\pi}^{0}(-\sin nx)\,dx + \dfrac{1}{\pi}\int_0^{\pi} 2\sin nx\,dx = \dfrac{3}{n\pi}[1-(-1)^n]$

$f(x) = \dfrac{1}{2} + \dfrac{3}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n}\sin nx$

Converges to $\frac{1}{2}$ at $x = 0$.

**3.** $a_0 = \displaystyle\int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} 1\,dx + \int_0^{1} x\,dx = \frac{3}{2}$

$a_n = \displaystyle\int_{-1}^{1} f(x)\cos n\pi x\,dx = \int_{-1}^{0}\cos n\pi x\,dx + \int_0^{1} x\cos n\pi x\,dx = \frac{1}{n^2\pi^2}[(-1)^n - 1]$

$b_n = \displaystyle\int_{-1}^{1} f(x)\sin n\pi x\,dx = \int_{-1}^{0}\sin n\pi x\,dx + \int_0^{1} x\sin n\pi x\,dx = -\frac{1}{n\pi}$

$$f(x) = \frac{3}{4} + \sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2\pi^2}\cos n\pi x - \frac{1}{n\pi}\sin n\pi x\right]$$

Converges to $\frac{1}{2}$ at $x = 0$.

**4.** $a_0 = \int_{-1}^{1} f(x)\,dx = \int_0^1 x\,dx = \frac{1}{2}$

$$a_n = \int_{-1}^{1} f(x)\cos n\pi x\,dx = \int_0^1 x\cos n\pi x\,dx = \frac{1}{n^2\pi^2}[(-1)^n - 1]$$

$$b_n = \int_{-1}^{1} f(x)\sin n\pi x\,dx = \int_0^1 x\sin n\pi x\,dx = \frac{(-1)^{n+1}}{n\pi}$$

$$f(x) = \frac{1}{4} + \sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2\pi^2}\cos n\pi x + \frac{(-1)^{n+1}}{n\pi}\sin n\pi x\right]$$

$f$ is continous on the interval

**5.** $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} x^2\,dx = \frac{1}{3}\pi^2$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{\pi} x^2\cos nx\,dx = \frac{1}{\pi}\left(\frac{x^2}{\pi}\sin nx\bigg|_0^{\pi} - \frac{2}{n}\int_0^{\pi} x\sin nx\,dx\right)$$

$$= \frac{2(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} x^2\sin nx\,dx = \frac{1}{\pi}\left(-\frac{x^2}{n}\cos nx\bigg|_0^{\pi} + \frac{2}{n}\int_0^{\pi} x\cos nx\,dx\right)$$

$$= \frac{\pi}{n}(-1)^{n+1} + \frac{2}{n^3\pi}[(-1)^n - 1]$$

$$f(x) = \frac{\pi^2}{6} + \sum_{n=1}^{\infty}\left[\frac{2(-1)^n}{n^2}\cos nx + \left(\frac{\pi}{n}(-1)^{n+1} + \frac{2[(-1)^n - 1]}{n^3\pi}\right)\sin nx\right]$$

$f$ is continous on the interval

**6.** $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} \pi^2\,dx + \frac{1}{\pi}\int_0^{\pi}\left(\pi^2 - x^2\right)\,dx = \frac{5}{3}\pi^2$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0}\pi^2\cos nx\,dx + \frac{1}{\pi}\int_0^{\pi}\left(\pi^2 - x^2\right)\cos nx\,dx$$

$$= \frac{1}{\pi}\left(\frac{\pi^2 - x^2}{n}\sin nx\bigg|_0^{\pi} + \frac{2}{n}\int_0^{\pi} x\sin nx\,dx\right) = \frac{2}{n^2}(-1)^{n+1}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0}\pi^2\sin nx\,dx + \frac{1}{\pi}\int_0^{\pi}\left(\pi^2 - x^2\right)\sin nx\,dx$$

$$= \frac{\pi}{n}[(-1)^n - 1] + \frac{1}{\pi}\left(\frac{x^2 - \pi^2}{n}\cos nx\bigg|_0^{\pi} - \frac{2}{n}\int_0^{\pi} x\cos nx\,dx\right) = \frac{\pi}{n}(-1)^n + \frac{2}{n^3\pi}[1 - (-1)^n]$$

$$f(x)=\frac{5\pi^2}{6}+\sum_{n=1}^{\infty}\left[\frac{2}{n^2}(-1)^{n+1}\cos nx+\left(\frac{\pi}{n}(-1)^n+\frac{2[1-(-1)^n]}{n^3\pi}\right)\sin nx\right]$$

$f$ is continous on the interval

**7.** $a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\,dx=\frac{1}{\pi}\int_{-\pi}^{\pi}(x+\pi)\,dx=2\pi$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx=\frac{1}{\pi}\int_{-\pi}^{\pi}(x+\pi)\cos nx\,dx=0$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx=\frac{2}{n}(-1)^{n+1}$$

$$f(x)=\pi+\sum_{n=1}^{\infty}\frac{2}{n}(-1)^{n+1}\sin nx$$

$f$ is continous on the interval

**8.** $a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\,dx=\frac{1}{\pi}\int_{-\pi}^{\pi}(3-2x)\,dx=6$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx=\frac{1}{\pi}\int_{-\pi}^{\pi}(3-2x)\cos nx\,dx=0$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}(3-2x)\sin nx\,dx=\frac{4}{n}(-1)^n$$

$$f(x)=3+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx$$

$f$ is continous on the interval

**9.** $a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\,dx=\frac{1}{\pi}\int_{0}^{\pi}\sin x\,dx=\frac{2}{\pi}$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx=\frac{1}{\pi}\int_{0}^{\pi}\sin x\,\cos nx\,dx=\frac{1}{2\pi}\int_{0}^{\pi}\left(\sin(1+n)x+\sin(1-n)x\right)\,dx$$

$$=\frac{1+(-1)^n}{\pi(1-n^2)}\quad \text{for} n=2,3,4,\ldots$$

$$a_1=\frac{1}{2\pi}\int_{0}^{\pi}\sin 2x\,dx=0$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx=\frac{1}{\pi}\int_{0}^{\pi}\sin x\,\sin nx\,dx$$

$$=\frac{1}{2\pi}\int_{0}^{\pi}\left(\cos(1-n)x-\cos(1+n)x\right)\,dx=0\quad \text{for} n=2,3,4,\ldots$$

$$b_1=\frac{1}{2\pi}\int_{0}^{\pi}(1-\cos 2x)\,dx=\frac{1}{2}$$

$$f(x)=\frac{1}{\pi}+\frac{1}{2}\sin x+\sum_{n=2}^{\infty}\frac{1+(-1)^n}{\pi(1-n^2)}\cos nx$$

$f$ is continous on the interval

**10.** $$a_0 = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi/2}\cos x\,dx = \frac{2}{\pi}$$

$$a_n = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} f(x)\cos 2nx\,dx = \frac{2}{\pi}\int_0^{\pi/2}\cos x\,\cos 2nx\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi/2}\left(\cos(2n-1)x + \cos(2n+1)x\right)\,dx = \frac{2(-1)^{n+1}}{\pi(4n^2-1)}$$

$$b_n = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} f(x)\sin 2nx\,dx = \frac{2}{\pi}\int_0^{\pi/2}\cos x\,\sin 2nx\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi/2}\left(\sin(2n-1)x + \sin(2n+1)x\right)\,dx = \frac{4n}{\pi(4n^2-1)}$$

$$f(x) = \frac{1}{\pi} + \sum_{n=1}^{\infty}\left[\frac{2(-1)^{n+1}}{\pi(4n^2-1)}\cos 2nx + \frac{4n}{\pi(4n^2-1)}\sin 2nx\right]$$

Converges to $\frac{1}{2}$ at $x = 0$.

**11.** $$a_0 = \frac{1}{2}\int_{-2}^{2} f(x)\,dx = \frac{1}{2}\left(\int_{-1}^{0} -2\,dx + \int_0^1 1\,dx\right) = -\frac{1}{2}$$

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi}{2}x\,dx = \frac{1}{2}\left(\int_{-1}^{0}(-2)\cos\frac{n\pi}{2}x\,dx + \int_0^1\cos\frac{n\pi}{2}x\,dx\right) = -\frac{1}{n\pi}\sin\frac{n\pi}{2}$$

$$b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{n\pi}{2}x\,dx = \frac{1}{2}\left(\int_{-1}^{0}(-2)\sin\frac{n\pi}{2}x\,dx + \int_0^1\sin\frac{n\pi}{2}x\,dx\right) = \frac{3}{n\pi}\left(1-\cos\frac{n\pi}{2}\right)$$

$$f(x) = -\frac{1}{4} + \sum_{n=1}^{\infty}\left[-\frac{1}{n\pi}\sin\frac{n\pi}{2}\,\cos\frac{n\pi}{2}x + \frac{3}{n\pi}\left(1-\cos\frac{n\pi}{2}\right)\sin\frac{n\pi}{2}x\right]$$

Converges to $-1$ at $x = -1$, $-\frac{1}{2}$ at $x = 0$, and $\frac{1}{2}$ at $x = 1$.

**12.** $$a_0 = \frac{1}{2}\int_{-2}^{2} f(x)\,dx = \frac{1}{2}\left(\int_0^1 x\,dx + \int_1^2 1\,dx\right) = \frac{3}{4}$$

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi}{2}x\,dx = \frac{1}{2}\left(\int_0^1 x\cos\frac{n\pi}{2}x\,dx + \int_1^2\cos\frac{n\pi}{2}x\,dx\right) = \frac{2}{n^2\pi^2}\left(\cos\frac{n\pi}{2}-1\right)$$

$$b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{n\pi}{2}x\,dx = \frac{1}{2}\left(\int_0^1 x\sin\frac{n\pi}{2}x\,dx + \int_1^2\sin\frac{n\pi}{2}x\,dx\right)$$

$$= \frac{2}{n^2\pi^2}\left(\sin\frac{n\pi}{2} + \frac{n\pi}{2}(-1)^{n+1}\right)$$

$$f(x) = \frac{3}{8} + \sum_{n=1}^{\infty}\left[\frac{2}{n^2\pi^2}\left(\cos\frac{n\pi}{2}-1\right)\cos\frac{n\pi}{2}x + \frac{2}{n^2\pi^2}\left(\sin\frac{n\pi}{2} + \frac{n\pi}{2}(-1)^{n+1}\right)\sin\frac{n\pi}{2}x\right]$$

$f$ is continous on the interval

**13.** $a_0 = \dfrac{1}{5}\displaystyle\int_{-5}^{5} f(x)\,dx = \dfrac{1}{5}\left(\int_{-5}^{0} 1\,dx + \int_{0}^{5} (1+x)\,dx\right) = \dfrac{9}{2}$

$$a_n = \frac{1}{5}\int_{-5}^{5} f(x)\cos\frac{n\pi}{5}x\,dx = \frac{1}{5}\left(\int_{-5}^{0} \cos\frac{n\pi}{5}x\,dx + \int_{0}^{5} (1+x)\cos\frac{n\pi}{5}x\,dx\right)$$

$$= \frac{5}{n^2\pi^2}[(-1)^n - 1]$$

$$b_n = \frac{1}{5}\int_{-5}^{5} f(x)\sin\frac{n\pi}{5}x\,dx = \frac{1}{5}\left(\int_{-5}^{0} \sin\frac{n\pi}{5}x\,dx + \int_{0}^{5} (1+x)\cos\frac{n\pi}{5}x\,dx\right) = \frac{5}{n\pi}(-1)^{n+1}$$

$$f(x) = \frac{9}{4} + \sum_{n=1}^{\infty}\left[\frac{5}{n^2\pi^2}[(-1)^n - 1]\cos\frac{n\pi}{5}x + \frac{5}{n\pi}(-1)^{n+1}\sin\frac{n\pi}{5}x\right]$$

$f$ is continous on the interval

**14.** $a_0 = \dfrac{1}{2}\displaystyle\int_{-2}^{2} f(x)\,dx = \dfrac{1}{2}\left(\int_{-2}^{0} (2+x)\,dx + \int_{0}^{2} 2\,dx\right) = 3$

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi}{2}x\,dx = \frac{1}{2}\left(\int_{-2}^{0} (2+x)\cos\frac{n\pi}{2}x\,dx + \int_{0}^{2} 2\cos\frac{n\pi}{2}x\,dx\right) = \frac{2}{n^2\pi^2}[1 - (-1)^n]$$

$$b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{n\pi}{2}x\,dx = \frac{1}{2}\left(\int_{-2}^{0} (2+x)\sin\frac{n\pi}{2}x\,dx + \int_{0}^{2} 2\sin\frac{n\pi}{2}x\,dx\right) = \frac{2}{n\pi}(-1)^{n+1}$$

$$f(x) = \frac{3}{2} + \sum_{n=1}^{\infty}\left[\frac{2}{n^2\pi^2}[1 - (-1)^n]\cos\frac{n\pi}{2}\,x + \frac{2}{n\pi}(-1)^{n+1}\sin\frac{n\pi}{2}\,x\right]$$

$f$ is continous on the interval

**15.** $a_0 = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\,dx = \dfrac{1}{\pi}\int_{-\pi}^{\pi} e^x\,dx = \dfrac{1}{\pi}(e^{\pi} - e^{-\pi})$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{(-1)^n(e^{\pi} - e^{-\pi})}{\pi(1+n^2)}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x\sin nx\,dx = \frac{(-1)^n n(e^{-\pi} - e^{\pi})}{\pi(1+n^2)}$$

$$f(x) = \frac{e^{\pi} - e^{-\pi}}{2\pi} + \sum_{n=1}^{\infty}\left[\frac{(-1)^n(e^{\pi} - e^{-\pi})}{\pi(1+n^2)}\cos nx + \frac{(-1)^n n(e^{-\pi} - e^{\pi})}{\pi(1+n^2)}\sin nx\right]$$

$f$ is continous on the interval

**16.** $a_0 = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\,dx = \dfrac{1}{\pi}\int_{0}^{\pi} (e^x - 1)\,dx = \dfrac{1}{\pi}(e^{\pi} - \pi - 1)$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{0}^{\pi} (e^x - 1)\cos nx\,dx = \frac{[e^{\pi}(-1)^n - 1]}{\pi(1+n^2)}$$

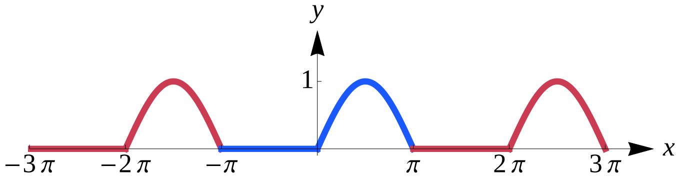
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{0}^{\pi} (e^x - 1)\sin nx\,dx$$

$$= \frac{1}{\pi}\left(\frac{ne^{\pi}(-1)^{n+1}}{1+n^2} + \frac{n}{1+n^2} + \frac{(-1)^n}{n} - \frac{1}{n}\right)$$

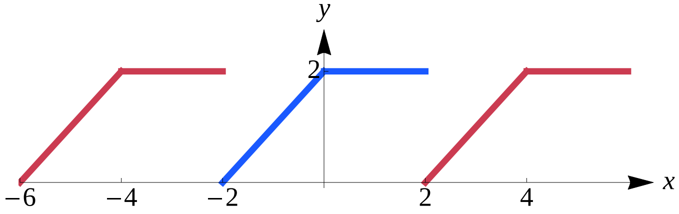
$$f(x) = \frac{e^{\pi} - \pi - 1}{2\pi} + \sum_{n=1}^{\infty} \left[ \frac{e^{\pi}(-1)^n - 1}{\pi(1+n^2)} \cos nx \right.$$
$$\left. + \left( \frac{n}{1+n^2} \left[ e^{\pi}(-1)^{n+1} + 1 \right] + \frac{(-1)^n - 1}{n} \right) \sin nx \right]$$

$f$ is continous on the interval

**17.**



**18.**



**19.** The function in Problem 5 is discontinuous at $x = \pi$, so the corresponding Fourier series converges to $\pi^2/2$ at $x = \pi$. That is,

$$\frac{\pi^2}{2} = \frac{\pi^2}{6} + \sum_{n=1}^{\infty} \left[ \frac{2(-1)^n}{n^2} \cos n\pi + \left( \frac{\pi}{n}(-1)^{n+1} + \frac{2[(-1)^n - 1]}{n^3\pi} \right) \sin n\pi \right]$$
$$= \frac{\pi^2}{6} + \sum_{n=1}^{\infty} \frac{2(-1)^n}{n^2}(-1)^n = \frac{\pi^2}{6} + \sum_{n=1}^{\infty} \frac{2}{n^2} = \frac{\pi^2}{6} + 2\left( 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \right)$$

and

$$\frac{\pi^2}{6} = \frac{1}{2}\left( \frac{\pi^2}{2} - \frac{\pi^2}{6} \right) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots .$$

At $x = 0$ the series converges to 0 and

$$0 = \frac{\pi^2}{6} + \sum_{n=1}^{\infty} \frac{2(-1)^n}{n^2} = \frac{\pi^2}{6} + 2\left( -1 + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \cdots \right)$$

so

$$\frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots .$$

**20.** From Problem 19

$$\frac{\pi^2}{8} = \frac{1}{2}\left( \frac{\pi^2}{6} + \frac{\pi^2}{12} \right) = \frac{1}{2}\left( 2 + \frac{2}{3^2} + \frac{2}{5^2} + \cdots \right) = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots .$$

**21.** The function in Problem 7 is continuous at $x = \pi/2$ so

$$\frac{3\pi}{2} = f\left( \frac{\pi}{2} \right) = \pi + \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1} \sin \frac{n\pi}{2} = \pi + 2\left( 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \right)$$

and

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots .$$

**22.** The function in Problem 9 is continuous at $x = \pi/2$ so

$$1 = f\left(\frac{\pi}{2}\right) = \frac{1}{\pi} + \frac{1}{2} + \sum_{n=2}^{\infty} \frac{1+(-1)^n}{\pi(1-n^2)} \cos\frac{n\pi}{2}$$

$$1 = \frac{1}{\pi} + \frac{1}{2} + \frac{2}{3\pi} - \frac{2}{3\cdot 5\pi} + \frac{2}{5\cdot 7\pi} - \cdots$$

and

$$\pi = 1 + \frac{\pi}{2} + \frac{2}{3} - \frac{2}{3\cdot 5} + \frac{2}{5\cdot 7} - \cdots$$

or

$$\frac{\pi}{4} = \frac{1}{2} + \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \cdots .$$

**23.** **(a)** Letting $c_0 = a_0/2$, $c_n = (a_n - ib_n)/2$, and $c_{-n} = (a_n + ib_n)/2$ we have

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{p}x + b_n \sin\frac{n\pi}{p}x\right) \\ &= c_0 + \sum_{n=1}^{\infty}\left(a_n \frac{e^{in\pi x/p} + e^{-in\pi x/p}}{2} - b_n \frac{ie^{in\pi x/p} - ie^{-in\pi x/p}}{2}\right) \\ &= c_0 + \sum_{n=1}^{\infty}\left(c_n e^{in\pi x/p} + c_{-n} e^{i(-n)\pi x/p}\right) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/p}. \end{aligned}$$

**(b)** From part (a) we have

$$c_n = \frac{1}{2}(a_n - ib_n) = \frac{1}{2p}\int_{-p}^{p} f(x)\left(\cos\frac{n\pi}{p}x - i\sin\frac{n\pi}{p}x\right)dx = \frac{1}{2p}\int_{-p}^{p} f(x)e^{-in\pi x/p}\,dx$$

and

$$c_{-n} = \frac{1}{2}(a_n + ib_n) = \frac{1}{2p}\int_{-p}^{p} f(x)\left(\cos\frac{n\pi}{p}x + i\sin\frac{n\pi}{p}x\right)dx = \frac{1}{2p}\int_{-p}^{p} f(x)e^{in\pi x/p}\,dx$$

for $n = 1, 2, 3, \ldots$. Thus, for $n = \pm 1, \pm 2, \pm 3, \ldots,$

$$c_n = \frac{1}{2p}\int_{-p}^{p} f(x)e^{-in\pi x/p}\,dx.$$

When $n = 0$ the above formula gives

$$c_0 = \frac{1}{2p}\int_{-p}^{p} f(x)\,dx,$$

which is $a_0/2$ where $a_0$ is (9) in the text. Therefore

$$c_n = \frac{1}{2p}\int_{-p}^{p} f(x)e^{-in\pi x/p}\,dx, \quad n = 0, \pm 1, \pm 2, \ldots .$$

**24.** Identifying $f(x) = e^{-x}$ and $p = \pi$, we have

$$\begin{aligned}
c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-x}e^{inx}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-(in+1)x}\,dx \\
&= -\frac{1}{2(in+1)\pi}\,e^{-(in+1)x}\Big|_{\pi}^{\pi} \\
&= -\frac{1}{2(in+1)\pi}\left[e^{-(in+1)\pi} - e^{(in+1)\pi}\right] \\
&= \frac{e^{(in+1)\pi} - e^{-(in+1)\pi}}{2(in+1)\pi} \\
&= \frac{e^{\pi}(\cos n\pi + i\sin n\pi) - e^{-\pi}(\cos n\pi - i\sin n\pi)}{2(in+1)\pi} \\
&= \frac{(e^{\pi} - e^{-\pi})\cos n\pi}{2(in+1)\pi} = \frac{(e^{\pi} - e^{-\pi})(-1)^n}{2(in+1)\pi}\,.
\end{aligned}$$

Thus

$$f(x) = \sum_{n=-\infty}^{\infty} (-1)^n\,\frac{e^{\pi} - e^{-\pi}}{2(in+1)\pi}\,e^{inx}.$$

## 11.3 Fourier Cosine and Sine Series

**1.** Since $f(-x) = \sin(-3x) = -\sin 3x = -f(x)$, $f(x)$ is an odd function.

**2.** Since $f(-x) = -x\cos(-x) = -x\cos x = -f(x)$, $f(x)$ is an odd function.

**3.** Since $f(-x) = (-x)^2 - x = x^2 - x$, $f(x)$ is neither even nor odd.

**4.** Since $f(-x) = (-x)^3 + 4x = -(x^3 - 4x) = -f(x)$, $f(x)$ is an odd function.

**5.** Since $f(-x) = e^{|-x|} = e^{|x|} = f(x)$, $f(x)$ is an even function.

**6.** Since $f(-x) = e^{-x} - e^{x} = -f(x)$, $f(x)$ is an odd function.

**7.** For $0 < x < 1$, $f(-x) = (-x)^2 = x^2 = -f(x)$, $f(x)$ is an odd function.

**8.** For $0 \le x < 2$, $f(-x) = -x + 5 = f(x)$, $f(x)$ is an even function.

**9.** Since $f(x)$ is not defined for $x < 0$, it is neither even nor odd.

**10.** Since $f(-x) = \left|(-x)^5\right| = \left|x^5\right| = f(x)$, $f(x)$ is an even function.

**11.** Since $f(x)$ is an odd function, we have

$$b_n = \frac{2}{\pi}\int_0^{\pi} 1 \cdot \sin nx\, dx = \frac{2}{n\pi}\left[1-(-1)^n\right]$$

Thus

$$f(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi}\left[1-(-1)^n\right]\sin nx.$$

**12.** Since $f(x)$ is an even function, we expand in a cosine series:

$$a_0 = \int_1^2 1\, dx = 1$$

$$a_n = \int_1^2 \cos\frac{n\pi}{2}x\, dx = -\frac{2}{n\pi}\sin\frac{n\pi}{2}.$$

Thus

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{-2}{n\pi}\sin\frac{n\pi}{2}\,\cos\frac{n\pi}{2}x.$$

**13.** Since $f(x)$ is an even function, we expand in a cosine series:

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x\, dx = \pi$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\cos nx\, dx = \frac{2}{n^2\pi}[(-1)^n - 1].$$

Thus

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi}[(-1)^n - 1]\cos nx.$$

**14.** Since $f(x)$ is an odd function, we expand in a sine series:

$$b_n = \frac{2}{\pi}\int_0^{\pi} x\sin nx\, dx = \frac{2}{n}(-1)^{n+1}.$$

Thus

$$f(x) = \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1}\sin nx.$$

**15.** Since $f(x)$ is an even function, we expand in a cosine series:

$$a_0 = 2\int_0^1 x^2\, dx = \frac{2}{3}$$

$$a_n = 2\int_0^1 x^2\cos n\pi x\, dx = 2\left(\frac{x^2}{n\pi}\sin n\pi x\bigg|_0^1 - \frac{2}{n\pi}\int_0^1 x\sin n\pi x\, dx\right) = \frac{4}{n^2\pi^2}(-1)^n.$$

Thus

$$f(x) = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}(-1)^n\cos n\pi x.$$

**16.** Since $f(x)$ is an odd function, we expand in a sine series:

$$b_n = 2\int_0^1 x^2 \sin n\pi x\,dx = 2\left(-\frac{x^2}{n\pi}\cos n\pi x\Big|_0^1 + \frac{2}{n\pi}\int_0^1 x\cos n\pi x\,dx\right)$$

$$= \frac{2(-1)^{n+1}}{n\pi} + \frac{4}{n^3\pi^3}[(-1)^n - 1].$$

Thus

$$f(x) = \sum_{n=1}^{\infty}\left(\frac{2(-1)^{n+1}}{n\pi} + \frac{4}{n^3\pi^3}[(-1)^n - 1]\right)\sin n\pi x.$$

**17.** Since $f(x)$ is an even function, we expand in a cosine series:

$$a_0 = \frac{2}{\pi}\int_0^{\pi}(\pi^2 - x^2)\,dx = \frac{4}{3}\pi^2$$

$$a_n = \frac{2}{\pi}\int_0^{\pi}(\pi^2 - x^2)\cos nx\,dx = \frac{2}{\pi}\left(\frac{\pi^2 - x^2}{n}\sin nx\Big|_0^{\pi} + \frac{2}{n}\int_0^{\pi} x\sin nx\,dx\right) = \frac{4}{n^2}(-1)^{n+1}.$$

Thus

$$f(x) = \frac{2}{3}\pi^2 + \sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^{n+1}\cos nx\,dx.$$

**18.** Since $f(x)$ is an odd function, we expand in a sine series:

$$b_n = \frac{2}{\pi}\int_0^{\pi} x^3\sin nx\,dx = \frac{2}{\pi}\left(-\frac{x^3}{n}\cos nx\Big|_0^{\pi} + \frac{3}{n}\int_0^{\pi} x^2\cos nx\,dx\right)$$

$$= \frac{2\pi^2}{n}(-1)^{n+1} - \frac{12}{n^2\pi}\int_0^{\pi} x\sin nx\,dx$$

$$= \frac{2\pi^2}{n}(-1)^{n+1} - \frac{12}{n^2\pi}\left(-\frac{x}{n}\cos nx\Big|_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos nx\,dx\right) = \frac{2\pi^2}{n}(-1)^{n+1} + \frac{12}{n^3}(-1)^n.$$

Thus

$$f(x) = \sum_{n=1}^{\infty}\left(\frac{2\pi^2}{n}(-1)^{n+1} + \frac{12}{n^3}(-1)^n\right)\sin nx.$$

**19.** Since $f(x)$ is an odd function, we expand in a sine series:

$$b_n = \frac{2}{\pi}\int_0^{\pi}(x+1)\sin nx\,dx = \frac{2(\pi+1)}{n\pi}(-1)^{n+1} + \frac{2}{n\pi}.$$

Thus

$$f(x) = \sum_{n=1}^{\infty}\left(\frac{2(\pi+1)}{n\pi}(-1)^{n+1} + \frac{2}{n\pi}\right)\sin nx.$$

**20.** Since $f(x)$ is an odd function, we expand in a sine series:

$$b_n = 2\int_0^1 (x-1)\sin n\pi x\,dx = 2\left[\int_0^1 x\sin n\pi x\,dx - \int_0^1 \sin n\pi x\,dx\right]$$

$$= 2\left[\frac{1}{n^2\pi^2}\sin n\pi x - \frac{x}{n\pi}\cos n\pi x + \frac{1}{n\pi}\cos n\pi x\right]_0^1 = -\frac{2}{n\pi}\,.$$

Thus

$$f(x) = -\sum_{n=1}^{\infty}\frac{2}{n\pi}\sin n\pi x.$$

**21.** Since $f(x)$ is an even function, we expand in a cosine series:

$$a_0 = \int_0^1 x\,dx + \int_1^2 1\,dx = \frac{3}{2}$$

$$a_n = \int_0^1 x\cos\frac{n\pi}{2}x\,dx + \int_1^2 \cos\frac{n\pi}{2}x\,dx = \frac{4}{n^2\pi^2}\left(\cos\frac{n\pi}{2} - 1\right).$$

Thus

$$f(x) = \frac{3}{4} + \sum_{n=1}^{\infty}\frac{4}{n^2\pi^2}\left(\cos\frac{n\pi}{2} - 1\right)\cos\frac{n\pi}{2}x.$$

**22.** Since $f(x)$ is an odd function, we expand in a sine series:

$$b_n = \frac{1}{\pi}\int_0^\pi x\sin\frac{n}{2}x\,dx + \int_\pi^{2\pi}\pi\sin\frac{n}{2}x\,dx = \frac{4}{n^2\pi}\sin\frac{n\pi}{2} + \frac{2}{n}(-1)^{n+1}.$$

Thus

$$f(x) = \sum_{n=1}^{\infty}\left(\frac{4}{n^2\pi}\sin\frac{n\pi}{2} + \frac{2}{n}(-1)^{n+1}\right)\sin\frac{n}{2}x.$$

**23.** Since $f(x)$ is an even function, we expand in a cosine series:

$$a_0 = \frac{2}{\pi}\int_0^\pi \sin x\,dx = \frac{4}{\pi}$$

$$a_n = \frac{2}{\pi}\int_0^\pi \sin x\,\cos nx\,dx = \frac{1}{\pi}\int_0^\pi (\sin(1+n)x + \sin(1-n)x)\,dx$$

$$= \frac{2}{\pi(1-n^2)}(1+(-1)^n) \qquad \text{for } n = 2, 3, 4, \ldots$$

$$a_1 = \frac{1}{\pi}\int_0^\pi \sin 2x\,dx = 0.$$

Thus

$$f(x) = \frac{2}{\pi} + \sum_{n=2}^{\infty}\frac{2[1+(-1)^n]}{\pi(1-n^2)}\cos nx.$$

**24.** Since $f(x)$ is an even function, we expand in a cosine series. [See the solution of Problem 10 in Exercises 11.2 for the computation of the integrals.]

$$a_0 = \frac{2}{\pi/2}\int_0^{\pi/2} \cos x\,dx = \frac{4}{\pi}$$

$$a_n = \frac{2}{\pi/2}\int_0^{\pi/2} \cos x \cos\frac{n\pi}{\pi/2}x\,dx = \frac{4(-1)^{n+1}}{\pi\,(4n^2-1)}$$

Thus

$$f(x) = \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{\pi\,(4n^2-1)}\cos 2nx.$$

**25.** $a_0 = 2\displaystyle\int_0^{1/2} 1\,dx = 1$

$$a_n = 2\int_0^{1/2} 1\cdot\cos n\pi x\,dx = \frac{2}{n\pi}\sin\frac{n\pi}{2}$$

$$b_n = 2\int_0^{1/2} 1\cdot\sin n\pi x\,dx = \frac{2}{n\pi}\left(1-\cos\frac{n\pi}{2}\right)$$

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty}\frac{2}{n\pi}\sin\frac{n\pi}{2}\cos n\pi x$$

$$f(x) = \sum_{n=1}^{\infty}\frac{2}{n\pi}\left(1-\cos\frac{n\pi}{2}\right)\sin n\pi x$$

**26.** $a_0 = 2\displaystyle\int_{1/2}^{1} 1\,dx = 1$

$$a_n = 2\int_{1/2}^{1} 1\cdot\cos n\pi x\,dx = -\frac{2}{n\pi}\sin\frac{n\pi}{2}$$

$$b_n = 2\int_{1/2}^{1} 1\cdot\sin n\pi x\,dx = \frac{2}{n\pi}\left(\cos\frac{n\pi}{2} + (-1)^{n+1}\right)$$

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty}\left(-\frac{2}{n\pi}\sin\frac{n\pi}{2}\right)\cos n\pi x$$

$$f(x) = \sum_{n=1}^{\infty}\frac{2}{n\pi}\left(\cos\frac{n\pi}{2} + (-1)^{n+1}\right)\sin n\pi x$$

**27.** $a_0 = \dfrac{4}{\pi}\displaystyle\int_0^{\pi/2}\cos x\,dx = \dfrac{4}{\pi}$

$$a_n = \frac{4}{\pi}\int_0^{\pi/2}\cos x\,\cos 2nx\,dx = \frac{2}{\pi}\int_0^{\pi/2}[\cos(2n+1)x + \cos(2n-1)x]\,dx = \frac{4(-1)^n}{\pi(1-4n^2)}$$

$$b_n = \frac{4}{\pi}\int_0^{\pi/2}\cos x\,\sin 2nx\,dx = \frac{2}{\pi}\int_0^{\pi/2}[\sin(2n+1)x + \sin(2n-1)x]\,dx = \frac{8n}{\pi(4n^2-1)}$$

$$f(x) = \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{\pi(1-4n^2)} \cos 2nx$$

$$f(x) = \sum_{n=1}^{\infty} \frac{8n}{\pi(4n^2-1)} \sin 2nx$$

**28.** $a_0 = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} \sin x\, dx = \frac{4}{\pi}$

$$a_n = \frac{2}{\pi}\int_0^{\pi} \sin x\, \cos nx\, dx = \frac{1}{\pi}\int_0^{\pi} [\sin(n+1)x - \sin(n-1)x]\, dx$$

$$= \frac{2[(-1)^n+1]}{\pi(1-n^2)} \qquad \text{for } n = 2, 3, 4, \ldots$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} \sin x\, \sin nx\, dx = \frac{1}{\pi}\int_0^{\pi} [\cos(n-1)x - \cos(n+1)x]\, dx = 0 \qquad \text{for } n = 2, 3, 4, \ldots$$

$$a_1 = \frac{1}{\pi}\int_0^{\pi} \sin 2x\, dx = 0$$

$$b_1 = \frac{2}{\pi}\int_0^{\pi} \sin^2 x\, dx = 1$$

$$f(x) = \sin x$$

$$f(x) = \frac{2}{\pi} + \frac{2}{\pi}\sum_{n=2}^{\infty} \frac{(-1)^n+1}{1-n^2} \cos nx$$

**29.** $a_0 = \dfrac{2}{\pi}\left(\displaystyle\int_0^{\pi/2} x\, dx + \int_{\pi/2}^{\pi} (\pi - x)\, dx\right) = \frac{\pi}{2}$

$$a_n = \frac{2}{\pi}\left(\int_0^{\pi/2} x\cos nx\, dx + \int_{\pi/2}^{\pi} (\pi - x)\cos nx\, dx\right) = \frac{2}{n^2\pi}\left(2\cos\frac{n\pi}{2} + (-1)^{n+1} - 1\right)$$

$$b_n = \frac{2}{\pi}\left(\int_0^{\pi/2} x\sin nx\, dx + \int_{\pi/2}^{\pi} (\pi - x)\sin nx\, dx\right) = \frac{4}{n^2\pi}\sin\frac{n\pi}{2}$$

$$f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi}\left(2\cos\frac{n\pi}{2} + (-1)^{n+1} - 1\right)\cos nx$$

$$f(x) = \sum_{n=1}^{\infty} \frac{4}{n^2\pi}\sin\frac{n\pi}{2}\, \sin nx$$

**30.** $a_0 = \dfrac{1}{\pi}\displaystyle\int_{\pi}^{2\pi} (x-\pi)\, dx = \frac{\pi}{2}$

$$a_n = \frac{1}{\pi}\int_{\pi}^{2\pi} (x-\pi)\cos\frac{n}{2}x\, dx = \frac{4}{n^2\pi}\left((-1)^n - \cos\frac{n\pi}{2}\right)$$

$$b_n = \frac{1}{\pi}\int_{\pi}^{2\pi} (x-\pi)\sin\frac{n}{2}x\, dx = \frac{2}{n}(-1)^{n+1} - \frac{4}{n^2\pi}\sin\frac{n\pi}{2}$$

$$f(x)=\frac{\pi}{4}+\sum_{n=1}^{\infty}\frac{4}{n^2\pi}\left((-1)^n-\cos\frac{n\pi}{2}\right)\cos\frac{n}{2}\,x$$

$$f(x)=\sum_{n=1}^{\infty}\left(\frac{2}{n}(-1)^{n+1}-\frac{4}{n^2\pi}\sin\frac{n\pi}{2}\right)\sin\frac{n}{2}\,x$$

**31.** $a_0=\int_0^1 x\,dx+\int_1^2 1\,dx=\dfrac{3}{2}$

$$a_n=\int_0^1 x\cos\frac{n\pi}{2}x\,dx=\frac{4}{n^2\pi^2}\left(\cos\frac{n\pi}{2}-1\right)$$

$$b_n=\int_0^1 x\sin\frac{n\pi}{2}x\,dx+\int_1^2 1\cdot\sin\frac{n\pi}{2}x\,dx=\frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}+\frac{2}{n\pi}(-1)^{n+1}$$

$$f(x)=\frac{3}{4}+\sum_{n=1}^{\infty}\frac{4}{n^2\pi^2}\left(\cos\frac{n\pi}{2}-1\right)\cos\frac{n\pi}{2}\,x$$

$$f(x)=\sum_{n=1}^{\infty}\left(\frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}+\frac{2}{n\pi}(-1)^{n+1}\right)\sin\frac{n\pi}{2}\,x$$

**32.** $a_0=\int_0^1 1\,dx+\int_1^2 (2-x)\,dx=\dfrac{3}{2}$

$$a_n=\int_0^1 1\cdot\cos\frac{n\pi}{2}x\,dx+\int_1^2 (2-x)\cos\frac{n\pi}{2}x\,dx=\frac{4}{n^2\pi^2}\left(\cos\frac{n\pi}{2}+(-1)^{n+1}\right)$$

$$b_n=\int_0^1 1\cdot\sin\frac{n\pi}{2}x\,dx+\int_1^2 (2-x)\sin\frac{n\pi}{2}x\,dx=\frac{2}{n\pi}+\frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$f(x)=\frac{3}{4}+\sum_{n=1}^{\infty}\frac{4}{n^2\pi^2}\left(\cos\frac{n\pi}{2}+(-1)^{n+1}\right)\cos\frac{n\pi}{2}\,x$$

$$f(x)=\sum_{n=1}^{\infty}\left(\frac{2}{n\pi}+\frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}\right)\sin\frac{n\pi}{2}\,x$$

**33.** $a_0=2\int_0^1 (x^2+x)\,dx=\dfrac{5}{3}$

$$a_n=2\int_0^1 (x^2+x)\cos n\pi x\,dx=\frac{2(x^2+x)}{n\pi}\sin n\pi x\Big|_0^1-\frac{2}{n\pi}\int_0^1 (2x+1)\sin n\pi x\,dx$$

$$=\frac{2}{n^2\pi^2}[3(-1)^n-1]$$

$$b_n=2\int_0^1 (x^2+x)\sin n\pi x\,dx=-\frac{2(x^2+x)}{n\pi}\cos n\pi x\Big|_0^1+\frac{2}{n\pi}\int_0^1 (2x+1)\cos n\pi x\,dx$$

$$=\frac{4}{n\pi}(-1)^{n+1}+\frac{4}{n^3\pi^3}[(-1)^n-1]$$

$$f(x) = \frac{5}{6} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2}[3(-1)^n - 1]\cos n\pi x$$

$$f(x) = \sum_{n=1}^{\infty} \left(\frac{4}{n\pi}(-1)^{n+1} + \frac{4}{n^3\pi^3}[(-1)^n - 1]\right)\sin n\pi x$$

**34.** $a_0 = \int_0^2 (2x - x^2)\,dx = \frac{4}{3}$

$$a_n = \int_0^2 (2x - x^2)\cos\frac{n\pi}{2}x\,dx = \frac{8}{n^2\pi^2}[(-1)^{n+1} - 1]$$

$$b_n = \int_0^2 (2x - x^2)\sin\frac{n\pi}{2}x\,dx = \frac{16}{n^3\pi^3}[1 - (-1)^n]$$

$$f(x) = \frac{2}{3} + \sum_{n=1}^{\infty} \frac{8}{n^2\pi^2}[(-1)^{n+1} - 1]\cos\frac{n\pi}{2}x$$

$$f(x) = \sum_{n=1}^{\infty} \frac{16}{n^3\pi^3}[1 - (-1)^n]\sin\frac{n\pi}{2}x$$

**35.** $a_0 = \frac{1}{\pi}\int_0^{2\pi} x^2\,dx = \frac{8}{3}\pi^2$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x^2\cos nx\,dx = \frac{4}{n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} x^2\sin nx\,dx = -\frac{4\pi}{n}$$

$$f(x) = \frac{4}{3}\pi^2 + \sum_{n=1}^{\infty}\left(\frac{4}{n^2}\cos nx - \frac{4\pi}{n}\sin nx\right)$$

**36.** $a_0 = \frac{2}{\pi}\int_0^{\pi} x\,dx = \pi$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\cos 2nx\,dx = 0$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} x\sin 2nx\,dx = -\frac{1}{n}$$

$$f(x) = \frac{\pi}{2} - \sum_{n=1}^{\infty}\frac{1}{n}\sin 2nx$$

**37.** $a_0 = 2\int_0^1 (x+1)\,dx = 3$

$$a_n = 2\int_0^1 (x+1)\cos 2n\pi x\,dx = 0$$

$$b_n = 2\int_0^1 (x+1)\sin 2n\pi x\,dx = -\frac{1}{n\pi}$$

$$f(x) = \frac{3}{2} - \sum_{n=1}^{\infty} \frac{1}{n\pi} \sin 2n\pi x$$

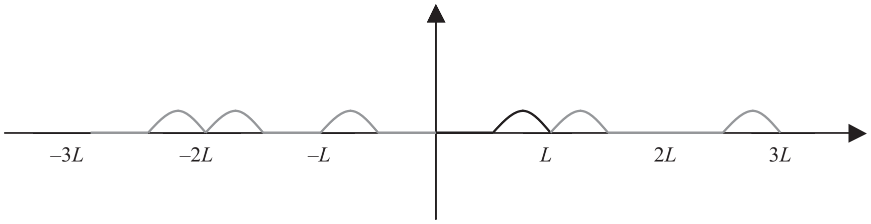
**38.** $a_0 = \dfrac{2}{2}\displaystyle\int_0^2 (2-x)\,dx = 2$

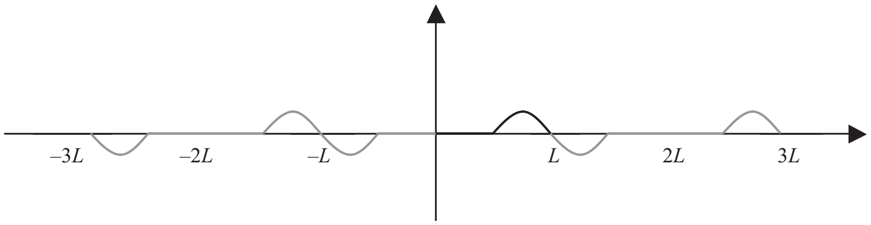
$$a_n = \frac{2}{2}\int_0^2 (2-x)\cos n\pi x\,dx = 0$$

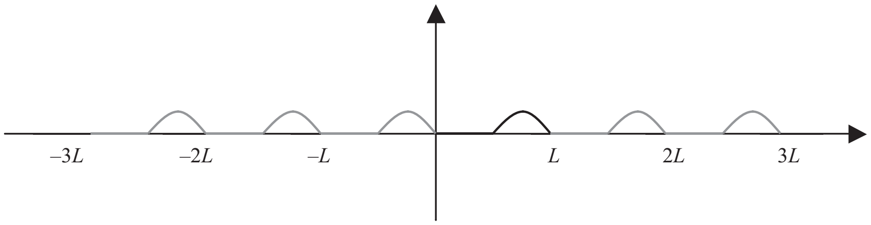
$$b_n = \frac{2}{2}\int_0^2 (2-x)\sin n\pi x\,dx = \frac{2}{n\pi}$$

$$f(x) = 1 + \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin n\pi x$$

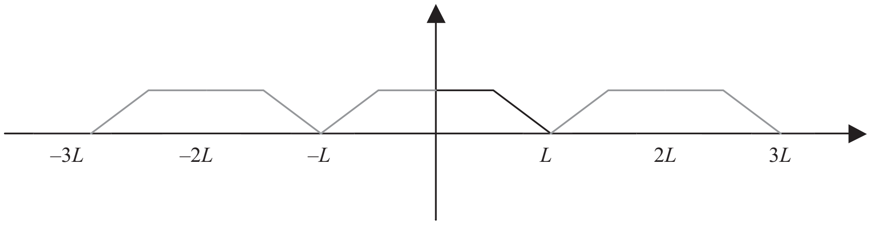
**39.** The periodic extensions for the cosine, sine, and Fourier series are shown below:

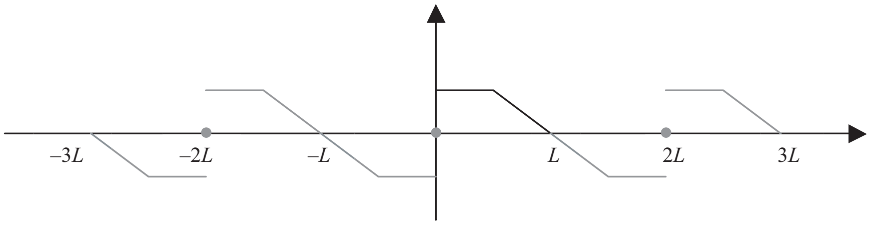


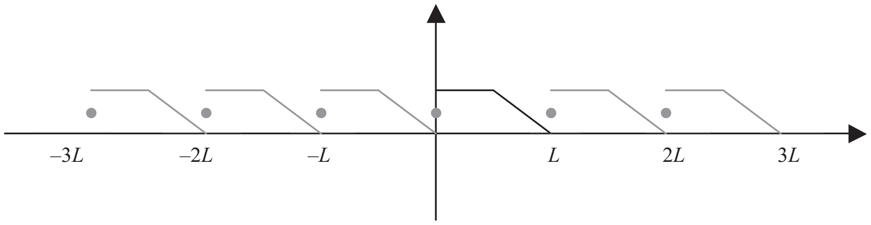




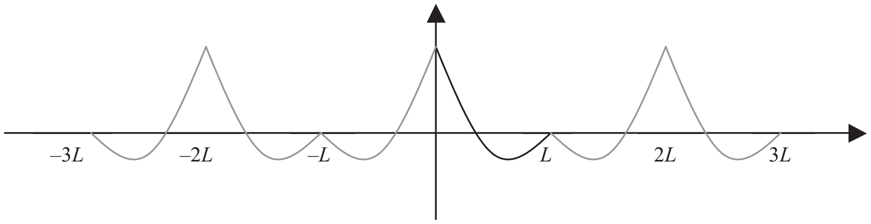
**40.** The periodic extensions for the cosine, sine, and Fourier series are shown below:

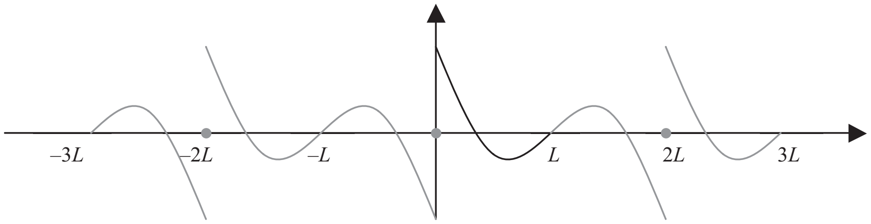


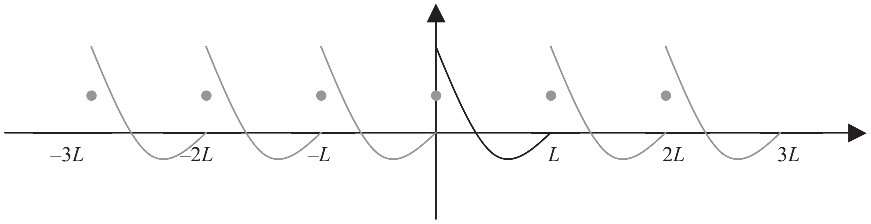




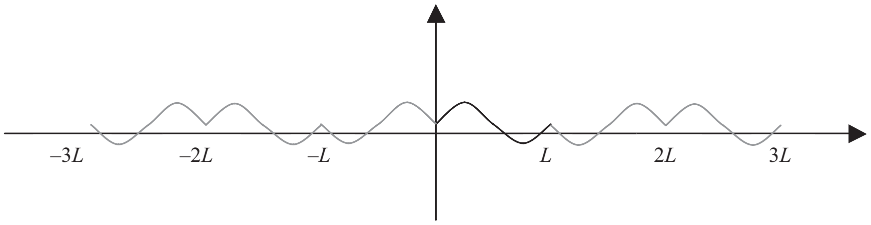
**41.** The periodic extensions for the cosine, sine, and Fourier series are shown below:

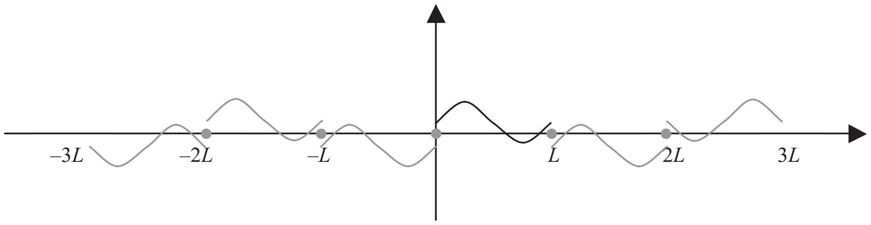


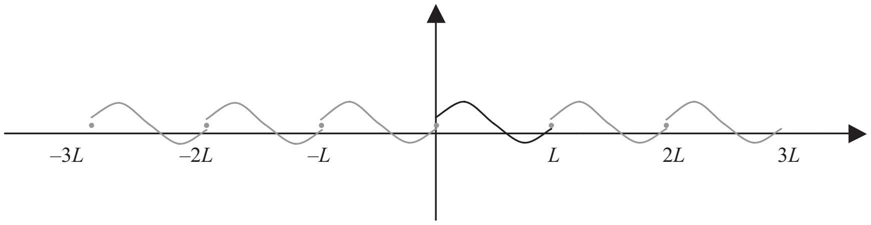




**42.** The periodic extensions for the cosine, sine, and Fourier series are shown below:







**43.** We have

$$b_n = \frac{2}{\pi}\int_0^{\pi} 5\sin nt\,dt = \frac{10}{n\pi}[1-(-1)^n]$$

so that

$$f(t) = \sum_{n=1}^{\infty} \frac{10[1-(-1)^n]}{n\pi}\,\sin nt.$$

Substituting the assumption $x_p(t) = \sum_{n=1}^{\infty} B_n \sin nt$ into the differential equation then gives

$$x_p'' + 10x_p = \sum_{n=1}^{\infty} B_n(10-n^2)\sin nt = \sum_{n=1}^{\infty} \frac{10[1-(-1)^n]}{n\pi}\,\sin nt$$

and so $B_n = 10[1-(-1)^n]/n\pi(10-n^2)$. Thus

$$x_p(t) = \frac{10}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n(10-n^2)}\sin nt.$$

**44.** We have

$$b_n = \frac{2}{\pi}\int_0^{1} (1-t)\sin n\pi t\,dt = \frac{2}{n\pi}$$

so that

$$f(t) = \sum_{n=1}^{\infty} \frac{2}{n\pi}\,\sin n\pi t.$$

Substituting the assumption $x_p(t) = \sum_{n=1}^{\infty} B_n \sin n\pi t$ into the differential equation then gives

$$x_p'' + 10x_p = \sum_{n=1}^{\infty} B_n(10 - n^2\pi^2)\sin n\pi t = \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin n\pi t$$

and so $B_n = 2/n\pi(10 - n^2\pi^2)$. Thus

$$x_p(t) = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{n(10 - n^2\pi^2)} \sin n\pi t.$$

**45.** We have

$$a_0 = \frac{2}{\pi}\int_0^{\pi} (2\pi t - t^2)\,dt = \frac{4}{3}\pi^2$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} (2\pi t - t^2)\cos nt\,dt = -\frac{4}{n^2}$$

so that

$$f(t) = \frac{2\pi^2}{3} - \sum_{n=1}^{\infty} \frac{4}{n^2} \cos nt.$$

Substituting the assumption

$$x_p(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos nt$$

into the differential equation then gives

$$\frac{1}{4}x_p'' + 12x_p = 6A_0 + \sum_{n=1}^{\infty} A_n\left(-\frac{1}{4}n^2 + 12\right)\cos nt = \frac{2\pi^2}{3} - \sum_{n=1}^{\infty} \frac{4}{n^2} \cos nt$$

and $A_0 = \pi^2/9$, $A_n = 16/n^2(n^2 - 48)$. Thus

$$x_p(t) = \frac{\pi^2}{18} + 16\sum_{n=1}^{\infty} \frac{1}{n^2(n^2 - 48)} \cos nt.$$

**46.** We have

$$a_0 = \frac{2}{1/2}\int_0^{1/2} t\,dt = \frac{1}{2}$$

$$a_n = \frac{2}{1/2}\int_0^{1/2} t\cos 2n\pi t\,dt = \frac{1}{n^2\pi^2}\left[(-1)^n - 1\right]$$

so that

$$f(t) = \frac{1}{4} + \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2\pi^2} \cos 2n\pi t.$$

Substituting the assumption

$$x_p(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos 2n\pi t$$

into the differential equation then gives

$$\frac{1}{4}x_p'' + 12x_p = 6A_0 + \sum_{n=1}^{\infty} A_n(12 - n^2\pi^2)\cos 2n\pi t = \frac{1}{4} + \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2\pi^2}\cos 2n\pi t$$

and $A_0 = 1/24,\ \ A_n = [(-1)^n - 1]/n^2\pi^2(12 - n^2\pi^2)$. Thus

$$x_p(t) = \frac{1}{48} + \frac{1}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2(12 - n^2\pi^2)}\cos 2n\pi t.$$

**47.** **(a)** The general solution is $x(t) = c_1\cos\sqrt{10}\,t + c_2\sin\sqrt{10}\,t + x_p(t)$, where

$$x_p(t) = \frac{10}{\pi}\sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n(10 - n^2)}\sin nt.$$

The initial condition $x(0) = 0$ implies $c_1 + x_p(0) = 0$. Since $x_p(0) = 0$, we have $c_1 = 0$ and $x(t) = c_2\sin\sqrt{10}\,t + x_p(t)$. Then $x'(t) = c_2\sqrt{10}\,\cos\sqrt{10}\,t + x_p'(t)$ and $x'(0) = 0$ implies

$$c_2\sqrt{10} + \frac{10}{\pi}\sum_{n=1}^{\infty} \frac{1 - (-1)^n}{10 - n^2}\cos 0 = 0.$$

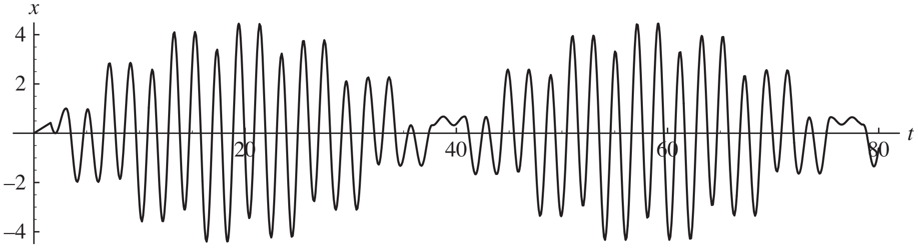
Thus

$$c_2 = -\frac{\sqrt{10}}{\pi}\sum_{n=1}^{\infty} \frac{1 - (-1)^n}{10 - n^2}$$

and

$$x(t) = \frac{10}{\pi}\sum_{n=1}^{\infty} \frac{1 - (-1)^n}{10 - n^2}\left[\frac{1}{n}\sin nt - \frac{1}{\sqrt{10}}\sin\sqrt{10}\,t\right].$$

**(b)** The graph is plotted using eight nonzero terms in the series expansion of $x(t)$.



**48.** **(a)** The general solution is $x(t) = c_1\cos 4\sqrt{3}\,t + c_2\sin 4\sqrt{3}t + x_p(t)$, where

$$x_p(t) = \frac{\pi^2}{18} + 16\sum_{n=1}^{\infty} \frac{1}{n^2(n^2 - 48)}\cos nt.$$

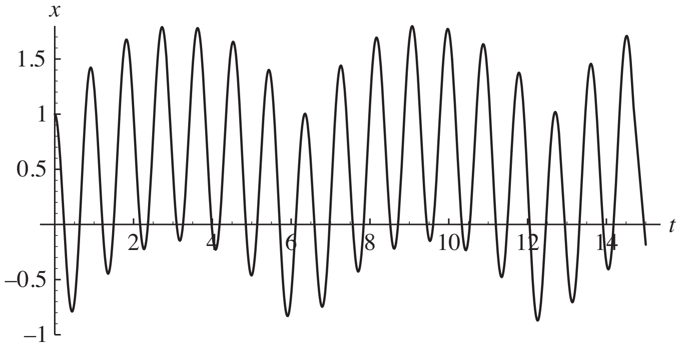
The initial condition $x(0) = 0$ implies $c_1 + x_p(0) = 1$ or

$$c_1 = 1 - x_p(0) = 1 - \frac{\pi^2}{18} - 16\sum_{n=1}^{\infty} \frac{1}{n^2(n^2 - 48)}\,.$$

Now $x'(t) = -4\sqrt{3}c_1 \sin 4\sqrt{3}\,t + 4\sqrt{3}c_2 \cos 4\sqrt{3}\,t + x_p'(t)$, so $x'(0) = 0$ implies $4\sqrt{3}c_2 + x_p'(0) = 0$. Since $x_p'(0) = 0$, we have $c_2 = 0$ and

$$\begin{aligned} x(t) &= \left(1 - \frac{\pi^2}{18} - 16\sum_{n=1}^{\infty} \frac{1}{n^2(n^2-48)}\right)\cos 4\sqrt{3}\,t + \frac{\pi^2}{18} + 16\sum_{n=1}^{\infty} \frac{1}{n^2(n^2-48)} \cos nt \\ &= \frac{\pi^2}{18} + \left(1 - \frac{\pi^2}{18}\right)\cos 4\sqrt{3}\,t + 16\sum_{n=1}^{\infty} \frac{1}{n^2(n^2-48)}\left[\cos nt - \cos 4\sqrt{3}\,t\right]. \end{aligned}$$

**(b)** The graph is plotted using five nonzero terms in the series expansion of $x(t)$.



**49.** **(a)** We have

$$b_n = \frac{2}{L}\int_0^L \frac{w_0 x}{L} \sin \frac{n\pi}{L} x\, dx = \frac{2w_0}{n\pi}(-1)^{n+1}$$

so that

$$w(x) = \sum_{n=1}^{\infty} \frac{2w_0}{n\pi}(-1)^{n+1} \sin \frac{n\pi}{L} x.$$

**(b)** If we assume $y_p(x) = \sum_{n=1}^{\infty} B_n \sin(n\pi x/L)$ then

$$y_p^{(4)} = \sum_{n=1}^{\infty} \frac{n^4\pi^4}{L^4} B_n \sin \frac{n\pi}{L} x$$

and so the differential equation $EIy_p^{(4)} = w(x)$ gives

$$B_n = \frac{2w_0(-1)^{n+1}L^4}{EIn^5\pi^5}.$$

Thus

$$y_p(x) = \frac{2w_0L^4}{EI\pi^5}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^5} \sin \frac{n\pi}{L} x.$$

**50.** We have

$$b_n = \frac{2}{L}\int_{L/3}^{2L/3} w_0 \sin \frac{n\pi}{L} x\, dx = \frac{2w_0}{n\pi}\left(\cos \frac{n\pi}{3} - \cos \frac{2n\pi}{3}\right)$$

so that

$$w(x) = \sum_{n=1}^{\infty} \frac{2w_0}{n\pi}\left(\cos \frac{n\pi}{3} - \cos \frac{2n\pi}{3}\right) \sin \frac{n\pi}{L} x.$$

If we assume $y_p(x) = \sum_{n=1}^{\infty} B_n \sin(n\pi x/L)$ then

$$y_p^{(4)}(x) = \sum_{n=1}^{\infty} \frac{n^4\pi^4}{L^4} B_n \sin\frac{n\pi}{L} x$$

and so the differential equation $EIy_p^{(4)}(x) = w(x)$ gives

$$B_n = 2w_0L^4 \frac{\cos\frac{n\pi}{3} - \cos\frac{2n\pi}{3}}{EIn^5\pi^5}.$$

Thus

$$y_p(x) = \frac{2w_0L^4}{EI\pi^5} \sum_{n=1}^{\infty} \frac{\cos\frac{n\pi}{3} - \cos\frac{2n\pi}{3}}{n^5} \sin\frac{n\pi}{L} x.$$

**51.** We note that $w(x)$is $2\pi$-periodic and even. With $p = \pi$ we find the cosine expansion of

$$f(x) = \begin{cases} w_0, & 0 < x < \pi/2 \\ 0, & \pi/2 < x < \pi \end{cases}$$

We have

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi/2} w_0\,dx = w_0$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi/2} w_0\cos nx\,dx = \frac{2w_0}{n\pi}\sin\frac{n\pi}{2}.$$

Thus,

$$w(x) = \frac{w_0}{2} + \frac{2w_0}{\pi}\sum_{n=1}^{\infty} \frac{1}{n}\sin\frac{n\pi}{2}\cos nx.$$

Now we assume a particular solution of the form $y_p(x) = A_0/2 + \sum_{n=1}^{\infty} A_n \cos nx$. Then $y_p^{(4)}(x) = \sum_{n=1}^{\infty} A_n n^4 \cos nx$ and substituting into the differential equation, we obtain

$$\begin{aligned} EIy_p^{(4)}(x) + ky_p(x) &= \frac{kA_0}{2} + \sum_{n=1}^{\infty} A_n(EIn^4 + k)\cos nx \\ &= \frac{w_0}{2} + \frac{2w_0}{\pi}\sum_{n=1}^{\infty} \frac{1}{n}\sin\frac{n\pi}{2}\cos nx. \end{aligned}$$

Thus

$$A_0 = \frac{w_0}{k} \qquad \text{and} \qquad A_n = \frac{2w_0}{\pi}\frac{\sin(n\pi/2)}{n(EIn^4 + k)},$$

and

$$y_p(x) = \frac{w_0}{2k} + \frac{2w_0}{\pi}\sum_{n=1}^{\infty} \frac{\sin(n\pi/2)}{n(EIn^4 + k)}\cos nx.$$

**52.** **(a)** If $f$ and $g$ are even and $h(x) = f(x)g(x)$ then

$$h(-x) = f(-x)g(-x) = f(x)g(x) = h(x),$$

and $h$ is even.

**(c)** If $f$ is even and $g$ is odd and $h(x) = f(x)g(x)$ then

$$h(-x) = f(-x)g(-x) = f(x)[-g(x)] = -h(x),$$

and $h$ is odd.

**(d)** Let $h(x) = f(x) \pm g(x)$ where $f$ and $g$ are even. Then

$$h(-x) = f(-x) \pm g(-x) = f(x) \pm g(x) = h(x),$$

and $h$ is even.

**(e)** If $f$ and $g$ are odd and $h(x) = f(x) \pm g(x)$ then

$$h(-x) = f(-x) \pm g(-x) = -f(x) \pm [-g(x)] = -[f(x) \pm g(x)] = -h(x)$$

and $h$ is odd.

**(f)** If $f$ is even then

$$\begin{aligned}\int_{-a}^{a} f(x)\,dx &= \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx \qquad \longleftarrow u = -x, \quad du = -dx\\ &= \int_{a}^{0} f(-u)(-du) + \int_{0}^{a} f(x)\,dx \qquad \longleftarrow f \text{ is an even function}\\ &= \int_{0}^{a} f(u)\,du + \int_{0}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.\end{aligned}$$

**(g)** If $f$ is odd then

$$\begin{aligned}\int_{-a}^{a} f(x)\,dx &= \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx \qquad \longleftarrow u = -x, \quad du = -dx\\ &= \int_{a}^{0} f(-u)(-du) + \int_{0}^{a} f(x)\,dxf \qquad \longleftarrow f \text{ is an odd function}\\ &= \int_{0}^{a} [-f(u)]\,du + \int_{0}^{a} f(x)\,dx = 0.\end{aligned}$$

**53.** If $f(x)$ is even then $f(-x) = f(x)$. If $f(x)$ is odd then $f(-x) = -f(x)$. Thus, if $f(x)$ is both even and odd $f(x) = f(-x) = -f(x)$, and $f(x) = 0$.

**54.** For $EIy^{(4)} + ky = 0$ the roots of the auxiliary equation are $m_1 = \alpha + \alpha i$, $m_2 = \alpha - \alpha i$, $m_3 = -\alpha + \alpha i$, and $m_4 = -\alpha - \alpha i$, where $\alpha = (k/EI)^{1/4}/\sqrt{2}$. Thus

$$y_c = e^{\alpha x}(c_1 \cos \alpha x + c_2 \sin \alpha x) + e^{-\alpha x}(c_3 \cos \alpha x + c_4 \sin \alpha x).$$

We expect $y(x)$ to be bounded as $x \to \infty$, so we must have $c_1 = c_2 = 0$. We also expect $y(x)$ to be bounded as $x \to -\infty$, so we must have $c_3 = c_4 = 0$. Thus, $y_c = 0$ and the solution of the differential equation in Problem 51 is $y_p(x)$.

## 11.4 Sturm–Liouville Problem

**1.** For $\lambda \le 0$ the only solution of the boundary-value problem is $y = 0$. For $\lambda = \alpha^2 > 0$ we have

$$y = c_1 \cos \alpha x + c_2 \sin \alpha x.$$

Now

$$y'(x) = -c_1 \alpha \sin \alpha x + c_2 \alpha \cos \alpha x$$

and $y'(0) = 0$ implies $c_2 = 0$, so

$$y(1) + y'(1) = c_1(\cos \alpha - \alpha \sin \alpha) = 0 \quad \text{or} \quad \cot \alpha = \alpha.$$

The eigenvalues are $\lambda_n = \alpha_n^2$ where $\alpha_1$, $\alpha_2$, $\alpha_3$, $\ldots$ are the consecutive positive solutions of $\cot \alpha = \alpha$. The corresponding eigenfunctions are $\cos \alpha_n x$ for $n = 1, 2, 3, \ldots$. Using a CAS we find that the first four eigenvalues are approximately 0.7402, 11.7349, 41.4388, and 90.8082 with corresponding approximate eigenfunctions $\cos 0.8603x$, $\cos 3.4256x$, $\cos 6.4373x$, and $\cos 9.5293x$.

**2.** For $\lambda < 0$ the only solution of the boundary-value problem is $y = 0$. For $\lambda = 0$ we have $y = c_1 x + c_2$. Now $y' = c_1$ and the boundary conditions both imply $c_1 + c_2 = 0$. Thus, $\lambda = 0$ is an eigenvalue with corresponding eigenfunction $y_0 = x - 1$.

For $\lambda = \alpha^2 > 0$ we have

$$y = c_1 \cos \alpha x + c_2 \sin \alpha x$$

and

$$y'(x) = -c_1 \alpha \sin \alpha x + c_2 \alpha \cos \alpha x.$$

The boundary conditions imply

$$\begin{aligned} c_1 + c_2 \alpha &= 0 \\ c_1 \cos \alpha + c_2 \sin \alpha &= 0 \end{aligned}$$

which gives

$$-c_2 \alpha \cos \alpha + c_2 \sin \alpha = 0 \qquad \text{or} \qquad \tan \alpha = \alpha.$$

The eigenvalues are $\lambda_n = \alpha_n^2$ where $\alpha_1$, $\alpha_2$, $\alpha_3$, $\ldots$ are the consecutive positive solutions of $\tan\alpha = \alpha$. The corresponding eigenfunctions are $\alpha\cos\alpha x - \sin\alpha x$ (obtained by taking $c_2 = -1$ in the first equation of the system.) Using a CAS we find that the first four positive eigenvalues are 20.1907, 59.6795, 118.9000, and 197.858 with corresponding eigenfunctions $4.4934\cos 4.4934x - \sin 4.4934x$, $7.7253\cos 7.7253x - \sin 7.7253x$, $10.9041\cos 10.9041x - \sin 10.9041x$, and $14.0662\cos 14.0662x - \sin 14.0662x$.

**3.** For $\lambda = 0$ the solution of $y'' = 0$ is $y = c_1x + c_2$. The condition $y'(0) = 0$ implies $c_1 = 0$, so $\lambda = 0$ is an eigenvalue with corresponding eigenfunction 1.

For $\lambda = -\alpha^2 < 0$ we have $y = c_1\cosh\alpha x + c_2\sinh\alpha x$ and $y' = c_1\alpha\sinh\alpha x + c_2\alpha\cosh\alpha x$. The condition $y'(0) = 0$ implies $c_2 = 0$ and so $y = c_1\cosh\alpha x$. Now the condition $y'(L) = 0$ implies $c_1 = 0$. Thus $y = 0$ and there are no negative eigenvalues.

For $\lambda = \alpha^2 > 0$ we have $y = c_1\cos\alpha x + c_2\sin\alpha x$ and $y' = -c_1\alpha\sin\alpha x + c_2\alpha\cos\alpha x$. The condition $y'(0) = 0$ implies $c_2 = 0$ and so $y = c_1\cos\alpha x$. Now the condition $y'(L) = 0$ implies $-c_1\alpha\sin\alpha L = 0$. For $c_1 \neq 0$ this condition will hold when $\alpha L = n\pi$ or $\lambda = \alpha^2 = n^2\pi^2/L^2$, where $n = 1, 2, 3, \ldots$. These are the positive eigenvalues with corresponding eigenfunctions $\cos(n\pi x/L)$, $n = 1, 2, 3, \ldots$.

**4.** For $\lambda = -\alpha^2 < 0$ we have

$$\begin{aligned} y &= c_1\cosh\alpha x + c_2\sinh\alpha x \\ y' &= c_1\alpha\sinh\alpha x + c_2\alpha\cosh\alpha x. \end{aligned}$$

Using the fact that $\cosh x$ is an even function and $\sinh x$ is odd we have

$$\begin{aligned} y(-L) &= c_1\cosh(-\alpha L) + c_2\sinh(-\alpha L) \\ &= c_1\cosh\alpha L - c_2\sinh\alpha L \end{aligned}$$

and

$$\begin{aligned} y'(-L) &= c_1\alpha\sinh(-\alpha L) + c_2\alpha\cosh(-\alpha L) \\ &= -c_1\alpha\sinh\alpha L + c_2\alpha\cosh\alpha L. \end{aligned}$$

The boundary conditions imply

$$c_1\cosh\alpha L - c_2\sinh\alpha L = c_1\cosh\alpha L + c_2\sinh\alpha L$$

or

$$2c_2\sinh\alpha L = 0$$

and

$$-c_1\alpha\sinh\alpha L + c_2\alpha\cosh\alpha L = c_1\alpha\sinh\alpha L + c_2\alpha\cosh\alpha L$$

or

$$2c_1\alpha\sinh\alpha L = 0.$$

Since $\alpha L \neq 0$, $c_1 = c_2 = 0$ and the only solution of the boundary-value problem in this case is $y = 0$.

For $\lambda = 0$ we have

$$\begin{aligned} y &= c_1 x + c_2 \\ y' &= c_1. \end{aligned}$$

From $y(-L) = y(L)$ we obtain

$$-c_1 L + c_2 = c_1 L + c_2.$$

Then $c_1 = 0$ and $y = 1$ is an eigenfunction corresponding to the eigenvalue $\lambda = 0$.

For $\lambda = \alpha^2 > 0$ we have

$$\begin{aligned} y &= c_1 \cos \alpha x + c_2 \sin \alpha x \\ y' &= -c_1 \alpha \sin \alpha x + c_2 \alpha \cos \alpha x. \end{aligned}$$

The first boundary condition implies

$$c_1 \cos \alpha L - c_2 \sin \alpha L = c_1 \cos \alpha L + c_2 \sin \alpha L$$

or

$$2c_2 \sin \alpha L = 0.$$

Thus, if $c_1 = 0$ and $c_2 \neq 0$,

$$\alpha L = n\pi \quad \text{or} \quad \lambda = \alpha^2 = \frac{n^2\pi^2}{L^2}\,,\ n = 1, 2, 3, \dots\,.$$

The corresponding eigenfunctions are $\sin(n\pi x/L)$, for $n = 1, 2, 3, \dots$. Similarly, the second boundary condition implies

$$2c_1 \alpha \sin \alpha L = 0.$$

If $c_1 \neq 0$ and $c_2 = 0$,

$$\alpha L = n\pi \quad \text{or} \quad \lambda = \alpha^2 = \frac{n^2\pi^2}{L^2}\,,\ n = 1, 2, 3, \dots,$$

and the corresponding eigenfunctions are $\cos(n\pi x/L)$, for $n = 1, 2, 3, \dots$.

**5.** The eigenfunctions are $\cos \alpha_n x$ where $\cot \alpha_n = \alpha_n$. Thus

$$\begin{aligned} \|\cos \alpha_n x\|^2 &= \int_0^1 \cos^2 \alpha_n x\, dx = \frac{1}{2}\int_0^1 (1 + \cos 2\alpha_n x)\, dx \\ &= \frac{1}{2}\left(x + \frac{1}{2\alpha_n}\sin 2\alpha_n x\right)\Bigg|_0^1 = \frac{1}{2}\left(1 + \frac{1}{2\alpha_n}\sin 2\alpha_n\right) \\ &= \frac{1}{2}\left[1 + \frac{1}{2\alpha_n}(2\sin\alpha_n\cos\alpha_n)\right] = \frac{1}{2}\left[1 + \frac{1}{\alpha_n}\sin\alpha_n\cot\alpha_n\sin\alpha_n\right] \\ &= \frac{1}{2}\left[1 + \frac{1}{\alpha_n}(\sin\alpha_n)\,\alpha_n\,(\sin\alpha_n)\right] = \frac{1}{2}\left(1 + \sin^2\alpha_n\right). \end{aligned}$$

**6.** The eigenfunctions are $\sin\alpha_n x$ where $\tan\alpha_n = -\alpha_n$. Thus

$$\begin{aligned}
\|\sin\alpha_n x\|^2 &= \int_0^1 \sin^2\alpha_n x\,dx = \frac{1}{2}\int_0^1 (1-\cos 2\alpha_n x)\,dx \\
&= \frac{1}{2}\left(x - \frac{1}{2\alpha_n}\sin 2\alpha_n x\right)\bigg|_0^1 = \frac{1}{2}\left(1 - \frac{1}{2\alpha_n}\sin 2\alpha_n\right) \\
&= \frac{1}{2}\left[1 - \frac{1}{2\alpha_n}(2\sin\alpha_n\cos\alpha_n)\right] = \frac{1}{2}\left[1 - \frac{1}{\alpha_n}\tan\alpha_n\cos\alpha_n\cos\alpha_n\right] \\
&= \frac{1}{2}\left[1 - \frac{1}{\alpha_n}\left(-\alpha_n\cos^2\alpha_n\right)\right] = \frac{1}{2}\left(1+\cos^2\alpha_n\right).
\end{aligned}$$

**7. (a)** If $\lambda \le 0$ the initial conditions imply $y = 0$. For $\lambda = \alpha^2 > 0$ the general solution of the Cauchy-Euler differential equation is $y = c_1\cos(\alpha\ln x) + c_2\sin(\alpha\ln x)$. The condition $y(1) = 0$ implies $c_1 = 0$, so that $y = c_2\sin(\alpha\ln x)$. The condition $y(5) = 0$ implies $\alpha\ln 5 = n\pi$, $n = 1, 2, 3, \ldots$. Thus, the eigenvalues are $n^2\pi^2/(\ln 5)^2$ for $n = 1, 2, 3, \ldots$, with corresponding eigenfunctions $\sin[(n\pi/\ln 5)\ln x]$.

**(b)** The self-adjoint form is

$$\frac{d}{dx}[xy'] + \frac{\lambda}{x}y = 0.$$

**(c)** An orthogonality relation is

$$\int_1^5 \frac{1}{x}\sin\left(\frac{m\pi}{\ln 5}\ln x\right)\sin\left(\frac{n\pi}{\ln 5}\ln x\right)dx = 0, \qquad m \ne n.$$

**8. (a)** The roots of the auxiliary equation $m^2 + m + \lambda = 0$ are $\frac{1}{2}(-1 \pm \sqrt{1-4\lambda}\,)$. When $\lambda = 0$ the general solution of the differential equation is $c_1 + c_2e^{-x}$. The boundary conditions imply $c_1 + c_2 = 0$ and $c_1 + c_2e^{-2} = 0$. Since the determinant of the coefficients is not 0, the only solution of this homogeneous system is $c_1 = c_2 = 0$, in which case $y = 0$. When $\lambda = \frac{1}{4}$, the general solution of the differential equation is $c_1e^{-x/2} + c_2xe^{-x/2}$. The boundary conditions imply $c_1 = 0$ and $c_1 + 2c_2 = 0$, so $c_1 = c_2 = 0$ and $y = 0$. Similarly, if $0 < \lambda < \frac{1}{4}$, the general solution is

$$y = c_1e^{\frac{1}{2}(-1+\sqrt{1-4\lambda}\,)x} + c_2e^{\frac{1}{2}(-1-\sqrt{1-4\lambda}\,)x}.$$

In this case the boundary conditions again imply $c_1 = c_2 = 0$, and so $y = 0$. Now, for $\lambda > \frac{1}{4}$, the general solution of the differential equation is

$$y = c_1e^{-x/2}\cos\sqrt{4\lambda-1}\,x + c_2e^{-x/2}\sin\sqrt{4\lambda-1}\,x.$$

The condition $y(0) = 0$ implies $c_1 = 0$ so $y = c_2e^{-x/2}\sin\sqrt{4\lambda-1}\,x$. From

$$y(2) = c_2e^{-1}\sin 2\sqrt{4\lambda-1} = 0$$

we see that the eigenvalues are determined by $2\sqrt{4\lambda - 1} = n\pi$ for $n = 1, 2, 3, \dots$. Thus, the eigenvalues are $n^2\pi^2/4^2 + 1/4$ for $n = 1, 2, 3, \dots$, with corresponding eigenfunctions $e^{-x/2}\sin(n\pi x/2)$.

**(b)** The self-adjoint form is

$$\frac{d}{dx}[e^x y'] + \lambda e^x y = 0.$$

**(c)** An orthogonality relation is

$$\int_0^2 e^x \left(e^{-x/2}\sin\frac{m\pi}{2}x\right)\left(e^{-x/2}\cos\frac{n\pi}{2}x\right)dx = \int_0^2 \sin\frac{m\pi}{2}x\,\cos\frac{n\pi}{2}x\,dx = 0.$$

**9.** We divide by lead coefficient of the differential equation to obtain the form $y'' + \left(-1 + \dfrac{1}{x}\right)y' + \dfrac{n}{x}y = 0$. The integrating factor is then

$$e^{\int(-1+1/x)\,dx} = e^{-x+\ln x} = e^{-x}e^{\ln x} = xe^{-x}.$$

Thus, the differential equation is

$$xe^{-x}y'' + (1-x)e^{-x}y' + ne^{-x}y = 0$$

and the self-adjoint form is

$$\frac{d}{dx}\left[xe^{-x}y'\right] + ne^{-x}y = 0.$$

Identifying the weight function $p(x) = e^{-x}$ and noting that since $r(x) = xe^{-x}$, $r(0) = 0$ and $\lim_{x\to\infty} r(x) = 0$, we have the orthogonality relation

$$\int_0^\infty e^{-x}L_m(x)L_n(x)\,dx = 0,\ m \neq n.$$

**10.** To obtain the self-adjoint form we note that an integrating factor is $e^{\int -2x\,dx} = e^{-x^2}$. Thus, the differential equation is

$$e^{-x^2}y'' - 2xe^{-x^2}y' + 2ne^{-x^2}y = 0$$

and the self-adjoint form is

$$\frac{d}{dx}\left[e^{-x^2}y'\right] + 2ne^{-x^2}y = 0.$$

Identifying the weight function $p(x) = e^{-x^2}$ and noting that since $r(x) = e^{-x^2}$, $\lim\limits_{x\to-\infty} r(x) = \lim\limits_{x\to\infty} r(x) = 0$, we have the orthogonality relation

$$\int_{-\infty}^\infty e^{-x^2}H_m(x)H_n(x)\,dx = 0,\ m \neq n.$$

**11. (a)** The differential equation is

$$(1+x^2)y'' + 2xy' + \frac{\lambda}{1+x^2}\,y = 0.$$

Letting $x = \tan\theta$ we have $\theta = \tan^{-1}x$ and

$$\frac{dy}{dx} = \frac{dy}{d\theta}\frac{d\theta}{dx} = \frac{1}{1+x^2}\frac{dy}{d\theta}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{1}{1+x^2}\frac{dy}{d\theta}\right] = \frac{1}{1+x^2}\left(\frac{d^2y}{d\theta^2}\frac{d\theta}{dx}\right) - \frac{2x}{(1+x^2)^2}\frac{dy}{d\theta}$$

$$= \frac{1}{(1+x^2)^2}\frac{d^2y}{d\theta^2} - \frac{2x}{(1+x^2)^2}\frac{dy}{d\theta}.$$

The differential equation can then be written in terms of $y(\theta)$ as

$$(1+x^2)\left[\frac{1}{(1+x^2)^2}\frac{d^2y}{d\theta^2} - \frac{2x}{(1+x^2)^2}\frac{dy}{d\theta}\right] + 2x\left[\frac{1}{1+x^2}\frac{dy}{d\theta}\right] + \frac{\lambda}{1+x^2}\,y$$

$$= \frac{1}{1+x^2}\frac{d^2y}{d\theta^2} + \frac{\lambda}{1+x^2}\,y = 0$$

or

$$\frac{d^2y}{d\theta^2} + \lambda y = 0.$$

The boundary conditions become $y(0) = y(\pi/4) = 0$. For $\lambda \le 0$ the only solution of the boundary-value problem is $y = 0$. For $\lambda = \alpha^2 > 0$ the general solution of the differential equation is $y = c_1\cos\alpha\theta + c_2\sin\alpha\theta$. The condition $y(0) = 0$ implies $c_1 = 0$ so $y = c_2\sin\alpha\theta$. Now the condition $y(\pi/4) = 0$ implies $c_2\sin\alpha\pi/4 = 0$. For $c_2 \ne 0$ this condition will hold when $\alpha\pi/4 = n\pi$ or $\lambda = \alpha^2 = 16n^2$, where $n = 1, 2, 3, \ldots$. These are the eigenvalues with corresponding eigenfunctions $\sin 4n\theta = \sin(4n\tan^{-1}x)$, for $n = 1$, 2, 3, ... .

**(b)** An orthogonality relation is

$$\int_0^1 \frac{1}{x^2+1}\sin\left(4m\tan^{-1}x\right)\sin\left(4n\tan^{-1}x\right)dx = 0, \qquad m \ne n.$$

**12. (a)** Letting $\lambda = \alpha^2$ the differential equation becomes $x^2y'' + xy' + (\alpha^2x^2 - 1)y = 0$. This is the parametric Bessel equation with $\nu = 1$. The general solution is

$$y = c_1J_1(\alpha x) + c_2Y_1(\alpha x).$$

Since $Y$ is unbounded at 0 we must have $c_2 = 0$, so that $y = c_1J_1(\alpha x)$. The condition $J_1(3\alpha) = 0$ defines the eigenvalues $\lambda_n = \alpha_n^2$ for $n = 1, 2, 3, \ldots$. The corresponding eigenfunctions are $J_1(\alpha_n x)$.

**(b)** Using a CAS or Table 5.1 in the text to solve $J_1(3\alpha) = 0$ we find $3\alpha_1 = 3.8317$, $3\alpha_2 = 7.0156$, $3\alpha_3 = 10.1735$, and $3\alpha_4 = 13.3237$. The corresponding eigenvalues are $\lambda_1 = \alpha_1^2 = 1.6313$, $\lambda_2 = \alpha_2^2 = 5.4687$, $\lambda_3 = \alpha_3^2 = 11.4999$, and $\lambda_4 = \alpha_4^2 = 19.7245$.

**13.** When $\lambda = 0$ the differential equation is $r(x)y'' + r'(x)y' = 0$. By inspection we see that $y = 1$ is a solution of the boundary-value problem. Thus, $\lambda = 0$ is an eigenvalue.

**14. (a)** An orthogonality relation is

$$\int_0^1 \cos x_m x \cos x_n x \, dx = 0$$

where $x_m \neq x_n$ are positive solutions of $\cot x = x$.

**(b)** Referring to Problem 1 we use a CAS to compute

$$\int_0^1 (\cos 0.8603x)(\cos 3.4256x) \, dx = -1.8771 \times 10^{-6} \approx 0.$$

**15. (a)** An orthogonality relation is

$$\int_0^1 (x_m \cos x_m x - \sin x_m x)(x_n \cos x_n x - \sin x_n x) \, dx = 0$$

where $x_m \neq x_n$ are positive solutions of $\tan x = x$.

**(b)** Referring to Problem 2 we use a CAS to compute

$$\int_0^1 (4.4934 \cos 4.4934x - \sin 4.4934x)(7.7253 \cos 7.7253x - \sin 7.7253x) \, dx = -2.5650 \times 10^{-4}$$

$$\approx 0.$$

## 11.5 Bessel and Legendre Series

**1.** Identifying $b = 3$, we have $\alpha_1 = 1.2772$, $\alpha_2 = 2.3385$, $\alpha_3 = 3.3912$, and $\alpha_4 = 4.4412$.

**2.** By (6) in the text $J_0'(2\alpha) = -J_1(2\alpha)$. Thus, $J_0'(2\alpha) = 0$ is equivalent to $J_1(2\alpha) = 0$. Then $\alpha_1 = 1.9159$, $\alpha_2 = 3.5078$, $\alpha_3 = 5.0867$, and $\alpha_4 = 6.6618$.

**3.** The boundary condition indicates that we use (15) and (16) in the text. With $b = 2$ we obtain

$$\begin{aligned} c_i &= \frac{2}{4J_1^2(2\alpha_i)} \int_0^2 x J_0(\alpha_i x) \, dx \qquad \boxed{t = \alpha_i x, \quad dt = \alpha_i \, dx} \\ &= \frac{1}{2J_1^2(2\alpha_i)} \cdot \frac{1}{\alpha_i^2} \int_0^{2\alpha_i} t J_0(t) \, dt \\ &= \frac{1}{2\alpha_i^2 J_1^2(2\alpha_i)} \int_0^{2\alpha_i} \frac{d}{dt}[tJ_1(t)] \, dt \qquad \text{[From (5) in the text]} \\ &= \frac{1}{2\alpha_i^2 J_1^2(2\alpha_i)} tJ_1(t) \Big|_0^{2\alpha_i} = \frac{1}{\alpha_i J_1(2\alpha_i)}. \end{aligned}$$

Thus

$$f(x)=\sum_{i=1}^{\infty}\frac{1}{\alpha_iJ_1(2\alpha_i)}J_0(\alpha_i x).$$

**4.** The boundary condition indicates that we use (19) and (20) in the text. With $b=2$ we obtain

$$\begin{aligned}
c_1&=\frac{2}{4}\int_0^2 x\,dx=\frac{2}{4}\,\frac{x^2}{2}\bigg|_0^2=1,\\
c_i&=\frac{2}{4J_0^2(2\alpha_i)}\int_0^2 xJ_0(\alpha_i x)\,dx && \boxed{t=\alpha_i x,\quad dt=\alpha_i\,dx}\\
&=\frac{1}{2J_0^2(2\alpha_i)}\cdot\frac{1}{\alpha_i^2}\int_0^{2\alpha_i}tJ_0(t)\,dt\\
&=\frac{1}{2\alpha_i^2J_0^2(2\alpha_i)}\int_0^{2\alpha_i}\frac{d}{dt}[tJ_1(t)]\,dt && \text{[From (5) in the text]}\\
&=\frac{1}{2\alpha_i^2J_0^2(2\alpha_i)}tJ_1(t)\bigg|_0^{2\alpha_i}=\frac{J_1(2\alpha_i)}{\alpha_iJ_0^2(2\alpha_i)}\,.
\end{aligned}$$

Now since $J_0'(2\alpha_i)=0$ is equivalent to $J_1(2\alpha_i)=0$ we conclude $c_i=0$ for $i=2,\,3,\,4,\,\ldots$ . Thus the expansion of $f$ on $0<x<2$ consists of a series with one nontrivial term:

$$f(x)=c_1=1.$$

**5.** The boundary condition indicates that we use (17) and (18) in the text. With $b=2$ and $h=1$ we obtain

$$\begin{aligned}
c_i&=\frac{2\alpha_i^2}{(4\alpha_i^2+1)J_0^2(2\alpha_i)}\int_0^2 xJ_0(\alpha_i x)\,dx && \boxed{t=\alpha_i x,\quad dt=\alpha_i\,dx}\\
&=\frac{2\alpha_i^2}{(4\alpha_i^2+1)J_0^2(2\alpha_i)}\cdot\frac{1}{\alpha_i^2}\int_0^{2\alpha_i}tJ_0(t)\,dt\\
&=\frac{2}{(4\alpha_i^2+1)J_0^2(2\alpha_i)}\int_0^{2\alpha_i}\frac{d}{dt}[tJ_1(t)]\,dt && \text{[From (5) in the text]}\\
&=\frac{2}{(4\alpha_i^2+1)J_0^2(2\alpha_i)}tJ_1(t)\bigg|_0^{2\alpha_i}=\frac{4\alpha_iJ_1(2\alpha_i)}{(4\alpha_i^2+1)J_0^2(2\alpha_i)}\,.
\end{aligned}$$

Thus

$$f(x)=4\sum_{i=1}^{\infty}\frac{\alpha_iJ_1(2\alpha_i)}{(4\alpha_i^2+1)J_0^2(2\alpha_i)}J_0(\alpha_i x).$$

**6.** Writing the boundary condition in the form

$$2J_0(2\alpha) + 2\alpha J_0'(2\alpha) = 0$$

we identify $b = 2$ and $h = 2$. Using (17) and (18) in the text we obtain

$$\begin{aligned}
c_i &= \frac{2\alpha_i^2}{(4\alpha_i^2+4)J_0^2(2\alpha_i)}\int_0^2 xJ_0(\alpha_i x)\,dx \qquad \boxed{t=\alpha_i x, \quad dt = \alpha_i\,dx} \\
&= \frac{\alpha_i^2}{2(\alpha_i^2+1)J_0^2(2\alpha_i)}\cdot\frac{1}{\alpha_i^2}\int_0^{2\alpha_i} tJ_0(t)\,dt \\
&= \frac{1}{2(\alpha_i^2+1)J_0^2(2\alpha_i)}\int_0^{2\alpha_i}\frac{d}{dt}[tJ_1(t)]\,dt \qquad \text{[From (5) in the text]} \\
&= \frac{1}{2(\alpha_i^2+1)J_0^2(2\alpha_i)}tJ_1(t)\Big|_0^{2\alpha_i} = \frac{\alpha_i J_1(2\alpha_i)}{(\alpha_i^2+1)J_0^2(2\alpha_i)}.
\end{aligned}$$

Thus

$$f(x) = \sum_{i=1}^{\infty}\frac{\alpha_i J_1(2\alpha_i)}{(\alpha_i^2+1)J_0^2(2\alpha_i)}J_0(\alpha_i x).$$

**7.** The boundary condition indicates that we use (17) and (18) in the text. With $n = 1$, $b = 4$, and $h = 3$ we obtain

$$\begin{aligned}
c_i &= \frac{2\alpha_i^2}{(16\alpha_i^2-1+9)J_1^2(4\alpha_i)}\int_0^4 xJ_1(\alpha_i x)5x\,dx \qquad \boxed{t=\alpha_i x, \quad dt = \alpha_i\,dx} \\
&= \frac{5\alpha_i^2}{4(2\alpha_i^2+1)J_1^2(4\alpha_i)}\cdot\frac{1}{\alpha_i^3}\int_0^{4\alpha_i} t^2J_1(t)\,dt \\
&= \frac{5}{4\alpha_i(2\alpha_i^2+1)J_1^2(4\alpha_i)}\int_0^{4\alpha_i}\frac{d}{dt}[t^2J_2(t)]\,dt \qquad \text{[From (5) in the text]} \\
&= \frac{5}{4\alpha_i(2\alpha_i^2+1)J_1^2(4\alpha_i)}t^2J_2(t)\Big|_0^{4\alpha_i} = \frac{20\alpha_i J_2(4\alpha_i)}{(2\alpha_i^2+1)J_1^2(4\alpha_i)}.
\end{aligned}$$

Thus

$$f(x) = 20\sum_{i=1}^{\infty}\frac{\alpha_i J_2(4\alpha_i)}{(2\alpha_i^2+1)J_1^2(4\alpha_i)}J_1(\alpha_i x).$$

**8.** The boundary condition indicates that we use (15) and (16) in the text. With $n = 2$ and $b = 1$ we obtain

$$\begin{aligned} c_1 &= \frac{2}{J_3^2(\alpha_i)} \int_0^1 x J_2(\alpha_i x) x^2\,dx \qquad\qquad \boxed{t = \alpha_i x, \quad dt = \alpha_i\,dx} \\ &= \frac{2}{J_3^2(\alpha_i)} \cdot \frac{1}{\alpha_i^4} \int_0^{\alpha_i} t^3 J_2(t)\,dt \\ &= \frac{2}{\alpha_i^4 J_3^2(\alpha_i)} \int_0^{\alpha_i} \frac{d}{dt}[t^3 J_3(t)]\,dt \qquad\qquad \text{[From (5) in the text]} \\ &= \frac{2}{\alpha_i^4 J_3^2(\alpha_i)} t^3 J_3(t) \Big|_0^{\alpha_i} = \frac{2}{\alpha_i J_3(\alpha_i)}. \end{aligned}$$

Thus

$$f(x) = 2 \sum_{i=1}^{\infty} \frac{1}{\alpha_i J_3(\alpha_i)} J_2(\alpha_i x).$$

**9.** The boundary condition indicates that we use (19) and (20) in the text. With $b = 3$ we obtain

$$\begin{aligned} c_1 &= \frac{2}{9} \int_0^3 x x^2\,dx = \frac{2}{9} \frac{x^4}{4} \Big|_0^3 = \frac{9}{2}, \\ c_i &= \frac{2}{9 J_0^2(3\alpha_i)} \int_0^3 x J_0(\alpha_i x) x^2\,dx \qquad\qquad \boxed{t = \alpha_i x, \quad dt = \alpha_i\,dx} \\ &= \frac{2}{9 J_0^2(3\alpha_i)} \cdot \frac{1}{\alpha_i^4} \int_0^{3\alpha_i} t^3 J_0(t)\,dt \\ &= \frac{2}{9\alpha_i^4 J_0^2(3\alpha_i)} \int_0^{3\alpha_i} t^2 \frac{d}{dt}\left[t J_1(t)\right]\,dt \qquad\qquad \boxed{\begin{array}{ll} u = t^2 & dv = \frac{d}{dt}\left[t J_1(t)\right]\,dt \\ du = 2t\,dt & v = t J_1(t) \end{array}} \\ &= \frac{2}{9\alpha_i^4 J_0^2(3\alpha_i)} \left( t^3 J_1(t) \Big|_0^{3\alpha_i} - 2 \int_0^{3\alpha_i} t^2 J_1(t)\,dt \right). \end{aligned}$$

With $n = 0$ in equation (6) in the text we have $J_0'(x) = -J_1(x)$, so the boundary condition $J_0'(3\alpha_i) = 0$ implies $J_1(3\alpha_i) = 0$. Then

$$\begin{aligned} c_i &= \frac{2}{9\alpha_i^4 J_0^2(3\alpha_i)} \left( -2 \int_0^{3\alpha_i} \frac{d}{dt}\left[t^2 J_2(t)\right]\,dt \right) = \frac{2}{9\alpha_i^4 J_0^2(3\alpha_i)} \left( -2t^2 J_2(t) \Big|_0^{3\alpha_i} \right) \\ &= \frac{2}{9\alpha_i^4 J_0^2(3\alpha_i)} \left[ -18\alpha_i^2 J_2(3\alpha_i) \right] = \frac{-4 J_2(3\alpha_i)}{\alpha_i^2 J_0^2(3\alpha_i)}. \end{aligned}$$

Thus

$$f(x) = \frac{9}{2} - 4 \sum_{i=1}^{\infty} \frac{J_2(3\alpha_i)}{\alpha_i^2 J_0^2(3\alpha_i)} J_0(\alpha_i x).$$

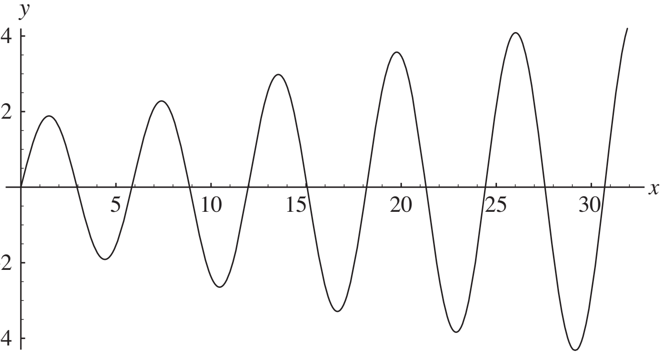
**10.** The boundary condition indicates that we use (15) and (16) in the text. With $b = 1$ it follows that

$$
\begin{aligned}
c_i &= \frac{2}{J_1^2(\alpha_i)} \int_0^1 x\left(1 - x^2\right) J_0(\alpha_i x)\,dx \\
&= \frac{2}{J_1^2(\alpha_i)} \left[\int_0^1 x J_0(\alpha_i x)\,dx - \int_0^1 x^3 J_0(\alpha_i x)\,dx\right] \qquad \boxed{t = \alpha_i x, \quad dt = \alpha_i\,dx} \\
&= \frac{2}{J_1^2(\alpha_i)} \left[\frac{1}{\alpha_i^2}\int_0^{\alpha_i} t J_0(t)\,dt - \frac{1}{\alpha_i^4}\int_0^{\alpha_i} t^3 J_0(t)\,dt\right] \\
&= \frac{2}{J_1^2(\alpha_i)} \left[\frac{1}{\alpha_i^2}\int_0^{\alpha_i} \frac{d}{dt}\left[tJ_1(t)\right]\,dt - \frac{1}{\alpha_i^4}\int_0^{\alpha_i} t^2\frac{d}{dt}\left[tJ_1(t)\right]\,dt\right] \quad \boxed{\begin{array}{ll} u = t^2 & dv = \frac{d}{dt}\left[tJ_1(t)\right]\,dt \\ du = 2t\,dt & v = tJ_1(t) \end{array}} \\
&= \frac{2}{J_1^2(\alpha_i)} \left[\frac{1}{\alpha_i^2} tJ_1(t)\Big|_0^{\alpha_i} - \frac{1}{\alpha_i^4}\left(t^3 J_1(t)\Big|_0^{\alpha_i} - 2\int_0^{\alpha_i} t^2 J_1(t)\,dt\right)\right] \\
&= \frac{2}{J_1^2(\alpha_i)} \left[\frac{J_1(\alpha_i)}{\alpha_i} - \frac{J_1(\alpha_i)}{\alpha_i} + \frac{2}{\alpha_i^4}\int_0^{\alpha_i} \frac{d}{dt}\left[t^2 J_2(t)\right]\,dt\right] = \frac{2}{J_1^2(\alpha_i)} \left[\frac{2}{\alpha_i^4} t^2 J_2(t)\Big|_0^{\alpha_i}\right] \\
&= \frac{4J_2(\alpha_i)}{\alpha_i^2 J_1^2(\alpha_i)}\,.
\end{aligned}
$$

Thus

$$f(x) = 4\sum_{i=1}^{\infty} \frac{J_2(\alpha_i)}{\alpha_i^2 J_1^2(\alpha_i)}\, J_0(\alpha_i x).$$

**11. (a)**



**(b)** Using **FindRoot** in *Mathematica* we find the roots $x_1 = 2.9496$, $x_2 = 5.8411$, $x_3 = 8.8727$, $x_4 = 11.9561$, and $x_5 = 15.0624$.

**(c)** Dividing the roots in part (**b**) by 4 we find the eigenvalues $\alpha_1 = 0.7374$, $\alpha_2 = 1.4603$, $\alpha_3 = 2.2182$, $\alpha_4 = 2.9890$, and $\alpha_5 = 3.7656$.

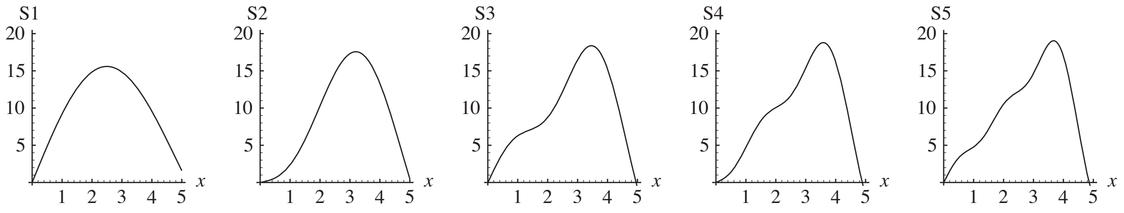
**(d)** The next five eigenvalues are $\alpha_6 = 4.5451$, $\alpha_7 = 5.3263$, $\alpha_8 = 6.1085$, $\alpha_9 = 6.8915$, and $\alpha_{10} = 7.6749$.

**12.** **(a)** From Problem 7, the coefficients of the Fourier-Bessel series are

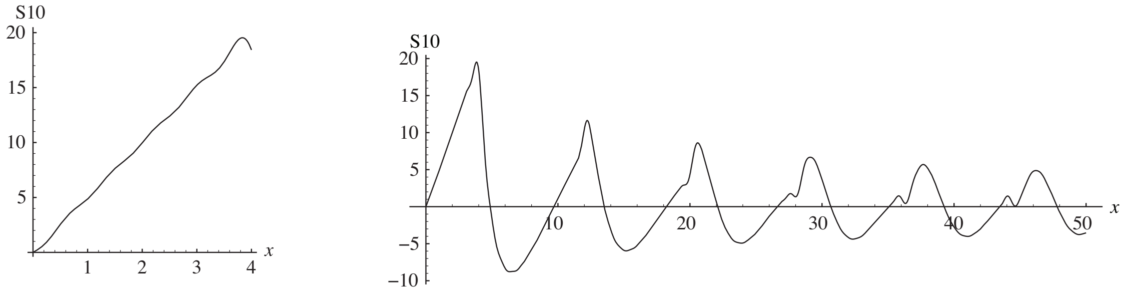
$$c_i = \frac{20\alpha_i J_2(4\alpha_i)}{(2\alpha_i^2+1)J_1^2(4\alpha_i)}.$$

Using a CAS we find $c_1 = 26.7896$, $c_2 = -12.4624$, $c_3 = 7.1404$, $c_4 = -4.68705$, and $c_5 = 3.35619$.

**(b)**

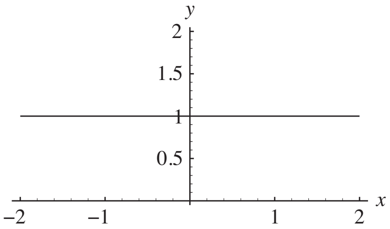


**(c)**



**13.** Since $f$ is expanded as a series of Bessel functions, $J_1(\alpha_i x)$ and $J_1$ is an odd function, the series should represent an odd function.

**14.** **(a)** Since $J_0$ is an even function, a series expansion of a function defined on $(0, 2)$ would converge to the even extension of the function on $(-2, 0)$.



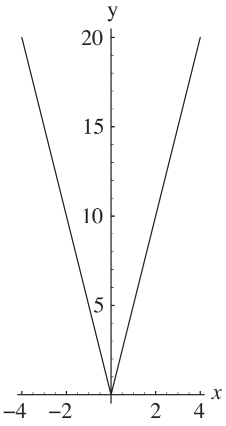
**(b)** In Section 6.3 we saw that $J_2'(x) = 2J_2(x)/x - J_3(x)$. Since $J_2$ is even and $J_3$ is odd we see that

$$\begin{aligned} J_2'(-x) &= 2J_2(-x)/(-x) - J_3(-x) \\ &= -2J_2(x)/x + J_3(x) = -J_2'(x), \end{aligned}$$

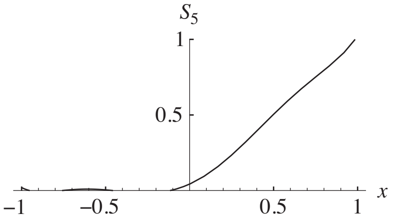
so that $J_2'$ is an odd function. Now, if $f(x) = 3J_2(x) + 2xJ_2'(x)$, we see that

$$\begin{aligned} f(-x) &= 3J_2(-x) - 2xJ_2'(-x) \\ &= 3J_2(x) + 2xJ_2'(x) = f(x), \end{aligned}$$

so that $f$ is an even function. Thus, a series expansion of a function defined on $(0, 4)$ would converge to the even extension of the function on $(-4, 0)$.



**15.** We compute



$$c_0 = \frac{1}{2}\int_0^1 xP_0(x)\,dx = \frac{1}{2}\int_0^1 x\,dx = \frac{1}{4}$$

$$c_1 = \frac{3}{2}\int_0^1 xP_1(x)\,dx = \frac{3}{2}\int_0^1 x^2\,dx = \frac{1}{2}$$

$$c_2 = \frac{5}{2}\int_0^1 xP_2(x)\,dx = \frac{5}{2}\int_0^1 \frac{1}{2}(3x^3 - x)\,dx = \frac{5}{16}$$

$$c_3 = \frac{7}{2}\int_0^1 xP_3(x)\,dx = \frac{7}{2}\int_0^1 \frac{1}{2}(5x^4 - 3x^2)\,dx = 0$$

$$c_4 = \frac{9}{2}\int_0^1 xP_4(x)\,dx = \frac{9}{2}\int_0^1 \frac{1}{8}(35x^5 - 30x^3 + 3x)\,dx = -\frac{3}{32}$$

$$c_5 = \frac{11}{2}\int_0^1 xP_5(x)\,dx = \frac{11}{2}\int_0^1 \frac{1}{8}(63x^6 - 70x^4 + 15x^2)\,dx = 0$$

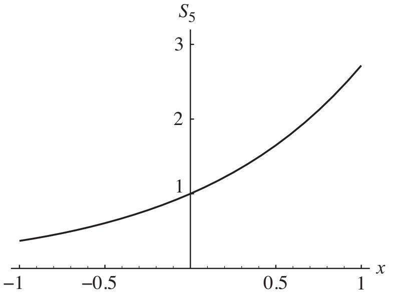
$$c_6 = \frac{13}{2}\int_0^1 xP_6(x)\,dx = \frac{13}{2}\int_0^1 \frac{1}{16}(231x^7 - 315x^5 + 105x^3 - 5x)\,dx = \frac{13}{256}\,.$$

Thus

$$f(x) = \frac{1}{4}P_0(x) + \frac{1}{2}P_1(x) + \frac{5}{16}P_2(x) - \frac{3}{32}P_4(x) + \frac{13}{256}P_6(x) + \cdots\,.$$

The figure above is the graph of $S_5(x) = \frac{1}{4}P_0(x) + \frac{1}{2}P_1(x) + \frac{5}{16}P_2(x) - \frac{3}{32}P_4(x) + \frac{13}{256}P_6(x)$.

**16.** We compute



$$c_0 = \frac{1}{2}\int_{-1}^1 e^x P_0(x)\,dx = \frac{1}{2}\int_{-1}^1 e^x\,dx = \frac{1}{2}(e - e^{-1})$$

$$c_1 = \frac{3}{2}\int_{-1}^1 e^x P_1(x)\,dx = \frac{3}{2}\int_{-1}^1 xe^x\,dx = 3e^{-1}$$

$$c_2 = \frac{5}{2}\int_{-1}^1 e^x P_2(x)\,dx = \frac{5}{2}\int_{-1}^1 \frac{1}{2}(3x^2e^x - e^x)\,dx$$

$$= \frac{5}{2}(e - 7e^{-1})$$

$$c_3 = \frac{7}{2}\int_{-1}^1 e^x P_3(x)\,dx = \frac{7}{2}\int_{-1}^1 \frac{1}{2}(5x^3e^x - 3xe^x)\,dx = \frac{7}{2}(-5e + 37e^{-1})$$

$$c_4 = \frac{9}{2}\int_{-1}^1 e^x P_4(x)\,dx = \frac{9}{2}\int_{-1}^1 \frac{1}{8}(35x^4e^x - 30x^2e^x + 3e^x)\,dx = \frac{9}{2}(36e - 266e^{-1}).$$

Thus

$$f(x) = \frac{1}{2}(e - e^{-1})P_0(x) + 3e^{-1}P_1(x) + \frac{5}{2}(e - 7e^{-1})P_2(x) + \frac{7}{2}(-5e + 37e^{-1})P_3(x) + \frac{9}{2}(36e - 266e^{-1})P_4(x) + \cdots .$$

The figure above is the graph of $S_5(x)$.

**17.** Using $\cos^2\theta = \frac{1}{2}(\cos 2\theta + 1)$ we have

$$P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1) = \frac{3}{2}\cos^2\theta - \frac{1}{2} = \frac{3}{4}(\cos 2\theta + 1) - \frac{1}{2} = \frac{3}{4}\cos 2\theta + \frac{1}{4} = \frac{1}{4}(3\cos 2\theta + 1).$$

**18.** From Problem 17 we have

$$P_2(\cos\theta) = \frac{1}{4}(3\cos 2\theta + 1) \qquad \text{or} \qquad \cos 2\theta = \frac{4}{3}P_2(\cos\theta) - \frac{1}{3}\,.$$

Then, using $P_0(\cos\theta) = 1$,

$$F(\theta) = 1 - \cos 2\theta = 1 - \left[\frac{4}{3}P_2(\cos\theta) - \frac{1}{3}\right] = \frac{4}{3} - \frac{4}{3}P_2(\cos\theta) = \frac{4}{3}P_0(\cos\theta) - \frac{4}{3}P_2(\cos\theta).$$

**19.** If $f$ is an even function on $(-1,1)$ then

$$\int_{-1}^{1} f(x)P_{2n}(x)\,dx = 2\int_0^1 f(x)P_{2n}(x)\,dx$$

and

$$\int_{-1}^{1} f(x)P_{2n+1}(x)\,dx = 0.$$

Thus

$$c_{2n} = \frac{2(2n)+1}{2}\int_{-1}^{1} f(x)P_{2n}(x)\,dx = \frac{4n+1}{2}\left(2\int_0^1 f(x)P_{2n}(x)\,dx\right) = (4n+1)\int_0^1 f(x)P_{2n}(x)\,dx,$$

$c_{2n+1} = 0$, and

$$f(x) = \sum_{n=0}^{\infty} c_{2n}P_{2n}(x).$$

**20.** If $f$ is an odd function on $(-1,1)$ then

$$\int_{-1}^{1} f(x)P_{2n}(x)\,dx = 0$$

and

$$\int_{-1}^{1} f(x)P_{2n+1}(x)\,dx = 2\int_0^1 f(x)P_{2n+1}(x)\,dx.$$

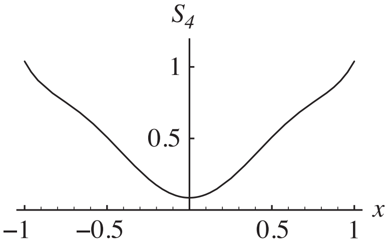
Thus

$$c_{2n+1} = \frac{2(2n+1)+1}{2}\int_{-1}^{1} f(x)P_{2n+1}(x)\,dx = \frac{4n+3}{2}\left(2\int_0^1 f(x)P_{2n+1}(x)\,dx\right)$$
$$= (4n+3)\int_0^1 f(x)P_{2n+1}(x)\,dx,$$

$c_{2n} = 0$, and

$$f(x) = \sum_{n=0}^{\infty} c_{2n+1}P_{2n+1}(x).$$

**21.** From (26) in Problem 19 in the text we find



$$c_0 = \int_0^1 xP_0(x)\,dx = \int_0^1 x\,dx = \frac{1}{2}$$

$$c_2 = 5\int_0^1 xP_2(x)\,dx = 5\int_0^1 \frac{1}{2}(3x^3 - x)\,dx = \frac{5}{8}$$

$$c_4 = 9\int_0^1 xP_4(x)\,dx = 9\int_0^1 \frac{1}{8}(35x^5 - 30x^3 + 3x)\,dx = -\frac{3}{16}$$

and

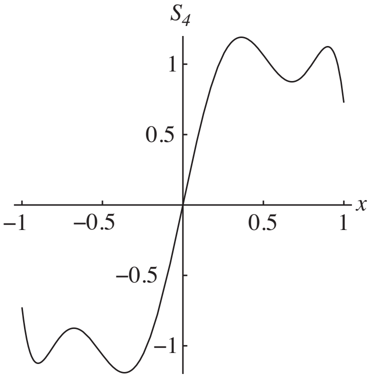
$$c_6 = 13\int_0^1 xP_6(x)\,dx = 13\int_0^1 \frac{1}{16}(231x^7 - 315x^5 + 105x^3 - 5x)\,dx = \frac{13}{128}.$$

Hence, from (25) in the text,

$$f(x) = \frac{1}{2}P_0(x) + \frac{5}{8}P_2(x) - \frac{3}{16}P_4(x) + \frac{13}{128}P_6 + \cdots.$$

On the interval $-1 < x < 1$ this series represents the function $f(x) = |x|$.

**22.** From (28) in Problem 20 in the text we find



$$c_1 = 3\int_0^1 P_1(x)\,dx = 3\int_0^1 x\,dx = \frac{3}{2},$$

$$c_3 = 7\int_0^1 P_3(x)\,dx = 7\int_0^1 \frac{1}{2}\left(5x^3 - 3x\right)\,dx = -\frac{7}{8},$$

$$c_5 = 11\int_0^1 P_5(x)\,dx = 11\int_0^1 \frac{1}{8}\left(63x^5 - 70x^3 + 15x\right)\,dx = \frac{11}{16}$$

and

$$c_7 = 15\int_0^1 P_7(x)\,dx = 15\int_0^1 \frac{1}{16}\left(429x^7 - 693x^5 + 315x^3 - 35x\right)\,dx = -\frac{75}{128}.$$

Hence, from (27) in the text,

$$f(x) = \frac{3}{2}P_1(x) - \frac{7}{8}P_3(x) + \frac{11}{16}P_5(x) - \frac{75}{128}P_7(x) + \cdots.$$

On the interval $-1 < x < 1$ this series represents the odd function

$$f(x) = \begin{cases} -1, & -1 < x < 0 \\ 1, & 0 < x < 1. \end{cases}$$

**23.** Since there is a Legendre polynomial of any specified degree, every polynomial can be represented as a finite linear combination of Legendre polynomials.

**24.** We want to express both $x^2$ and $x^3$ as linear combinations of $P_0(x) = 1, P_1(x) = x$, $P_2(x) = \frac{1}{2}(3x^2 - 1)$, and $P_3(x) = \frac{1}{2}(5x^3 - 3x)$. Setting

$$x^2 = c_0P_0(x) + c_1P_1(x) + c_2P_2(x) = c_0 + c_1x + c_2\left[\frac{1}{2}(3x^2 - 1)\right] = \left(c_0 - \frac{3}{2}c_2\right) + c_1x + \frac{3}{2}c_2x^2,$$

we obtain the system

$$\begin{aligned} c_0 - \frac{1}{2}c_2 &= 0 \\ c_1 &= 0 \\ \frac{3}{2}c_2 &= 1. \end{aligned}$$

The solution is $c_0 = \frac{1}{3}, c_1 = 0, c_2 = \frac{2}{3}$. Thus, $x^2 = \frac{1}{3}P_0(x) + \frac{2}{3}P_2(x)$. Setting

$$x^3 = c_0P_0(x) + c_1P_1(x) + c_2P_2(x) + c_3P_3(x) = c_0 + c_1x + c_2\left[\frac{1}{2}(3x^2 - 1)\right] + c_3\left[\frac{1}{2}(5x^3 - 3x)\right]$$

$$= \left(c_0 - \frac{1}{2}c_2\right) + \left(c_1 - \frac{3}{2}c_3\right)x + \frac{3}{2}c_2x^2 + \frac{5}{2}c_3x^3,$$

we obtain the system

$$\begin{aligned} c_0 - \frac{1}{2}c_2 &= 0 \\ c_1 - \frac{3}{2}c_3 &= 0 \\ \frac{3}{2}c_2 &= 0 \\ \frac{5}{2}c_3 &= 1. \end{aligned}$$

The solution is $c_0 = 0, c_1 = \frac{3}{5}, c_2 = 0, c_3 = \frac{2}{5}$. Thus $x^3 = \frac{3}{5}P_1(x) + \frac{2}{5}P_3(x)$.

## Chapter 11 in Review

**1.** True, since $\int_{-\pi}^{\pi}(x^2-1)x^5\,dx = 0$

**2.** Even, since if $f$ and $g$ are odd then $h(-x) = f(-x)g(-x) = -f(x)[-g(x)] = f(x)g(x) = h(x)$

**3.** cosine, since $f$ is even

**4.** True

**5.** False; the Sturm-Liouville problem,

$$\frac{d}{dx}[r(x)y'] + \lambda p(x)y = 0, \qquad y'(a) = 0, \quad y'(b) = 0,$$

on the interval $[a,b]$, has eigenvalue $\lambda = 0$.

**6.** Periodically extending the function we see that at $x=-1$ the function converges to $\frac{1}{2}(-1+0) = -\frac{1}{2}$; at $x=0$ it converges to $\frac{1}{2}(0+1) = \frac{1}{2}$, and at $x=1$ it converges to $\frac{1}{2}(-1+0) = -\frac{1}{2}$.

**7.** The Fourier series will converge to 1, the cosine series to 1, and the sine series to 0 at $x=0$. Respectively, this is because the rule $(x^2+1)$ defining $f(x)$ determines a continuous function on $(-3,3)$, the even extension of $f$ to $(-3,0)$ is continuous at 0, and the odd extension of $f$ to $(-3,0)$ approaches $-1$ as $x$ approaches 0 from the left.

**8.** $\cos 5x$, since the general solution is $y = c_1\cos\alpha x + c_2\sin\alpha x$ and $y'(0) = 0$ implies $c_2 = 0$.

**9.** Since the coefficient of $y$ in the differential equation is $n^2$, the weight function is the integrating factor

$$\frac{1}{a(x)}e^{\int (b/a)\,dx} = \frac{1}{1-x^2}e^{\int -\frac{x}{1-x^2}\,dx} = \frac{1}{1-x^2}e^{\frac{1}{2}\ln(1-x^2)} = \frac{\sqrt{1-x^2}}{1-x^2} = \frac{1}{\sqrt{1-x^2}}$$

on the interval $[-1,1]$. The orthogonality relation is

$$\int_{-1}^{1}\frac{1}{\sqrt{1-x^2}}T_m(x)T_n(x)\,dx = 0, \quad m \neq n.$$

**10.** Since $P_n(x)$ is orthogonal to $P_0(x) = 1$ for $n > 0$,

$$\int_{-1}^{1}P_n(x)\,dx = \int_{-1}^{1}P_0(x)P_n(x)\,dx = 0.$$

**11.** We know from a half-angle formula in trigonometry that $\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$, which is a cosine series.

**12.** **(a)** For $m \neq n$

$$\int_0^L \sin\frac{(2n+1)\pi}{2L}x\,\sin\frac{(2m+1)\pi}{2L}x\,dx = \frac{1}{2}\int_0^L\left(\cos\frac{n-m}{L}\pi x - \cos\frac{n+m+1}{L}\pi x\right)dx$$

$$= 0.$$

**(b)** From

$$\int_0^L \sin^2 \frac{(2n+1)\pi}{2L} x\,dx = \int_0^L \left(\frac{1}{2} - \frac{1}{2}\cos\frac{(2n+1)\pi}{L}x\right) dx = \frac{L}{2}$$

we see that

$$\left\|\sin\frac{(2n+1)\pi}{2L}x\right\| = \sqrt{\frac{L}{2}}\,.$$

**13.** Since

$$a_0 = \int_{-1}^{0} (-2x)\,dx = 1,$$

$$a_n = \int_{-1}^{0} (-2x)\cos n\pi x\,dx = \frac{2}{n^2\pi^2}[(-1)^n - 1],$$

and

$$b_n = \int_{-1}^{0} (-2x)\sin n\pi x\,dx = \frac{2}{n\pi}(-1)^n$$

for $n = 1, 2, 3, \ldots$ we have

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{2}{n^2\pi^2}[(-1)^n - 1]\cos n\pi x + \frac{2}{n\pi}(-1)^n \sin n\pi x\right).$$

**14.** Since

$$a_0 = \int_{-1}^{1} (2x^2 - 1)\,dx = -\frac{2}{3},$$

$$a_n = \int_{-1}^{1} (2x^2 - 1)\cos n\pi x\,dx = \frac{8}{n^2\pi^2}(-1)^n,$$

and

$$b_n = \int_{-1}^{1} (2x^2 - 1)\sin n\pi x\,dx = 0$$

for $n = 1, 2, 3, \ldots$ we have

$$f(x) = -\frac{1}{3} + \sum_{n=1}^{\infty}\frac{8}{n^2\pi^2}(-1)^n\cos n\pi x.$$

**15.** **(a)** Since

$$a_0 = \frac{2}{1}\int_0^1 e^{-x}\,dx = 2(1 - e^{-1})$$

and

$$a_n = \frac{2}{1}\int_{-1}^{1} e^{-x}\cos n\pi x\,dx = \frac{2}{1+n^2\pi^2}[1 - (-1)^n e^{-1}],$$

for $n = 1, 2, 3, \ldots$ , we have the cosine series

$$f(x) = 1 - e^{-1} + 2\sum_{n=1}^{\infty}\frac{1 - (-1)^n e^{-1}}{1+n^2\pi^2}\cos n\pi x.$$

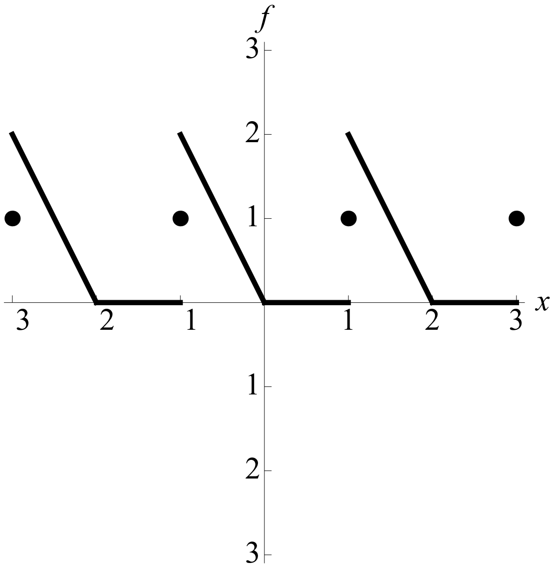
**(b)** Since

$$b_n = \frac{2}{1}\int_0^1 e^{-x}\sin n\pi x\,dx = \frac{2n\pi}{1+n^2\pi^2}[1-(-1)^n e^{-1}],$$

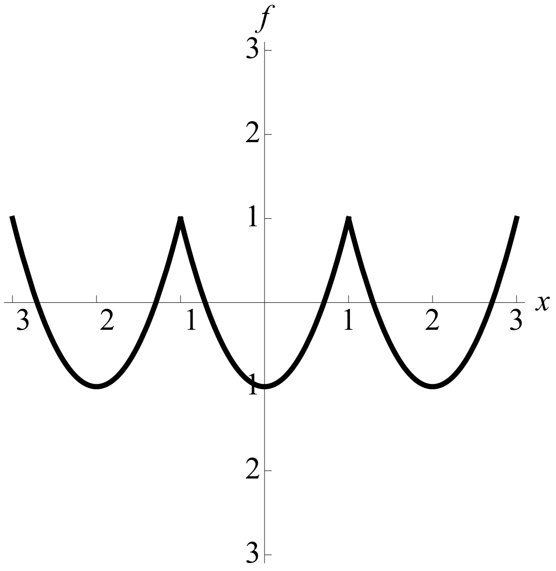
for $n = 1, 2, 3, \ldots$ , we have the sine series

$$f(x) = 2\pi\sum_{n=1}^{\infty}\frac{n[1-(-1)^n e^{-1}]}{1+n^2\pi^2}\sin n\pi x.$$

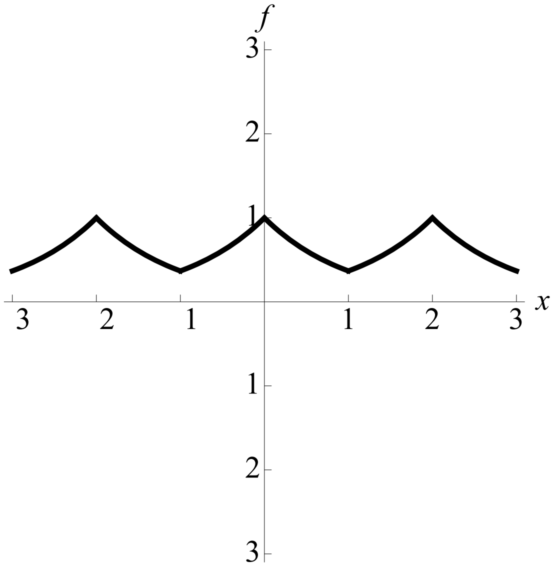
**16.**



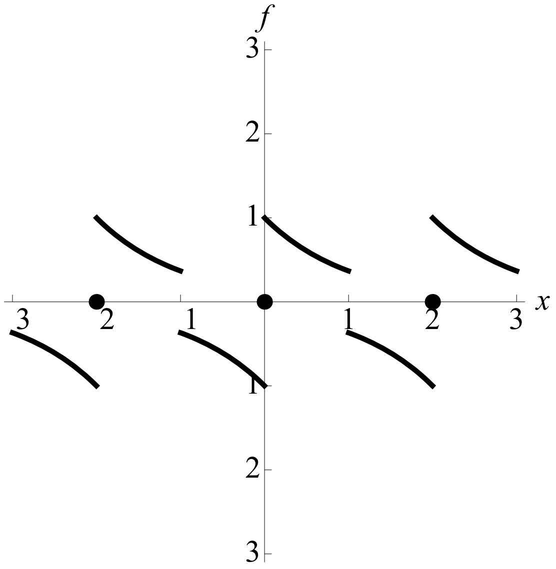
$f(x) = |x| - x, \ \ -1 < x < 1$



$f(x) = 2x^2 - 1, \ \ -1 < x < 1$



$f(x) = e^{-|x|}$



$$f(x) = \begin{cases} -e^{-x}, & 0 < x < 1 \\ 0, & x = 0 \\ -e^{x}, & -1 < x < 0 \end{cases}$$

**17.** The cosine series of $f$ in Problem 15 converges to $F(x)$ on the interval $-1 < x < 1$ since $F$ is the even extension of $f$ to the interval.

**18.** Expanding in a full Fourier series we have

$$a_0 = \frac{1}{2}\left(\int_0^2 x\,dx + \int_2^4 2\,dx\right) = 3$$

$$a_n = \frac{1}{2}\left(\int_0^2 x\cos\frac{n\pi x}{2}\,dx + \int_2^4 2\cos\frac{n\pi x}{2}\,dx\right) = 2\,\frac{(-1)^n - 1}{n^2\pi^2}$$

$$b_n = \frac{1}{2}\left(\int_0^2 x\sin\frac{n\pi x}{2}\,dx + \int_2^4 2\sin\frac{n\pi x}{2}\,dx\right) = -\frac{2}{n\pi}$$

so

$$f(x) = \frac{3}{2} + 2\sum_{n=1}^{\infty}\left(\frac{(-1)^n - 1}{n^2\pi^2}\cos\frac{n\pi x}{2} - \frac{1}{n\pi}\sin\frac{n\pi x}{2}\right).$$

**19.** For $\lambda = \alpha^2 > 0$ a general solution of the given differential equation is

$$y = c_1\cos(3\alpha\ln x) + c_2\sin(3\alpha\ln x)$$

and

$$y' = -\frac{3c_1\alpha}{x}\sin(3\alpha\ln x) + \frac{3c_2\alpha}{x}\cos(3\alpha\ln x).$$

Since $\ln 1 = 0$, the boundary condition $y'(1) = 0$ implies $c_2 = 0$. Therefore

$$y = c_1\cos(3\alpha\ln x).$$

Using $\ln e = 1$ we find that $y(e) = 0$ implies $c_1\cos 3\alpha = 0$ or $3\alpha = (2n-1)\pi/2$, for $n = 1, 2, 3, \ldots$. The eigenvalues are $\lambda = \alpha^2 = (2n-1)^2\pi^2/36$ with corresponding eigenfunctions $\cos[(2n-1)\pi(\ln x)/2]$ for $n = 1, 2, 3, \ldots$.

**20.** To obtain the self-adjoint form of the differential equation in Problem 19 we note that an integrating factor is $(1/x^2)e^{\int dx/x} = 1/x$. Thus the weight function is $1/x$ and an orthogonality relation is

$$\int_1^e \frac{1}{x}\cos\left(\frac{2n-1}{2}\pi\ln x\right)\cos\left(\frac{2m-1}{2}\pi\ln x\right)dx = 0,\ m \neq n.$$

**21.** The boundary condition indicates that we use (15) and (16) of Section 11.5 in the text. With $b = 4$ we obtain

$$\begin{aligned}
c_i &= \frac{2}{16J_1^2(4\alpha_i)}\int_0^4 xJ_0(\alpha_i x)f(x)\,dx \\
&= \frac{1}{8J_1^2(4\alpha_i)}\int_0^2 xJ_0(\alpha_i x)\,dx \qquad \boxed{t=\alpha_i x, \quad dt=\alpha_i\,dx} \\
&= \frac{1}{8J_1^2(4\alpha_i)}\cdot\frac{1}{\alpha_i^2}\int_0^{2\alpha_i} tJ_0(t)\,dt \\
&= \frac{1}{8J_1^2(4\alpha_i)}\int_0^{2\alpha_i}\frac{d}{dt}[tJ_1(t)]\,dt \qquad \text{[From (5) in 11.5 in the text]} \\
&= \frac{1}{8J_1^2(4\alpha_i)}tJ_1(t)\Big|_0^{2\alpha_i} = \frac{J_1(2\alpha_i)}{4\alpha_i J_1^2(4\alpha_i)}\,.
\end{aligned}$$

Thus

$$f(x) = \frac{1}{4}\sum_{i=1}^{\infty}\frac{J_1(2\alpha_i)}{\alpha_i J_1^2(4\alpha_i)}J_0(\alpha_i x.$$

**22.** Since $f(x) = x^4$ is a polynomial in $x$, an expansion of $f$ in Legendre polynomials in $x$ must terminate with the term having the same degree as $f$. Using the fact that $x^4P_1(x)$ and $x^4P_3(x)$ are odd functions, we see immediately that $c_1 = c_3 = 0$. Now

$$\begin{aligned}
c_0 &= \frac{1}{2}\int_{-1}^1 x^4P_0(x)\,dx = \frac{1}{2}\int_{-1}^1 x^4\,dx = \frac{1}{5} \\
c_2 &= \frac{5}{2}\int_{-1}^1 x^4P_2(x)\,dx = \frac{5}{2}\int_{-1}^1 \frac{1}{2}(3x^6 - x^4)\,dx = \frac{4}{7} \\
c_4 &= \frac{9}{2}\int_{-1}^1 x^4P_4(x)\,dx = \frac{9}{2}\int_{-1}^1 \frac{1}{8}(35x^8 - 30x^6 + 3x^4)\,dx = \frac{8}{35}\,.
\end{aligned}$$

Thus

$$f(x) = \frac{1}{5}P_0(x) + \frac{4}{7}P_2(x) + \frac{8}{35}P_4(x).$$

**23.** **(a)**

$$f_e(x) + f_o(x) = \frac{f(x) + f(-x) + f(x) - f(-x)}{2} = \frac{2f(x)}{2} = f(x)$$

**(b)**

$$\begin{aligned}
f_e(-x) &= \frac{f(-x) + f(-(-x))}{2} = \frac{f(-x) + f(x)}{2} = f_e(x) \\
f_o(-x) &= \frac{f(-x) - f(-(-x))}{2} = \frac{f(-x) - f(x)}{2} = -f_o(x)
\end{aligned}$$

**24.**

$$f_e(x) = \frac{f(x)+f(-x)}{2} = \frac{e^x + e^{-x}}{2} = \cosh x$$

$$f_o(x) = \frac{f(x)-f(-x)}{2} = \frac{e^x - e^{-x}}{2} = \sinh x$$

**25.** If we let $u = x + 2p$ then $dx = du$ and we get

$$\begin{aligned}\int_0^a f(x)\,dx &= \int_{2p}^{a+2p} f(u-2p)\,du \quad \longleftarrow \text{ since } f \text{ is } 2p-\text{periodic, } f(u-2p) = f(u)\\ &= \int_{2p}^{a+2p} f(u)\,du\\ &= \int_{2p}^{a+2p} f(x)\,dx\end{aligned}$$

Therefore we get

$$\begin{aligned}\int_a^{a+2p} f(x)\,dx &= \int_a^{2p} f(x)\,dx + \int_{2p}^{a+2p} f(x)\,dx\\ &= \int_a^{2p} f(x)\,dx + \left(\int_0^a f(x)\,dx\right)\\ &= \int_0^a f(x)\,dx + \int_a^{2p} f(x)\,dx\\ &= \int_0^{2p} f(x)\,dx\end{aligned}$$

# Chapter 12

## Boundary-Value Problems in Rectangular Coordinates

### 12.1 Separable Partial Differential Equations

**1.** Substituting $u(x,y) = X(x)Y(y)$ into the partial differential equation yields $X'Y = XY'$. Separating variables and using the separation constant $-\lambda$, where $\lambda \neq 0$, we obtain

$$\frac{X'}{X} = \frac{Y'}{Y} = -\lambda.$$

When $\lambda \neq 0$

$$X' + \lambda X = 0 \qquad \text{and} \qquad Y' + \lambda Y = 0$$

so that

$$X = c_1 e^{-\lambda x} \qquad \text{and} \qquad Y = c_2 e^{-\lambda y}.$$

A particular product solution of the partial differential equation is

$$u = XY = c_3 e^{-\lambda(x+y)}, \quad \lambda \neq 0.$$

When $\lambda = 0$ the differential equations become $X' = 0$ and $Y' = 0$, so in this case $X = c_4$, $Y = c_5$, and $u = XY = c_6$.

**2.** Substituting $u(x,y) = X(x)Y(y)$ into the partial differential equation yields $X'Y + 3XY' = 0$. Separating variables and using the separation constant $-\lambda$ we obtain

$$\frac{X'}{-3X} = \frac{Y'}{Y} = -\lambda.$$

When $\lambda \neq 0$

$$X' - 3\lambda X = 0 \qquad \text{and} \qquad Y' + \lambda Y = 0$$

so that

$$X = c_1 e^{3\lambda x} \qquad \text{and} \qquad Y = c_2 e^{-\lambda y}.$$

A particular product solution of the partial differential equation is

$$u = XY = c_3 e^{\lambda(3x-y)}.$$

When $\lambda = 0$ the differential equations become $X' = 0$ and $Y' = 0$, so in this case $X = c_4$, $Y = c_5$, and $u = XY = c_6$.

**3.** Substituting $u(x, y) = X(x)Y(y)$ into the partial differential equation yields $X'Y + XY' = XY$. Separating variables and using the separation constant $-\lambda$ we obtain

$$\frac{X'}{X} = \frac{Y - Y'}{Y} = -\lambda.$$

Then

$$X' + \lambda X = 0 \qquad \text{and} \qquad Y' - (1 + \lambda)Y = 0$$

so that

$$X = c_1 e^{-\lambda x} \qquad \text{and} \qquad Y = c_2 e^{(1+\lambda)y}.$$

A particular product solution of the partial differential equation is

$$u = XY = c_3 e^{y + \lambda(y - x)}.$$

**4.** Substituting $u(x, y) = X(x)Y(y)$ into the partial differential equation yields $X'Y = XY' + XY$. Separating variables and using the separation constant $-\lambda$ we obtain

$$\frac{X'}{X} = \frac{Y + Y'}{Y} = -\lambda.$$

Then

$$X' + \lambda X = 0 \qquad \text{and} \qquad y' + (1 + \lambda)Y = 0$$

so that

$$X = c_1 e^{-\lambda x} \qquad \text{and} \qquad Y = c_2 e^{-(1+\lambda)y} = 0.$$

A particular product solution of the partial differential equation is

$$u = XY = c_3 e^{-y - \lambda(x + y)}.$$

**5.** Substituting $u(x, y) = X(x)Y(y)$ into the partial differential equation yields $xX'Y = yXY'$. Separating variables and using the separation constant $-\lambda$ we obtain

$$\frac{xX'}{X} = \frac{yY'}{Y} = -\lambda.$$

When $\lambda \neq 0$

$$xX' + \lambda X = 0 \qquad \text{and} \qquad yY' + \lambda Y = 0$$

so that

$$X = c_1 x^{-\lambda} \qquad \text{and} \qquad Y = c_2 y^{-\lambda}.$$

A particular product solution of the partial differential equation is

$$u = XY = c_3 (xy)^{-\lambda}.$$

When $\lambda = 0$ the differential equations become $X' = 0$ and $Y' = 0$, so in this case $X = c_4$, $Y = c_5$, and $u = XY = c_6$.

**6.** Substituting $u(x,y) = X(x)Y(y)$ into the partial differential equation yields $yX'Y + xXY' = 0$. Separating variables and using the separation constant $-\lambda$ we obtain

$$\frac{X'}{xX} = -\frac{Y'}{yY} = -\lambda.$$

When $\lambda \neq 0$

$$X' + \lambda xX = 0 \qquad \text{and} \qquad Y' - \lambda yY = 0$$

so that

$$X = c_1 e^{\lambda x^2/2} \qquad \text{and} \qquad Y = c_2 e^{-\lambda y^2/2}.$$

A particular product solution of the partial differential equation is

$$u = XY = c_3 e^{\lambda(x^2-y^2)/2}.$$

When $\lambda = 0$ the differential equations become $X' = 0$ and $Y' = 0$, so in this case $X = c_4$, $Y = c_5$, and $u = XY = c_6$.

**7.** Substituting $u(x,y) = X(x)Y(y)$ into the partial differential equation yields $X''Y + X'Y' + XY'' = 0'$, which is not separable.

**8.** Substituting $u(x,y) = X(x)Y(y)$ into the partial differential equation yields $yX'Y' + XY = 0$. Separating variables and using the separation constant $-\lambda$ we obtain

$$\frac{X'}{X} = -\frac{Y}{yY'} = -\lambda.$$

When $\lambda \neq 0$

$$X' + \lambda X = 0 \qquad \text{and} \qquad \lambda yY' - Y = 0$$

so that

$$X = c_1 e^{-\lambda x} \qquad \text{and} \qquad Y = c_2 y^{1/\lambda}.$$

A particular product solution of the partial differential equation is

$$u = XY = c_3 e^{-\lambda x} y^{1/\lambda}.$$

In this case $\lambda = 0$ yields no solution.

**9.** Substituting $u(x,t) = X(x)T(t)$ into the partial differential equation yields $kX''T - XT = XT'$. Separating variables and using the separation constant $-\lambda$ we obtain

$$\frac{kX'' - X}{X} = \frac{T'}{T} = -\lambda.$$

Then

$$X'' + \frac{\lambda - 1}{k} X = 0 \qquad \text{and} \qquad T' + \lambda T = 0.$$

The second differential equation implies $T(t) = c_1 e^{-\lambda t}$. For the first differential equation we consider three cases:

**I.** If $(\lambda - 1)/k = 0$ then $\lambda = 1$, $X'' = 0$, and $X(x) = c_2 x + c_3$, so

$$u = XT = e^{-t}(A_1 x + A_2).$$

**II.** If $(\lambda - 1)/k = -\alpha^2 < 0$, then $\lambda = 1 - k\alpha^2$, $X'' - \alpha^2 X = 0$, and $X(x) = c_4 \cosh \alpha x + c_5 \sinh \alpha x$, so

$$u = XT = (A_3 \cosh \alpha x + A_4 \sinh \alpha x)e^{-(1-k\alpha^2)t}.$$

**III.** If $(\lambda - 1)/k = \alpha^2 > 0$, then $\lambda = 1 + k\alpha^2$, $X'' + \alpha^2 X = 0$, and $X(x) = c_6 \cos \alpha x + c_7 \sin \alpha x$, so

$$u = XT = (A_5 \cos \alpha x + A_6 \sin \alpha x)e^{-(1+\lambda\alpha^2)t}.$$

**10.** Substituting $u(x,t) = X(x)T(t)$ into the partial differential equation yields $kX''T = XT'$. Separating variables and using the separation constant $-\lambda$ we obtain

$$\frac{X''}{X} = \frac{T'}{kT} = -\lambda.$$

Then

$$X'' + \lambda X = 0 \qquad \text{and} \qquad T' + \lambda k T = 0.$$

The second differential equation implies $T(t) = c_1 e^{-\lambda k t}$. For the first differential equation we consider three cases:

**I.** If $\lambda = 0$ then $X'' = 0$ and $X(x) = c_2 x + c_3$, so

$$u = XT = A_1 x + A_2.$$

**II.** If $\lambda = -\alpha^2 < 0$, then $X'' - \alpha^2 X = 0$, and $X(x) = c_4 \cosh \alpha x + c_5 \sinh \alpha x$, so

$$u = XT = (A_3 \cosh \alpha x + A_4 \sinh \alpha x)e^{k\alpha^2 t}.$$

**III.** If $\lambda = \alpha^2 > 0$, then $X'' + \alpha^2 X = 0$, and $X(x) = c_6 \cos \alpha x + c_7 \sin \alpha x$, so

$$u = XT = (A_5 \cos \alpha x + A_6 \sin \alpha x)e^{-k\alpha^2 t}.$$

**11.** Substituting $u(x,t) = X(x)T(t)$ into the partial differential equation yields $a^2 X''T = XT''$. Separating variables and using the separation constant $-\lambda$ we obtain

$$\frac{X''}{X} = \frac{T''}{a^2 T} = -\lambda.$$

Then

$$X'' + \lambda X = 0 \qquad \text{and} \qquad T'' + a^2 \lambda T = 0.$$

We consider three cases:

**I.** If $\lambda = 0$ then $X'' = 0$ and $X(x) = c_1x + c_2$. Also, $T'' = 0$ and $T(t) = c_3t + c_4$, so

$$u = XT = (c_1x + c_2)(c_3t + c_4).$$

**II.** If $\lambda = -\alpha^2 < 0$, then $X'' - \alpha^2X = 0$, and $X(x) = c_5 \cosh \alpha x + c_6 \sinh \alpha x$. Also, $T'' - \alpha^2a^2T = 0$ and $T(t) = c_7 \cosh \alpha at + c_8 \sinh \alpha at$, so

$$u = XT = (c_5 \cosh \alpha x + c_6 \sinh \alpha x)(c_7 \cosh \alpha at + c_8 \sinh \alpha at).$$

**III.** If $\lambda = \alpha^2 > 0$, then $X'' + \alpha^2X = 0$, and $X(x) = c_9 \cos \alpha x + c_{10} \sin \alpha x$. Also, $T'' + \alpha^2a^2T = 0$ and $T(t) = c_{11} \cos \alpha at + c_{12} \sin \alpha at$, so

$$u = XT = (c_9 \cos \alpha x + c_{10} \sin \alpha x)(c_{11} \cos \alpha at + c_{12} \sin \alpha at).$$

**12.** Substituting $u(x,t) = X(x)T(t)$ into the partial differential equation yields $a^2X''T = XT'' + 2kXT'$. Separating variables and using the separation constant $-\lambda$ we obtain

$$\frac{X''}{X} = \frac{T'' + 2kT'}{a^2T} = -\lambda.$$

Then

$$X'' + \lambda X = 0 \qquad \text{and} \qquad T'' + 2kT' + a^2\lambda T = 0.$$

We consider three cases:

**I.** If $\lambda = 0$ then $X'' = 0$ and $X(x) = c_1x + c_2$. Also, $T'' + 2kT' = 0$ and $T(t) = c_3 + c_4e^{-2kt}$, so

$$u = XT = (c_1x + c_2)(c_3 + c_4e^{-2kt}).$$

**II.** If $\lambda = -\alpha^2 < 0$, then $X'' - \alpha^2X = 0$, and $X(x) = c_5 \cosh \alpha x + c_6 \sinh \alpha x$. The auxiliary equation of $T'' + 2kT' - \alpha^2a^2T = 0$ is $m^2 + 2km - \alpha^2a^2 = 0$. Solving for $m$ we obtain $m = -k \pm \sqrt{k^2 + \alpha^2a^2}$, so $T(t) = c_7e^{(-k+\sqrt{k^2+\alpha^2a^2})t} + c_8e^{(-k-\sqrt{k^2+\alpha^2a^2})t}$. Then

$$u = XT = (c_5 \cosh \alpha x + c_6 \sinh \alpha x)\left(c_7e^{(-k+\sqrt{k^2+\alpha^2a^2})t} + c_8e^{(-k-\sqrt{k^2+\alpha^2a^2})t}\right).$$

**III.** If $\lambda = \alpha^2 > 0$, then $X'' + \alpha^2X = 0$, and $X(x) = c_9 \cos \alpha x + c_{10} \sin \alpha x$. The auxiliary equation of $T'' + 2kT' + \alpha^2a^2T = 0$ is $m^2 + 2km + \alpha^2a^2 = 0$. Solving for $m$ we obtain $m = -k \pm \sqrt{k^2 - \alpha^2a^2}$. We consider three possibilities for the discriminant $k^2 - \alpha^2a^2$:

**(i)** If $k^2 - \alpha^2a^2 = 0$ then $T(t) = c_{11}e^{-kt} + c_{12}te^{-kt}$ and

$$u = XT = (c_9 \cos \alpha x + c_{10} \sin \alpha x)(c_{11}e^{-kt} + c_{12}te^{-kt}).$$

From $k^2 - \alpha^2a^2 = 0$ we have $\alpha = k/a$ so the solution can be written

$$u = XT = (c_9 \cos kx/a + c_{10} \sin kx/a)(c_{11}e^{-kt} + c_{12}te^{-kt}).$$

**(ii)** If $k^2 - \alpha^2 a^2 < 0$ then $T(t) = e^{-kt}\left(c_{13}\cos\sqrt{\alpha^2 a^2 - k^2}\,t + c_{14}\sin\sqrt{\alpha^2 a^2 - k^2}\,t\right)$ and

$$u = XT = (c_9\cos\alpha x + c_{10}\sin\alpha x)e^{-kt}\left(c_{13}\cos\sqrt{\alpha^2 a^2 - k^2}\,t + c_{14}\sin\sqrt{\alpha^2 a^2 - k^2}\,t\right).$$

**(iii)** If $k^2 - \alpha^2 a^2 > 0$ then $T(t) = c_{15}e^{(-k+\sqrt{k^2-\alpha^2 a^2})t} + c_{16}e^{(-k-\sqrt{k^2-\alpha^2 a^2})t}$ and

$$u = XT = (c_9\cos\alpha x + c_{10}\sin\alpha x)\left(c_{15}e^{(-k+\sqrt{k^2-\alpha^2 a^2})t} + c_{16}e^{(-k-\sqrt{k^2-\alpha^2 a^2})t}\right).$$

**13.** Substituting $u(x,y) = X(x)Y(y)$ into the partial differential equation yields $X''Y + XY'' = 0$. Separating variables and using the separation constant $-\lambda$ we obtain

$$-\frac{X''}{X} = \frac{Y''}{Y} = -\lambda.$$

Then

$$X'' - \lambda X = 0 \qquad \text{and} \qquad Y'' + \lambda Y = 0.$$

We consider three cases:

**I.** If $\lambda = 0$ then $X'' = 0$ and $X(x) = c_1x + c_2$. Also, $Y'' = 0$ and $Y(y) = c_3y + c_4$ so

$$u = XY = (c_1x + c_2)(c_3x + c_4).$$

**II.** If $\lambda = -\alpha^2 < 0$ then $X'' + \alpha^2 X = 0$ and $X(x) = c_5\cos\alpha x + c_6\sin\alpha x$. Also, $Y'' - \alpha^2 Y = 0$ and $Y(y) = c_7\cosh\alpha y + c_8\sinh\alpha y$ so

$$u = XY = (c_5\cos\alpha x + c_6\sin\alpha x)(c_7\cosh\alpha y + c_8\sinh\alpha y).$$

**III.** If $\lambda = \alpha^2 > 0$ then $X'' - \alpha^2 X = 0$ and $X(x) = c_9\cosh\alpha x + c_{10}\sinh\alpha x$. Also, $Y'' + \alpha^2 Y = 0$ and $Y(y) = c_{11}\cos\alpha y + c_{12}\sin\alpha y$ so

$$u = XY = (c_9\cosh\alpha x + c_{10}\sinh\alpha x)(c_{11}\cos\alpha y + c_{12}\sin\alpha y).$$

**14.** Substituting $u(x,y) = X(x)Y(y)$ into the partial differential equation yields $x^2X''Y + XY'' = 0$. Separating variables and using the separation constant $-\lambda$ we obtain

$$-\frac{x^2X''}{X} = \frac{Y''}{Y} = -\lambda.$$

Then

$$x^2X'' - \lambda X = 0 \qquad \text{and} \qquad Y'' + \lambda Y = 0.$$

We consider three cases:

**I.** If $\lambda = 0$ then $x^2X'' = 0$ and $X(x) = c_1x + c_2$. Also, $Y'' = 0$ and $Y(y) = c_3y + c_4$ so

$$u = XY = (c_1x + c_2)(c_3y + c_4).$$

**II.** If $\lambda = -\alpha^2 < 0$ then $x^2X'' + \alpha^2X = 0$ and $Y'' - \alpha^2Y = 0$. The solution of the second differential equation is $Y(y) = c_5\cosh\alpha y + c_6\sinh\alpha y$. The first equation is Cauchy-Euler with auxiliary equation $m^2 - m + \alpha^2 = 0$. Solving for $m$ we obtain $m = \frac{1}{2} \pm \frac{1}{2}\sqrt{1-4\alpha^2}$. We consider three possibilities for the discriminant $1 - 4\alpha^2$.

**(i)** If $1 - 4\alpha^2 = 0$ then $X(x) = c_7x^{1/2} + c_8x^{1/2}\ln x$ and

$$u = XY = x^{1/2}(c_7 + c_8\ln x)(c_5\cosh\alpha y + c_6\sinh\alpha y).$$

**(ii)** If $1 - 4\alpha^2 < 0$ then

$$X(x) = x^{1/2}\left[c_9\cos\left(\frac{\sqrt{4\alpha^2-1}}{2}\ln x\right) + c_{10}\sin\left(\frac{\sqrt{4\alpha^2-1}}{2}\ln x\right)\right]$$

and

$$u = XY = x^{1/2}\left[c_9\cos\left(\frac{\sqrt{4\alpha^2-1}}{2}\ln x\right)\right.$$
$$\left.+c_{10}\sin\left(\frac{\sqrt{4\alpha^2-1}}{2}\ln x\right)\right](c_5\cosh\alpha y + c_6\sinh\alpha y).$$

**(iii)** If $1 - 4\alpha^2 > 0$ then $X(x) = x^{1/2}\left(c_{11}x^{\sqrt{1-4\alpha^2}/2} + c_{12}x^{-\sqrt{1-4\alpha^2}/2}\right)$ and

$$u = XY = x^{1/2}\left(c_{11}x^{\sqrt{1-4\alpha^2}/2} + c_{12}x^{-\sqrt{1-4\alpha^2}/2}\right)(c_5\cosh\alpha y + c_6\sinh\alpha y).$$

**III.** If $\lambda = \alpha^2 > 0$ then $x^2X'' - \alpha^2X = 0$ and $Y'' + \alpha^2Y = 0$. The solution of the second differential equation is $Y(y) = c_{13}\cos\alpha y + c_{14}\sin\alpha y$. The first equation is Cauchy-Euler with auxiliary equation $m^2 - m - \alpha^2 = 0$. Solving for $m$ we obtain $m = \frac{1}{2} \pm \frac{1}{2}\sqrt{1+4\alpha^2}$. In this case the discriminant is always positive so the solution of the differential equation is $X(x) = x^{1/2}\left(c_{15}x^{\sqrt{1+4\alpha^2}/2} + c_{16}x^{-\sqrt{1+4\alpha^2}/2}\right)$ and

$$u = XY = x^{1/2}\left(c_{15}x^{\sqrt{1+4\alpha^2}/2} + c_{16}x^{-\sqrt{1+4\alpha^2}/2}\right)(c_{13}\cos\alpha y + c_{14}\sin\alpha y).$$

**15.** Substituting $u(x,y) = X(x)Y(y)$ into the partial differential equation yields $X''Y + XY'' = XY$. Separating variables and using the separation constant $-\lambda$ we obtain

$$\frac{X''}{X} = \frac{Y - Y''}{Y} = -\lambda.$$

Then

$$X'' + \lambda X = 0 \qquad \text{and} \qquad Y'' - (1+\lambda)Y = 0.$$

We consider three cases:

**I.** If $\lambda = 0$ then $X'' = 0$ and $X(x) = c_1x + c_2$. Also $Y'' - Y = 0$ and $Y(y) = c_3 \cosh y + c_4 \sinh y$ so

$$u = XY = (c_1x + c_2)(c_3 \cosh y + c_4 \sinh y).$$

**II.** If $\lambda = -\alpha^2 < 0$ then $X'' - \alpha^2 X = 0$ and $Y'' + (\alpha^2 - 1)Y = 0$. The solution of the first differential equation is $X(x) = c_5 \cosh \alpha x + c_6 \sinh \alpha x$. The solution of the second differential equation depends on the nature of $\alpha^2 - 1$. We consider three cases:

**(i)** If $\alpha^2 - 1 = 0$, or $\alpha^2 = 1$, then $Y(y) = c_7 y + c_8$ and

$$u = XY = (c_5 \cosh \alpha x + c_6 \sinh \alpha x)(c_7 y + c_8).$$

**(ii)** If $\alpha^2 - 1 < 0$, or $0 < \alpha^2 < 1$, then $Y(y) = c_9 \cosh \sqrt{1-\alpha^2}\, y + c_{10} \sinh \sqrt{1-\alpha^2}\, y$ and

$$u = XY = (c_5 \cosh \alpha x + c_6 \sinh \alpha x)\left(c_9 \cosh \sqrt{1-\alpha^2}\, y + c_{10} \sinh \sqrt{1-\alpha^2}\, y\right).$$

**(iii)** If $\alpha^2 - 1 > 0$, or $\alpha^2 > 1$, then $Y(y) = c_{11} \cos \sqrt{\alpha^2-1}\, y + c_{12} \sin \sqrt{\alpha^2-1}\, y$ and

$$u = XY = (c_5 \cosh \alpha x + c_6 \sinh \alpha x)\left(c_{11} \cos \sqrt{\alpha^2-1}\, y + c_{12} \sin \sqrt{\alpha^2-1}\, y\right).$$

**III.** If $\lambda = \alpha^2 > 0$, then $X'' + \alpha^2 X = 0$ and $X(x) = c_{13} \cos \alpha x + c_{14} \sin \alpha x$. Also, $Y'' - (1+\alpha^2)Y = 0$ and $Y(y) = c_{15} \cosh \sqrt{1+\alpha^2}\, y + c_{16} \sinh \sqrt{1+\alpha^2}\, y$ so

$$u = XY = (c_{13} \cos \alpha x + c_{14} \sin \alpha x)\left(c_{15} \cosh \sqrt{1+\alpha^2}\, y + c_{16} \sinh \sqrt{1+\alpha^2}\, y\right).$$

**16.** Substituting $u(x,t) = X(x)T(t)$ into the partial differential equation yields $a^2X''T - g = XT''$, which is not separable.

**17.** Identifying $A = B = C = 1$, we compute $B^2 - 4AC = -3 < 0$. The equation is elliptic.

**18.** Identifying $A = 3$, $B = 5$, and $C = 1$, we compute $B^2 - 4AC = 13 > 0$. The equation is hyperbolic.

**19.** Identifying $A = 1$, $B = 6$, and $C = 9$, we compute $B^2 - 4AC = 0$. The equation is parabolic.

**20.** Identifying $A = 1$, $B = -1$, and $C = -3$, we compute $B^2 - 4AC = 13 > 0$. The equation is hyperbolic.

**21.** Identifying $A = 1$, $B = -9$, and $C = 0$, we compute $B^2 - 4AC = 81 > 0$. The equation is hyperbolic.

**22.** Identifying $A = 0$, $B = 1$, and $C = 0$, we compute $B^2 - 4AC = 1 > 0$. The equation is hyperbolic.

**23.** Identifying $A = 1$, $B = 2$, and $C = 1$, we compute $B^2 - 4AC = 0$. The equation is parabolic.

**24.** Identifying $A = 1$, $B = 0$, and $C = 1$, we compute $B^2 - 4AC = -4 < 0$. The equation is elliptic.

**25.** Identifying $A = a^2$, $B = 0$, and $C = -1$, we compute $B^2 - 4AC = 4a^2 > 0$. The equation is hyperbolic.

**26.** Identifying $A = k > 0$, $B = 0$, and $C = 0$, we compute $B^2 - 4AC = 0$. The equation is parabolic.

**27.** Substituting $u(r, t) = R(r)T(t)$ into the partial differential equation yields

$$k\left(R''T + \frac{1}{r}R'T\right) = RT'.$$

Separating variables and using the separation constant $-\lambda$ we obtain

$$\frac{rR'' + R'}{rR} = \frac{T'}{kT} = -\lambda.$$

Then

$$rR'' + R' + \lambda rR = 0 \qquad \text{and} \qquad T' + \lambda kT = 0.$$

Letting $\lambda = \alpha^2$ and writing the first equation as $r^2R'' + rR' = \alpha^2r^2R = 0$ we see that it is a parametric Bessel equation of order 0. As discussed in Chapter 5 of the text, it has solution $R(r) = c_1J_0(\alpha r) + c_2Y_0(\alpha r)$. Since a solution of $T' + \alpha^2kT$ is $T(t) = e^{-k\alpha^2t}$, we see that a solution of the partial differential equation is

$$u = RT = e^{-k\alpha^2t}[c_1J_0(\alpha r) + c_2Y_0(\alpha r)].$$

**28.** Substituting $u(r, \theta) = R(r)\Theta(\theta)$ into the partial differential equation yields

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0.$$

Separating variables and using the separation constant $-\lambda$ we obtain

$$\frac{r^2R'' + rR'}{R} = -\frac{\Theta''}{\Theta} = -\lambda.$$

Then

$$r^2R'' + rR' + \lambda R = 0 \qquad \text{and} \qquad \Theta'' - \lambda\Theta = 0.$$

Letting $\lambda = -\alpha^2$ we have the Cauchy-Euler equation $r^2R'' + rR' - \alpha^2R = 0$ whose solution is $R(r) = c_3r^\alpha + c_4r^{-\alpha}$. Since the solution of $\Theta'' + \alpha^2\Theta = 0$ is $\Theta(\theta) = c_1\cos\alpha\theta + c_2\sin\alpha\theta$ we see that a solution of the partial differential equation is

$$u = R\Theta = (c_1\cos\alpha\theta + c_2\sin\alpha\theta)(c_3r^\alpha + c_4r^{-\alpha}).$$

**29.** For $u = A_1 + B_1 x$ we compute $\partial^2 u/\partial x^2 = 0 = \partial u/\partial y$. Then $\partial^2 u/\partial x^2 = 4\,\partial u/\partial y$.

For $u = A_2 e^{\alpha^2 y}\cos 2\alpha x + B_2 e^{\alpha^2 y}\sin 2\alpha x$ we compute

$$\frac{\partial u}{\partial x} = 2\alpha A_2 e^{\alpha^2 y}\sinh 2\alpha x + 2\alpha B_2 e^{\alpha^2 y}\cosh 2\alpha x$$

$$\frac{\partial^2 u}{\partial x^2} = 4\alpha^2 A_2 e^{\alpha^2 y}\cosh 2\alpha x + 4\alpha^2 B_2 e^{\alpha^2 y}\sinh 2\alpha x$$

and

$$\frac{\partial u}{\partial y} = \alpha^2 A_2 e^{\alpha^2 y}\cosh 2\alpha x + \alpha^2 B_2 e^{\alpha^2 y}\sinh 2\alpha x.$$

Then $\partial^2 u/\partial x^2 = 4\,\partial u/\partial y$.

For $u = A_3 e^{-\alpha^2 y}\cosh 2\alpha x + B_3 e^{-\alpha^2 y}\sinh 2\alpha x$ we compute

$$\frac{\partial u}{\partial x} = -2\alpha A_3 e^{-\alpha^2 y}\sin 2\alpha x + 2\alpha B_3 e^{-\alpha^2 y}\cos 2\alpha x$$

$$\frac{\partial^2 u}{\partial x^2} = -4\alpha^2 A_3 e^{-\alpha^2 y}\cos 2\alpha x - 4\alpha^2 B_3 e^{-\alpha^2 y}\sin 2\alpha x$$

and

$$\frac{\partial u}{\partial y} = -\alpha^2 A_3 e^{-\alpha^2 y}\cos 2\alpha x - \alpha^2 B_3 e^{-\alpha^2 y}\sin 2\alpha x.$$

Then $\partial^2 u/\partial x^2 = 4\,\partial u/\partial y$.

**30.** We identify $A = xy + 1$, $B = x + 2y$, and $C = 1$. Then $B^2 - 4AC = x^2 + 4y^2 - 4$. The equation $x^2 + 4y^2 = 4$ defines an ellipse. The partial differential equation is hyperbolic outside the ellipse, parabolic on the ellipse, and elliptic inside the ellipse.

**31.** Assuming $u(x, y) = X(x)Y(y)$ and substituting into $\partial^2 u/\partial x^2 - u = 0$ we get $X''Y - XY = 0$ or $Y(X'' - X) = 0$. This implies $X(x) = c_1 e^x$ or $X(x) = c_2 e^{-x}$. For these choices of $X$, $Y$ can be any function of $y$. Two solutions of the partial differential equation are then

$$u_1(x, y) = A(y)e^x \quad \text{and} \quad u_2(x, y) = B(y)e^{-x}.$$

Since the partial differential equation is linear and homogeneous the superposition principle indicates that another solution is

$$u(x, y) = u_1(x, y) + u_2(x, y) = A(y)e^x + B(y)e^{-x}.$$

**32.** Assuming $u(x, y) = X(x)Y(y)$ and substituting into $\partial^2 u/\partial x\partial y + \partial u/\partial x = 0$ we get $X'Y' + X'Y = 0$ or $X'(Y' + Y) = 0$. This implies $Y(y) = c_1 e^{-y}$. For this choice of $Y$, $X$ can be any function of $x$. A solution of the partial differential equation is then $u(x, y) = A(x)e^{-y}$. In addition, noting that the partial differential equation can be written

$$\frac{\partial}{\partial x}\left[\frac{\partial u}{\partial y} + u\right] = 0,$$

any function $u_2(x,y) = B(y)$ will satisfy the partial differential equation since, in this case, $\partial u_2/\partial y + u_2 = B'(y) + B(y)$ and the $x$-partial of $B'(y) + B(y)$ is 0. Thus, using the superposition principle, a solution of the partial differential equation is

$$u(x,y) = u_1(x,y) + u_2(x,y) = A(x)e^{-y} + B(y).$$

## 12.2 Classical PDEs and Boundary-Value Problems

**1.** $k\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}, \quad 0 < x < L,\ t > 0$

$u(0,t) = 0, \quad \left.\dfrac{\partial u}{\partial x}\right|_{x=L} = 0, \quad t > 0$

$u(x,0) = f(x), \quad 0 < x < L$

**2.** $k\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}, \quad 0 < x < L,\ t > 0$

$u(0,t) = u_0, \quad u(L,t) = u_1, \quad t > 0$

$u(x,0) = 0, \quad 0 < x < L$

**3.** $k\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}, \quad 0 < x < L,\ t > 0$

$u(0,t) = 100, \quad \left.\dfrac{\partial u}{\partial x}\right|_{x=L} = -hu(L,t), \quad t > 0$

$u(x,0) = f(x), \quad 0 < x < L$

**4.** $k\dfrac{\partial^2 u}{\partial x^2} + h(u - 50) = \dfrac{\partial u}{\partial t}, \quad 0 < x < L,\ t > 0$

$\left.\dfrac{\partial u}{\partial x}\right|_{x=0} = 0, \quad \left.\dfrac{\partial u}{\partial x}\right|_{x=L} = 0, \quad t > 0$

$u(x,0) = 100, \quad 0 < x < L$

**5.** $k\dfrac{\partial^2 u}{\partial x^2} - hu = \dfrac{\partial u}{\partial t}, \quad 0 < x < L,\ t > 0$, $h$ a constant

$u(0,t) = \sin\dfrac{\pi t}{L}, \quad u(L,t) = 0, \quad t > 0$

$u(x,0) = f(x), \quad 0 < x < L$

**6.** $k\dfrac{\partial^2 u}{\partial x^2} + h(u - 50) = \dfrac{\partial u}{\partial t}, \quad 0 < x < L,\ t > 0$

$\left.\dfrac{\partial u}{\partial x}\right|_{x=0} = 0, \quad \left.\dfrac{\partial u}{\partial x}\right|_{x=L} = 0, \quad t > 0$

$u(x,0) = 100, \quad 0 < x < L$

**7.** $a^2\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2}, \quad 0 < x < L,\ t > 0$

$u(0,t) = 0, \quad u(L,t) = 0, \quad t > 0$

$u(x,0) = x(L-x), \quad \left.\dfrac{\partial u}{\partial t}\right|_{t=0} = 0, \quad 0 < x < L$

**8.** $a^2\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2}, \quad 0 < x < L,\ t > 0$

$u(0,t) = 0, \quad u(L,t) = 0, \quad t > 0$

$u(x,0) = 0, \quad \left.\dfrac{\partial u}{\partial t}\right|_{t=0} = \sin\dfrac{\pi x}{L}, \quad 0 < x < L$

**9.** $a^2\dfrac{\partial^2 u}{\partial x^2} - 2\beta\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial t^2}, \quad 0 < x < L,\ t > 0$

$u(0,t) = 0, \quad u(L,t) = \sin\pi t, \quad t > 0$

$u(x,0) = f(x), \quad \left.\dfrac{\partial u}{\partial t}\right|_{t=0} = 0, \quad 0 < x < L$

**10.** $a^2\dfrac{\partial^2 u}{\partial x^2} + Ax = \dfrac{\partial^2 u}{\partial t^2}, \quad 0 < x < L,\ t > 0,\ A \text{ a constant}$

$u(0,t) = 0, \quad u(L,t) = 0, \quad t > 0$

$u(x,0) = 0, \quad \left.\dfrac{\partial u}{\partial t}\right|_{t=0} = 0, \quad 0 < x < L$

**11.** $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < 4,\ 0 < y < 2$

$\left.\dfrac{\partial u}{\partial x}\right|_{x=0} = 0, \quad u(4,y) = f(y), \quad 0 < y < 2$

$\left.\dfrac{\partial u}{\partial y}\right|_{y=0} = 0, \quad u(x,2) = 0, \quad 0 < x < 4$

**12.** $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < \pi,\ y > 0$

$u(0,y) = e^{-y}, \quad u(\pi,y) = \begin{cases} 100, & 0 < y \le 1 \\ 0, & y > 1 \end{cases}$

$u(x,0) = f(x), \quad 0 < x < \pi$

## 12.3 Heat Equation

**1.** Using $u = XT$ and $-\lambda$ as a separation constant we obtain

$$\begin{aligned} X'' + \lambda X &= 0, \\ X(0) &= 0, \\ X(L) &= 0, \end{aligned}$$

and

$$T' + k\lambda T = 0.$$

This leads to

$$X = c_1 \sin \frac{n\pi}{L} x \qquad \text{and} \qquad T = c_2 e^{-kn^2\pi^2 t/L^2}$$

for $n = 1, 2, 3, \ldots$ so that

$$u = \sum_{n=1}^{\infty} A_n e^{-kn^2\pi^2 t/L^2} \sin \frac{n\pi}{L} x.$$

Imposing

$$u(x, 0) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{L} x$$

gives

$$A_n = \frac{2}{L} \int_0^{L/2} \sin \frac{n\pi}{L} x \, dx = \frac{2}{n\pi} \left( 1 - \cos \frac{n\pi}{2} \right)$$

for $n = 1, 2, 3, \ldots$ so that

$$u(x, t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1 - \cos \frac{n\pi}{2} n}{} e^{-kn^2\pi^2 t/L^2} \sin \frac{n\pi}{L} x.$$

**2.** Using $u = XT$ and $-\lambda$ as a separation constant we obtain

$$\begin{aligned} X'' + \lambda X &= 0, \\ X(0) &= 0, \\ X(L) &= 0, \end{aligned}$$

and

$$T' + k\lambda T = 0.$$

This leads to

$$X = c_1 \sin \frac{n\pi}{L} x \qquad \text{and} \qquad T = c_2 e^{-kn^2\pi^2 t/L^2}$$

for $n = 1, 2, 3, \ldots$ so that

$$u = \sum_{n=1}^{\infty} A_n e^{-kn^2\pi^2 t/L^2} \sin \frac{n\pi}{L} x.$$

Imposing

$$u(x,0)=\sum_{n=1}^{\infty}A_n\sin\frac{n\pi}{L}x$$

gives

$$A_n=\frac{2}{L}\int_0^L x(L-x)\sin\frac{n\pi}{L}x\,dx=\frac{4L^2}{n^3\pi^3}[1-(-1)^n]$$

for $n=1,\,2,\,3,\,\ldots$ so that

$$u(x,t)=\frac{4L^2}{\pi^3}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^3}\,e^{-kn^2\pi^2t/L^2}\sin\frac{n\pi}{L}x.$$

**3.** Using $u=XT$ and $-\lambda$ as a separation constant we obtain

$$X''+\lambda X=0,$$
$$X'(0)=0,$$
$$X'(L)=0,$$

and

$$T'+k\lambda T=0.$$

This leads to

$$X=c_1\cos\frac{n\pi}{L}x\qquad\text{and}\qquad T=c_2e^{-kn^2\pi^2t/L^2}$$

for $n=0,\,1,\,2,\,\ldots$ ($\lambda=0$ is an eigenvalue in this case) so that

$$u=\sum_{n=0}^{\infty}A_ne^{-kn^2\pi^2t/L^2}\cos\frac{n\pi}{L}x.$$

Imposing

$$u(x,0)=f(x)=A_0+\sum_{n=1}^{\infty}A_n\cos\frac{n\pi}{L}x$$

gives

$$u(x,t)=\frac{1}{L}\int_0^L f(x)\,dx+\frac{2}{L}\sum_{n=1}^{\infty}\left(\int_0^L f(x)\cos\frac{n\pi}{L}x\,dx\right)e^{-kn^2\pi^2t/L^2}\cos\frac{n\pi}{L}x.$$

**4.** If $L=2$ and $f(x)$ is $x$ for $0<x<1$ and $f(x)$ is 0 for $1<x<2$ then

$$u(x,t)=\frac{1}{4}+4\sum_{n=1}^{\infty}\left[\frac{1}{2n\pi}\sin\frac{n\pi}{2}+\frac{1}{n^2\pi^2}\left(\cos\frac{n\pi}{2}-1\right)\right]e^{-kn^2\pi^2t/4}\cos\frac{n\pi}{2}x.$$

**5.** Using $u=XT$ and $-\lambda$ as a separation constant leads to

$$X''+\lambda X=0,$$
$$X'(0)=0,$$
$$X'(L)=0,$$

and

$$T' + (h + k\lambda)T = 0.$$

Then

$$X = c_1 \cos \frac{n\pi}{L} x \qquad \text{and} \qquad T = c_2 e^{-ht - kn^2\pi^2 t/L^2}$$

for $n = 0, 1, 2, \ldots$ ($\lambda = 0$ is an eigenvalue in this case) so that

$$u = A_0 e^{-ht} + e^{-ht} \sum_{n=1}^{\infty} A_n e^{-kn^2\pi^2 t/L^2} \cos \frac{n\pi}{L} x.$$

Imposing

$$u(x,0) = f(x) = \sum_{n=0}^{\infty} A_n \cos \frac{n\pi}{L} x$$

gives

$$u(x,t) = \frac{e^{-ht}}{L} \int_0^L f(x)\,dx + \frac{2e^{-ht}}{L} \sum_{n=1}^{\infty} \left( \int_0^L f(x) \cos \frac{n\pi}{L} x\,dx \right) e^{-kn^2\pi^2 t/L^2} \cos \frac{n\pi}{L} x.$$

**6.** In Problem 5 we instead find that $X(0) = 0$ and $X(L) = 0$ so that

$$X = c_1 \sin \frac{n\pi}{L} x$$

and

$$u = \frac{2e^{-ht}}{L} \sum_{n=1}^{\infty} \left( \int_0^L f(x) \sin \frac{n\pi}{L} x\,dx \right) e^{-kn^2\pi^2 t/L^2} \sin \frac{n\pi}{L} x.$$

**7.** Using $-\lambda$ as the separation constant implies $X'' + \lambda X = 0$ and $T' + k\lambda T = 0$. The boundary conditions are then $X(-L) = X(L)$ and $X'(-L) = X'(L)$.

For $\lambda = 0$, $X(x) = c_1 + c_2 x$. The condition $X(-L) = X(L)$ implies $c_2 = 0$. Therefore an eigenfunction is $X(x) = c_1 \neq 0$. The boundary condition $X'(-L) = X'(L)$ is automatically satisfied.

For $\lambda = -\alpha^2 < 0$, $X(x) = c_3 \cosh \alpha x + c_4 \sinh \alpha x$. From the condition $X(-L) = X(L)$ we obtain

$$c_3 \cosh \alpha L - c_4 \sinh \alpha L = c_3 \cosh \alpha L + c_4 \sinh \alpha L \quad \text{so} \quad 2c_4 \sinh \alpha L = 0.$$

This implies $c_4 = 0$ and so $X(x) = c_3 \cosh \alpha x$. The boundary condition $X'(-L) = X'(L)$ implies

$$-c_3 \alpha \sinh \alpha L = c_3 \sinh \alpha L \quad \text{so} \quad 2\alpha c_3 \sinh \alpha L = 0.$$

Therefore $c_3 = 0$ and $X(x) = 0$.

For $\lambda = \alpha^2 > 0$, $X(x) = c_5 \cos \alpha x + c_6 \sin \alpha x$. The boundary condition $X(-L) = X(L)$ implies

$$c_5 \cos \alpha L - c_6 \sin \alpha L = c_5 \cos \alpha L + c_6 \sin \alpha L \quad \text{so} \quad 2c_6 \sin \alpha L = 0.$$

If $c_6 \neq 0$, then $\alpha = n\pi/L$ for $n = 1,\ 2,\ \dots$. The boundary condition $X'(-L) = X'(L)$ implies

$$-c_5\alpha \sin\alpha L + c_6\alpha\cos\alpha L = \alpha c_5 \sin\alpha L + c_6\alpha\cos\alpha L \quad \text{so} \quad 2\alpha c_5 \sin\alpha L = 0.$$

Then, for $c_5 \neq 0$,

$$\sin\alpha L = 0 \quad \text{and} \quad \alpha = n\pi/L,\ n = 1,\ 2,\ \dots .$$

Thus the coefficients $c_5$ and $c_6$ are arbitrary but nonzero. Therefore the eigenvalues are $\lambda_n = (n\pi/L)^2,\ n = 0,\ 1,\ 2,\ \dots$, and the corresponding eigenfunctions are

$$1,\ \cos\frac{n\pi}{L}x,\ \sin\frac{n\pi}{L}x \quad \text{for} \quad n = 0,\ 1,\ 2,\ \dots .$$

Forming product solutions with

$$T(t) = \begin{cases} c_7, & \lambda = 0 \\ c_7 e^{-k(n\pi/L)^2 t}, & \lambda > 0, \end{cases}$$

relabeling constants, and summing gives

$$u(x,\ t) = A_0 + \sum_{k=1}^{\infty} e^{-k(n\pi/L)^2 t}\left(A_n\cos\frac{n\pi}{L}x + B_n\sin\frac{n\pi}{L}x\right).$$

When $t = 0$ we get the full Fourier series of $f$ on $(-L, L)$,

$$f(x) = A_0 + \sum_{k=1}^{\infty}\left(A_n\cos\frac{n\pi}{L}x + B_n\sin\frac{n\pi}{L}x\right).$$

The coefficients are then $A_0 = \frac{1}{2}a_0$, $A_n = a_n$, $B_n = b_n$, or

$$A_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx, \quad A_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi}{L}\,dx, \quad B_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi}{L}\,dx.$$

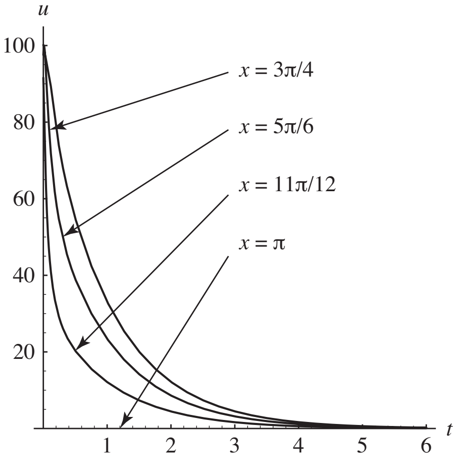
**8.** In this case we have from (13) of this section in the text that

$$u_n(x,0) = A_n\sin\frac{n\pi x}{L} = f(x) = 10\sin\frac{5\pi x}{L},$$

so we can take $n = 5$ and $A_5 = 10$. All other values of $A_n$ are 0. Therefore, we can take the solution of the boundary-value problem to be

$$u(x,t) = 10e^{-k(25\pi^2/L^2)t}\sin\frac{5\pi x}{L}.$$

**9.**



**10. (a)** The solution is

$$u(x,t)=\sum_{n=1}^{\infty}A_n e^{-kn^2\pi^2t/100^2}\sin\frac{n\pi}{100}\,x,$$

where

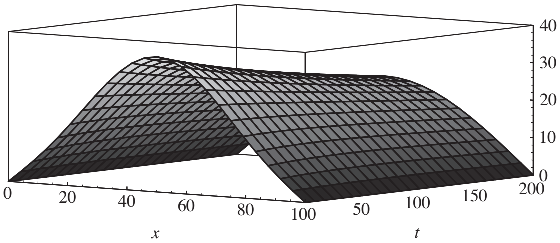
$$A_n=\frac{2}{100}\left[\int_0^{50}0.8x\sin\frac{n\pi}{100}\,x\,dx+\int_{50}^{100}0.8(100-x)\sin\frac{n\pi}{100}\,x\,dx\right]=\frac{320}{n^2\pi^2}\sin\frac{n\pi}{2}\,.$$

Thus,

$$u(x,t)=\frac{320}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left(\sin\frac{n\pi}{2}\right)e^{-kn^2\pi^2t/100^2}\sin\frac{n\pi}{100}\,x.$$

**(b)** Since $A_n=0$ for $n$ even, the first five nonzero terms correspond to $n=1,\ 3,\ 5,\ 7,\ 9$. In this case $\sin(n\pi/2)=\sin(2p-1)/2=(-1)^{p+1}$ for $p=1,\ 2,\ 3,\ 4,\ 5$, and

$$u(x,t)=\frac{320}{\pi^2}\sum_{p=1}^{\infty}\frac{(-1)^{p+1}}{(2p-1)^2}e^{(-1.6352(2p-1)^2\pi^2/100^2)t}\sin\frac{(2p-1)\pi}{100}\,x.$$



## 12.4 Wave Equation

*For Problems 1–10, recall that the solution to the wave equation is given by*

$$u(x,t)=\sum_{n=1}^{\infty}\left(A_n\cos\frac{n\pi a}{L}t+B_n\sin\frac{n\pi a}{L}t\right)\sin\frac{n\pi}{L}x$$

*where* $A_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx$ *and* $B_n=\frac{2}{n\pi a}\int_0^L g(x)\sin\frac{n\pi}{L}x\,dx$.

*Here,* $f(x)=u(x,0)$ *and* $g(x)=\left.\frac{\partial u}{\partial t}\right|_{t=0}$.

**1.** By the discussion in Section 12.4, pages 474 and 475,

$$u(x,t)=\sum_{n=1}^{\infty}\left(A_n\cos\frac{n\pi a}{L}t+B_n\sin\frac{n\pi a}{L}t\right)\sin\frac{n\pi}{L}x$$

The coefficients are given by

$$A_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx=\frac{2}{L}\int_0^L\left(\frac{1}{4}x(L-x)\right)\sin\frac{n\pi}{L}x\,dx=\frac{L^2\left[1-(-1)^n\right]}{n^3\pi^3}$$

$$B_n=\frac{2}{n\pi a}\int_0^L g(x)\sin\frac{n\pi}{L}x\,dx=0$$

Therefore the solution to the problem is

$$u(x,t)=\sum_{n=1}^{\infty}\left(\frac{L^2\left[1-(-1)^n\right]}{n^3\pi^3}\cdot\cos\frac{n\pi a}{L}t\right)\sin\frac{n\pi}{L}x$$

**2.** By the discussion in Section 12.4, pages 474 and 475,

$$u(x,t)=\sum_{n=1}^{\infty}\left(A_n\cos\frac{n\pi a}{L}t+B_n\sin\frac{n\pi a}{L}t\right)\sin\frac{n\pi}{L}x$$

The coefficients are given by

$$A_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx=0$$

$$B_n=\frac{2}{n\pi a}\int_0^L g(x)\sin\frac{n\pi}{L}x\,dx=\frac{2}{n\pi a}\int_0^L x(L-x)\sin\frac{n\pi}{L}x\,dx=\frac{4L^3\left[1-(-1)^n\right]}{an^4\pi^4}$$

Therefore the solution to the problem is

$$u(x,t)=\sum_{n=1}^{\infty}\left(\frac{4L^3\left[1-(-1)^n\right]}{an^4\pi^4}\cdot\sin\frac{n\pi a}{L}t\right)\sin\frac{n\pi}{L}x$$

**3.** By the discussion in Section 12.4, pages 474 and 475,

$$u(x,t)=\sum_{n=1}^{\infty}(A_n\cos nat+B_n\sin nat)\sin nx$$

The coefficients are given by

$$A_n=\frac{2}{\pi}\int_0^{\pi}f(x)\sin nx\,dx=0$$

$$B_n=\frac{2}{n\pi a}\int_0^{\pi}g(x)\sin nx\,dx=\frac{1}{n\pi a}\int_0^{\pi}\sin x\sin nx\,dx=0\quad\text{for } n=2,3,4,\ldots$$

$$B_1=\frac{2}{\pi a}\int_0^{\pi}\sin x\sin x\,dx=\frac{1}{a}$$

Therefore the solution to the problem is

$$u(x,t)=B_1\sin at\sin x=\frac{1}{a}\sin at\sin x$$

**4.** By the discussion in Section 12.4, pages 474 and 475,

$$u(x,t)=\sum_{n=1}^{\infty}(A_n\cos nat+B_n\sin nat)\sin nx$$

The coefficients are given by

$$A_n=\frac{2}{\pi}\int_0^{\pi}f(x)\sin nx\,dx=\frac{2}{\pi}\int_0^{\pi}\frac{1}{6}x\left(\pi^2-x^2\right)\sin nx\,dx=\frac{2(-1)^{n+1}}{n^3}$$

$$B_n=\frac{2}{n\pi a}\int_0^{\pi}g(x)\sin nx\,dx=0$$

Therefore the solution to the problem is

$$u(x,t)=\sum_{n=1}^{\infty}\left(\frac{2(-1)^{n+1}}{n^3}\cdot\cos nat\right)\sin nx$$

**5.** By the discussion in Section 12.4, pages 474 and 475,

$$u(x,t)=\sum_{n=1}^{\infty}(A_n\cos n\pi at+B_n\sin n\pi at)\sin n\pi x$$

The coefficients are given by

$$A_n=\frac{2}{1}\int_0^{1}f(x)\sin n\pi x\,dx=\frac{2}{1}\int_0^{1}x(1-x)\sin n\pi x\,dx=\frac{4\left[1-(-1)^n\right]}{n^3\pi^3}$$

$$B_n=\frac{2}{n\pi a}\int_0^{1}g(x)\sin n\pi x\,dx=\frac{2}{n\pi a}\int_0^{1}x(1-x)\sin n\pi x\,dx=\frac{4\left[1-(-1)^n\right]}{an^4\pi^4}$$

Therefore the solution to the problem is

$$u(x,t)=\sum_{n=1}^{\infty}\left(\frac{4\left[1-(-1)^n\right]}{n^3\pi^3}\cos n\pi at+\frac{4\left[1-(-1)^n\right]}{an^4\pi^4}\sin n\pi at\right)\sin n\pi x$$

**6.** By the discussion in Section 12.4, pages 474 and 475,

$$u(x,t) = \sum_{n=1}^{\infty} (A_n \cos n\pi at + B_n \sin n\pi at) \sin n\pi x$$

The coefficients are

$$A_n = \frac{2}{1}\int_0^1 f(x) \sin n\pi x\, dx = \frac{2}{1}\int_0^1 0.01 \sin 3\pi x \sin n\pi x\, dx = 0 \quad \text{for } n = 1, 2, 4, 5, 6, \ldots$$

$$A_3 = \frac{2}{1}\int_0^1 0.01 \sin 3\pi x \sin 3\pi x\, dx = 0.01$$

$$B_n = \frac{2}{n\pi a}\int_0^1 g(x) \sin n\pi x\, dx = 0$$

Therefore the solution to the problem is

$$u(x,t) = (A_3 \cos 3\pi at) \sin 3\pi x = 0.01 \cos 3\pi at \sin 3\pi x$$

*For Problems 7–10, we have $g(x) = 0$ because the string is released from the rest, so $B_n = 0$. So, our general solution is of the form*

$$u(x,t) = \sum_{n=1}^{\infty} A_n \cos \frac{n\pi at}{L} \sin \frac{n\pi x}{L}$$

*where* $A_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi}{L} x dx$ *as before.*

**7.** By the discussion in Section 12.4, pages 474 and 475,

$$u(x,t) = \sum_{n=1}^{\infty} \left( A_n \cos \frac{n\pi a}{L} t + B_n \sin \frac{n\pi a}{L} t \right) \sin \frac{n\pi}{L} x$$

The coefficients are

$$A_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi}{L} x\, dx = \frac{2}{L}\left[ \int_0^{L/2} \frac{2hx}{L} \sin \frac{n\pi}{L} x\, dx + \int_{L/2}^{L} 2h\left(1 - \frac{x}{L}\right) \sin \frac{n\pi}{L} x\, dx \right]$$

$$= \frac{8h \sin\left(\frac{n\pi}{2}\right)}{n^2\pi^2}$$

$$B_n = \frac{2}{n\pi a}\int_0^L g(x) \sin \frac{n\pi}{L} x\, dx = 0$$

Therefore the solution to the problem is

$$u(x,t) = \sum_{n=1}^{\infty} \left( \frac{8h \sin\left(\frac{n\pi}{2}\right)}{n^2\pi^2} \cos \frac{n\pi a}{L} t \right) \sin \frac{n\pi}{L} x$$

**8.** By the discussion in Section 12.4, pages 474 and 475,

$$u(x,t)=\sum_{n=1}^{\infty}\left(A_n\cos\frac{n\pi a}{L}t+B_n\sin\frac{n\pi a}{L}t\right)\sin\frac{n\pi}{L}x$$

The coefficients are

$$\begin{aligned}A_n&=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx=\frac{2}{L}\left[\int_0^{L/3}\frac{3hx}{L}\sin\frac{n\pi}{L}x\,dx+\int_{L/3}^{L}\frac{3h}{2}\left(1-\frac{x}{L}\right)\sin\frac{n\pi}{L}x\,dx\right]\\&=\frac{9h\sin\left(\frac{n\pi}{3}\right)}{n^2\pi^2}\\B_n&=\frac{2}{n\pi a}\int_0^L g(x)\sin\frac{n\pi}{L}x\,dx=0\end{aligned}$$

Therefore the solution to the problem is

$$u(x,t)=\sum_{n=1}^{\infty}\left(\frac{9h\sin\left(\frac{n\pi}{3}\right)}{n^2\pi^2}\cos\frac{n\pi a}{L}t\right)\sin\frac{n\pi}{L}x$$

**9.** By the discussion in Section 12.4, pages 474 and 475,

$$u(x,t)=\sum_{n=1}^{\infty}\left(A_n\cos\frac{n\pi a}{L}t+B_n\sin\frac{n\pi a}{L}t\right)\sin\frac{n\pi}{L}x$$

The coefficients are

$$\begin{aligned}A_n&=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx\\&=\frac{2}{L}\left[\int_0^{L/3}\frac{3hx}{L}\sin\frac{n\pi}{L}x\,dx+\int_{L/3}^{2L/3}h\sin\frac{n\pi}{L}x\,dx+\int_{2L/3}^{L}3h\left(1-\frac{x}{L}\right)\sin\frac{n\pi}{L}x\,dx\right]\\&=\frac{6h\left[\sin\left(\frac{2n\pi}{3}\right)+\sin\left(\frac{n\pi}{3}\right)\right]}{n^2\pi^2}\\B_n&=\frac{2}{n\pi a}\int_0^L g(x)\sin\frac{n\pi}{L}x\,dx=0\end{aligned}$$

Therefore the solution to the problem is

$$u(x,t)=\sum_{n=1}^{\infty}\left(\frac{6h\left[\sin\left(\frac{2n\pi}{3}\right)+\sin\left(\frac{n\pi}{3}\right)\right]}{n^2\pi^2}\cos\frac{n\pi a}{L}t\right)\sin\frac{n\pi}{L}x$$

Using the trigonometric identity

$$\sin\frac{2n\pi}{3}=\sin\left(n\pi-\frac{n\pi}{3}\right)=\sin n\pi\cos\frac{n\pi}{3}-\cos n\pi\sin\frac{n\pi}{3}=-(-1)^n\sin\frac{n\pi}{3}$$

we have

$$u(x,t)=\frac{6h}{\pi^2}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^2}\sin\frac{n\pi}{3}\cos\frac{n\pi a}{L}t\sin\frac{n\pi}{L}x.$$

**10.** By the discussion in Section 12.4, pages 474 and 475,

$$u(x,t)=\sum_{n=1}^{\infty}\left(A_n\cos\frac{n\pi a}{L}t+B_n\sin\frac{n\pi a}{L}t\right)\sin\frac{n\pi}{L}x$$

The coefficients are

$$B_n=\frac{2}{n\pi a}\int_0^L g(x)\sin\frac{n\pi}{L}x\,dx=0$$

and

$$A_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx=\frac{18h}{n^2\pi^2}\left(\sin\frac{n\pi}{3}-\sin\frac{2n\pi}{3}\right).$$

Using $\sin\frac{2n\pi}{3}=-(-1)^n\frac{n\pi}{3}$ we have

$$u(x,t)=\frac{18h}{\pi^2}\sum_{n=1}^{\infty}\frac{1+(-1)^n}{n^2}\sin\frac{n\pi}{3}\cos\frac{n\pi a}{L}t\sin\frac{n\pi}{L}x$$

**11.** By the discussion in Section 12.4, pages 474 and 475, we get

$$\begin{cases}X''+\lambda X=0,\\ X'(0)=0,\\ X'(L)=0,\end{cases}$$

For $\lambda=0$,

$$\begin{cases}X''=0,\\ X'(0)=0,\\ X'(L)=0,\end{cases}$$

Thus $X=c_1+c_2x$ and therefore $X=c_1$. Moreover $\{T''=0$ implies $T=c_3+c_4t$.

Therefore we have $u_0(x,t)=c_1(c_3+c_4t)=A_0+B_0t$. Using the given conditions and the results from Problem 3 in Section 12.5, the rest of the eigenvalues are $\lambda_n=n^2\pi^2/L^2$ with corresponding eigenfunctions $X_n=\cos\frac{n\pi}{L}x$, $n=1,2,3,\ldots$ therefore
$T=c_3\cos\frac{n\pi a}{L}t+c_4\sin\frac{n\pi a}{L}t$ and we now have

$$u(x,t)=(A_0+B_0t)+\sum_{n=1}^{\infty}\left(A_n\cos\frac{n\pi a}{L}t+B_n\sin\frac{n\pi a}{L}t\right)\cos\frac{n\pi}{L}x$$

To complete the problem we need only to find the coefficients. At $t=0$ we have

$$u(x,0)=x=A_0+\sum_{n=1}^{\infty}A_n\cos\frac{n\pi}{L}x\qquad\longleftarrow\text{ A Fourier cosine series for }x$$

where

$$A_0=\frac{1}{L}\int_0^L x\,dx=\frac{L}{2}\quad\text{and}\quad A_n=\frac{2}{L}\int_0^L x\cos\frac{n\pi}{L}x\,dx=\frac{2L\left[(-1)^n-1\right]}{n^2\pi^2}$$

Similarly at $t = 0$,

$$u_t(x,0) = 0 = B_0 + \sum_{n=1}^{\infty}\left(B_n \cdot \frac{n\pi a}{L}\right)\cos\frac{n\pi}{L}x$$

and so we get $B_n = 0$ for $n = 0, 1, 2, 3, \ldots$ therefore the solution to the problem is

$$u(x,t) = \frac{L}{2} + \sum_{n=1}^{\infty}\left(\frac{2L\left[(-1)^n - 1\right]}{n^2\pi^2}\cos\frac{n\pi a}{L}t\right)\cos\frac{n\pi}{L}x$$

**12.** By the discussion in Section 12.4, pages 474 and 475, we get

$$\begin{cases} X'' + \lambda X = 0, \\ X'(0) = 0, \\ X'(L) = 0. \end{cases}$$

For $\lambda = 0$,

$$\begin{cases} X'' = 0, \\ X'(0) = 0, \\ X'(L) = 0. \end{cases}$$

Thus $X = c_1 + c_2x$ and therefore $X = c_1$. Moreover, $T'' = 0$ implies $T = c_3 + c_4t$.

Therefore we have $u_0(x,t) = c_1(c_3 + c_4t) = A_0 + B_0t$. Using the given conditions and the results from Problem 3 in Section 12.5, the rest of the eigenvalues are $\lambda_n = n^2\pi^2/L^2$ with corresponding eigenfunctions $X_n = \cos\frac{n\pi}{L}x$, $n = 1, 2, 3, \ldots$ therefore
$T = c_3\cos\frac{n\pi a}{L}t + c_4\sin\frac{n\pi a}{L}t$ and we now have

$$u(x,t) = (A_0 + B_0t) + \sum_{n=1}^{\infty}\left(A_n\cos\frac{n\pi a}{L}t + B_n\sin\frac{n\pi a}{L}t\right)\cos\frac{n\pi}{L}x$$

To complete the problem we need only to find the coefficients. At $t = 0$ we have

$$u(x,0) = f(x) = A_0 + \sum_{n=1}^{\infty}A_n\cos\frac{n\pi}{L}x \qquad \longleftarrow \text{A Fourier cosine series for } f(x)$$

where

$$A_0 = \frac{1}{L}\int_0^L f(x)\,dx \qquad \text{and} \qquad A_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi}{L}x\,dx$$

Similarly at $t = 0$,

$$u_t(x,0) = g(x) = B_0 + \sum_{n=1}^{\infty}\left(B_n \cdot \frac{n\pi a}{L}\right)\cos\frac{n\pi}{L}x$$

and so we get

$$B_0 = \frac{1}{L}\int_0^L g(x)\,dx \qquad \text{and} \qquad B_n = \frac{2}{n\pi a}\int_0^L g(x)\cos\frac{n\pi}{L}x\,dx$$

**13.** From Figure 12.4.6 we expect $u(L/2,t)=0$ for $t\ge 0$. To prove this we have from Problem 10,

$$u(L/2,t)=\frac{18h}{\pi^2}\sum_{n=1}^{\infty}\frac{1+(-1)^n}{n^2}\sin\frac{n\pi}{3}\cos\frac{n\pi a}{L}t\sin\frac{n\pi}{2}$$

$$=\begin{cases}0, & n=1,3,5,\dots,2m+1,\dots,\ \text{ we have } 1+(-1)^{2m+1}=1-1=0\\[2ex] 0, & n=2,4,6,\dots,2m,\dots,\ \text{ we have } \sin\dfrac{n\pi}{2}=\sin m\pi=0.\end{cases}$$

**14.** For Problem 1,

$$u_1(x,t)=\sum_{n=1}^{\infty}\left(A_n\cos\frac{n\pi a}{L}t\right)\sin\frac{n\pi}{L}x, \qquad \text{where} \qquad A_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx$$

For Problem 2,

$$u_2(x,t)=\sum_{n=1}^{\infty}\left(B_n\sin\frac{n\pi a}{L}t\right)\sin\frac{n\pi}{L}x \qquad \text{where} \qquad B_n=\frac{2}{n\pi a}\int_0^L g(x)\sin\frac{n\pi}{L}x\,dx$$

By superposition,

$$u(x,t)=u_1(x,t)+u_2(x,t)=\sum_{n=1}^{\infty}\left(A_n\cos\frac{n\pi a}{L}t\right)\sin\frac{n\pi}{L}x+\sum_{n=1}^{\infty}\left(B_n\sin\frac{n\pi a}{L}t\right)\sin\frac{n\pi}{L}x$$

$$=\sum_{n=1}^{\infty}\left(A_n\cos\frac{n\pi a}{L},t+B_n\sin\frac{n\pi a}{L}t\right)\sin\frac{n\pi}{L}x$$

**15.** Using $u=XT$ and $-\lambda$ as a separation constant we obtain

$$\begin{aligned}X''+\lambda X&=0,\\ X(0)&=0,\\ X(\pi)&=0,\end{aligned}$$

and

$$\begin{aligned}T''+2\beta T'+\lambda T&=0,\\ T'(0)&=0.\end{aligned}$$

Solving the differential equations we get

$$X=c_1\sin nx+c_2\cos nx \qquad \text{and} \qquad T=e^{-\beta t}\left(c_3\cos\sqrt{n^2-\beta^2}\,t+c_4\sin\sqrt{n^2-\beta^2}\,t\right)$$

The boundary conditions on $X$ imply $c_2=0$ so

$$X=c_1\sin nx \qquad \text{and} \qquad T=e^{-\beta t}\left(c_3\cos\sqrt{n^2-\beta^2}\,t+c_4\sin\sqrt{n^2-\beta^2}\,t\right)$$

and

$$u = \sum_{n=1}^{\infty} e^{-\beta t}\left(A_n \cos\sqrt{n^2-\beta^2}\,t + B_n \sin\sqrt{n^2-\beta^2}\,t\right)\sin nx.$$

Imposing

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} A_n \sin nx$$

and

$$u_t(x,0) = 0 = \sum_{n=1}^{\infty}\left(B_n\sqrt{n^2-\beta^2} - \beta A_n\right)\sin nx$$

gives

$$u(x,t) = e^{-\beta t}\sum_{n=1}^{\infty} A_n\left(\cos\sqrt{n^2-\beta^2}\,t + \frac{\beta}{\sqrt{n^2-\beta^2}}\sin\sqrt{n^2-\beta^2}\,t\right)\sin nx,$$

where

$$A_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx.$$

**16.** Using $u = XT$ and $-\lambda =$ as a separation constant leads to $X'' + \alpha^2 X = 0$, $X(0) = 0$, $X(\pi) = 0$ and $T'' + (1+\alpha^2)T = 0$, $T'(0) = 0$. Then $X = c_2 \sin nx$ and $T = c_3 \cos\sqrt{n^2+1}\,t$ for $n = 1, 2, 3, \ldots$ so that

$$u = \sum_{n1}^{\infty} B_n \cos\sqrt{n^2+1}\,t \sin nx.$$

Imposing $u(x,0) = \sum_{n=1}^{\infty} B_n \sin nx$ gives

$$\begin{aligned} B_n &= \frac{2}{\pi}\int_0^{\pi/2} x\sin nx\,dx + \frac{2}{\pi}\int_{\pi/2}^{\pi}(\pi - x)\sin nx\,dx = \frac{4}{\pi n^2}\sin\frac{n\pi}{2} \\ &= \begin{cases} 0, & n \text{ even,} \\ \dfrac{4}{\pi n^2}(-1)^{(n+3)/2}, & n = 2k-1,\ k = 1,2,3,\ldots \end{cases} \end{aligned}$$

Thus with $n = 2k-1$,

$$\begin{aligned} u(x,t) &= \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi}{2}}{n^2}\cos\sqrt{n^2+1}\,t\sin nx \\ &= \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(2k-1)^2}\cos\sqrt{(2k-1)^2+1}\,t\,\sin(2k-1)\,x. \end{aligned}$$

**17.** Separating variables in the partial differential equation and using the separation constant $-\lambda = \alpha^4$ gives

$$\frac{X^{(4)}}{X} = -\frac{T''}{a^2T} = \alpha^4$$

so that

$$X^{(4)} - \alpha^4 X = 0$$
$$T'' + a^2\alpha^4 T = 0$$

and

$$X = c_1 \cosh \alpha x + c_2 \sinh \alpha x + c_3 \cos \alpha x + c_4 \sin \alpha x$$
$$T = c_5 \cos a\alpha^2 t + c_6 \sin a\alpha^2 t.$$

The boundary conditions translate into $X(0) = X(L) = 0$ and $X''(0) = X''(L) = 0$. From $X(0) = X''(0) = 0$ we find $c_1 = c_3 = 0$. From

$$X(L) = c_2 \sinh \alpha L + c_4 \sin \alpha L = 0$$
$$X''(L) = \alpha^2 c_2 \sinh \alpha L - \alpha^2 c_4 \sin \alpha L = 0$$

we see by subtraction that $c_2 = 0$ and $c_4 \sin \alpha L = 0$. This equation yields the eigenvalues $\alpha = n\pi L$ for $n = 1, 2, 3, \ldots$. The corresponding eigenfunctions are

$$X = c_4 \sin \frac{n\pi}{L} x.$$

Thus

$$u(x,t) = \sum_{n=1}^{\infty} \left( A_n \cos \frac{n^2\pi^2}{L^2} at + B_n \sin \frac{n^2\pi^2}{L^2} at \right) \sin \frac{n\pi}{L} x.$$

From

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{L} x$$

we obtain

$$A_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x \, dx.$$

From

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} \left( -A_n \frac{n^2\pi^2 a}{L^2} \sin \frac{n^2\pi^2}{L^2} at + B_n \frac{n^2\pi^2 a}{L^2} \cos \frac{n^2\pi^2}{L^2} at \right) \sin \frac{n\pi}{L} x$$

and

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x) = \sum_{n=1}^{\infty} B_n \frac{n^2\pi^2 a}{L^2} \sin \frac{n\pi}{L} x$$

we obtain

$$B_n \frac{n^2\pi^2 a}{L^2} = \frac{2}{L} \int_0^L g(x) \sin \frac{n\pi}{L} x \, dx$$

and

$$B_n = \frac{2L}{n^2\pi^2 a} \int_0^L g(x) \sin \frac{n\pi}{L} x \, dx.$$

**18.** **(a)** Write the differential equation in $X$ from Problem 23 as $X^{(4)} - \alpha^4 X = 0$ where the eigenvalues are $\lambda = \alpha^2$. Then

$$X = c_1 \cosh \alpha x + c_2 \sinh \alpha x + c_3 \cos \alpha x + c_4 \sin \alpha x$$

and using $X(0) = 0$ and $X'(0) = 0$ we find $c_3 = -c_1$ and $c_4 = -c_2$. The conditions $X(L) = 0$ and $X'(L) = 0$ yield the system of equations

$$\begin{aligned} c_1(\cosh \alpha L - \cos \alpha L) + c_2(\sinh \alpha L - \sin \alpha L) &= 0 \\ c_1(\alpha \sinh \alpha L + \alpha \sin \alpha L) + c_2(\alpha \cosh \alpha L - \alpha \cos \alpha L) &= 0. \end{aligned}$$

In order for this system to have nontrivial solutions the determinant of the coefficients must be zero. That is,

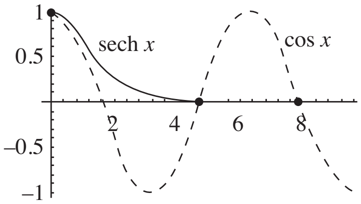
$$\alpha(\cosh \alpha L - \cos \alpha L)^2 - \alpha(\sinh^2 \alpha L - \sin^2 \alpha L) = 0.$$

Since $\alpha = 0$ leads to $X = 0$, $\lambda = \alpha^2 = 0^2 = 0$ is not an eigenvalue. Then, dividing the above equation by $\alpha$, we have

$$\begin{aligned} &(\cosh \alpha L - \cos \alpha L)^2 - (\sinh^2 \alpha L - \sin^2 \alpha L) \\ &\qquad = \cosh^2 \alpha L - 2 \cosh \alpha L \cos \alpha L + \cos^2 \alpha L - \sinh^2 \alpha L + \sin^2 \alpha L \\ &\qquad = -2 \cosh \alpha L \cos \alpha L + 2 = 0 \end{aligned}$$

or $\cosh \alpha L \cos \alpha L = 1$. Letting $x = \alpha L$ we see that the eigenvalues are $\lambda_n = \alpha_n^2 = x_n^2/L^2$ where $x_n$, $n = 1, 2, 3, \ldots$, are the positive roots of the equation $\cosh x \cos x = 1$.

**(b)** The equation $\cosh x \cos x = 1$ is the same as $\cos x = \operatorname{sech} x$. The figure indicates that the equation has an infinite number of roots.



**(c)** Using a CAS we find the first four positive roots of $\cosh x \cos x = 1$ to be $x_1 = 4.7300$, $x_2 = 7.8532$, $x_3 = 10.9956$, and $x_4 = 14.1372$. Thus the first four eigenvalues are $\lambda_1 = x_1^2/L = 22.3733/L$, $\lambda_2 = x_2^2/L = 61.6728/L$, $\lambda_3 = x_3^2/L = 120.9034/L$, and $\lambda_4 = x_4^2/L = 199.8594/L$.

**19.** From (8) in the text we have

$$u(x,t) = \sum_{n=1}^{\infty} \left( A_n \cos \frac{n\pi a}{L} t + B_n \sin \frac{n\pi a}{L} t \right) \sin \frac{n\pi}{L} x.$$

Since $u_t(x,0) = g(x) = 0$ we have $B_n = 0$ and

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} A_n \cos \frac{n\pi a}{L} t \sin \frac{n\pi}{L} x = \sum_{n=1}^{\infty} A_n \frac{1}{2} \left[ \sin\left( \frac{n\pi}{L} x + \frac{n\pi a}{L} t \right) + \sin\left( \frac{n\pi}{L} x - \frac{n\pi a}{L} t \right) \right] \\ &= \frac{1}{2} \sum_{n=1}^{\infty} A_n \left[ \sin \frac{n\pi}{L}(x + at) + \sin \frac{n\pi}{L}(x - at) \right]. \end{aligned}$$

From

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{L} x$$

we identify

$$f(x+at) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{L}(x+at)$$

and

$$f(x-at) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{L}(x-at),$$

so that

$$u(x,t) = \frac{1}{2}[f(x+at) + f(x-at)].$$

**20. (a)** We note that $\xi_x = \eta_x = 1$, $\xi_t = a$, and $\eta_t = -a$. Then

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = u_\xi + u_\eta$$

and

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}(u_\xi + u_\eta) = \frac{\partial u_\xi}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u_\xi}{\partial \eta}\frac{\partial \eta}{\partial x} + \frac{\partial u_\eta}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u_\eta}{\partial \eta}\frac{\partial \eta}{\partial x}$$
$$= u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}.$$

Similarly

$$\frac{\partial^2 u}{\partial t^2} = a^2(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}).$$

Thus

$$a^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \qquad \text{becomes} \qquad \frac{\partial^2 u}{\partial \xi \partial \eta} = 0.$$

**(b)** Integrating

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = \frac{\partial}{\partial \eta} u_\xi = 0$$

we obtain

$$\int \frac{\partial}{\partial \eta} u_\xi \, d\eta = \int 0 \, d\eta$$
$$u_\xi = f(\xi).$$

Integrating this result with respect to $\xi$ we obtain

$$\int \frac{\partial u}{\partial \xi} d\xi = \int f(\xi)\, d\xi$$
$$u = F(\xi) + G(\eta).$$

Since $\xi = x + at$ and $\eta = x - at$, we then have

$$u = F(\xi) + G(\eta) = F(x + at) + G(x - at).$$

Next, we have

$$\begin{aligned} u(x,t) &= F(x+at) + G(x-at) \\ u(x,0) &= F(x) + G(x) = f(x) \\ u_t(x,0) &= aF'(x) - aG'(x) = g(x) \end{aligned}$$

Integrating the last equation with respect to $x$ gives

$$F(x) - G(x) = \frac{1}{a}\int_{x_0}^{x} g(s)\,ds + c_1.$$

Substituting $G(x) = f(x) - F(x)$ we obtain

$$F(x) = \frac{1}{2}\,f(x) + \frac{1}{2a}\int_{x_0}^{x} g(s)\,ds + c$$

where $c = c_1/2$. Thus

$$G(x) = \frac{1}{2}\,f(x) - \frac{1}{2a}\int_{x_0}^{x} g(s)\,ds - c.$$

**(c)** From the expressions for $F$ and $G$,

$$\begin{aligned} F(x+at) &= \frac{1}{2}f(x+at) + \frac{1}{2a}\int_{x_0}^{x+at} g(s)\,ds + c \\ G(x-at) &= \frac{1}{2}\,f(x-at) - \frac{1}{2a}\int_{x_0}^{x-at} g(s)\,ds - c. \end{aligned}$$

Thus,

$$u(x,t) = F(x+at) + G(x-at) = \frac{1}{2}[f(x+at) + f(x-at)] + \frac{1}{2a}\int_{x-at}^{x+at} g(s)\,ds.$$

Here we have used $-\int_{x_0}^{x-at} g(s)\,ds = \int_{x-at}^{x_0} g(s)\,ds$.

**21.** $u(x,t) = \dfrac{1}{2}\left[\sin(x+at) + \sin(x-at)\right] + \dfrac{1}{2a}\displaystyle\int_{x-at}^{x+at} ds$

$$= \frac{1}{2}\left[\sin x\cos at + \cos x\sin at + \sin x\cos at - \cos x\sin at\right] + \frac{1}{2a}s\Big|_{x-at}^{x+at} = \sin x\cos at + t$$

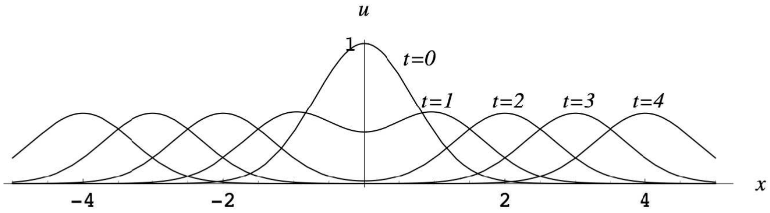
**22.** $u(x,t) = \dfrac{1}{2}\left[\sin(x+at) + \sin(x-at)\right] + \dfrac{1}{2a}\displaystyle\int_{x-at}^{x+at} \cos s\,ds$

$$= \sin x\cos at + \frac{1}{2a}\left[\sin(x+at) - \sin(x-at)\right] = \sin x\cos at + \frac{1}{a}\cos x\sin at$$

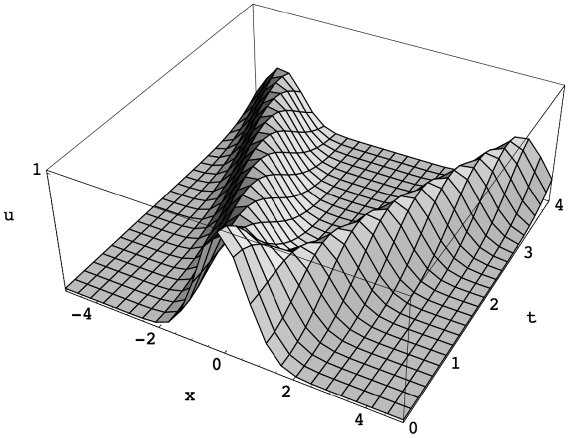
**23.** $$u(x,t) = 0 + \frac{1}{2a}\int_{x-at}^{x+at} \sin 2s\,ds = \frac{1}{2a}\left[\frac{-\cos(2x+2at)+\cos(2x-2at)}{2}\right]$$
$$= \frac{1}{4a}\left[-\cos 2x\cos 2at + \sin 2x\sin 2at + \cos 2x\cos 2at + \sin 2x\sin 2at\right]$$
$$= \frac{1}{2a}\sin 2x\sin 2at$$

**24.** $$u(x,t) = \frac{1}{2}\left[e^{-(x+at)^2} + e^{-(x-at)^2}\right] = \frac{1}{2}\left[e^{-(x^2+2axt+a^2t^2)} + e^{-(x^2-2axt+a^2t^2)}\right]$$
$$= e^{-(x^2+a^2t^2)}\left[\frac{e^{-2axt}+e^{2axt}}{2}\right] = e^{-(x^2+a^2t^2)}\cosh 2axt$$

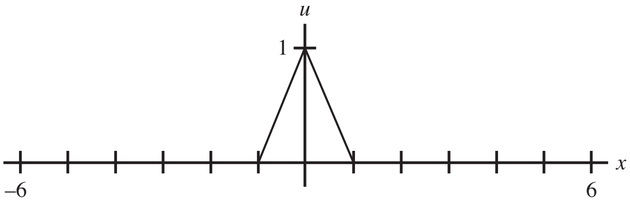
**25.** **(a)**



**(b)**



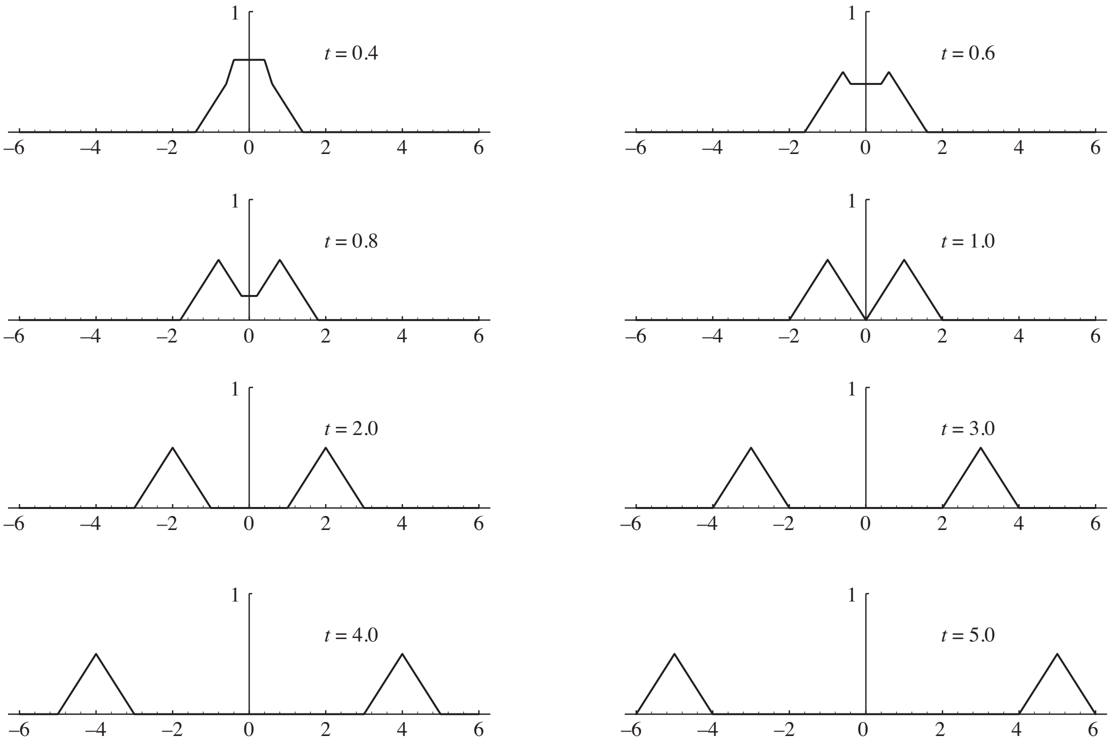
**26.** **(a)**



**(b)** Since $g(x) = 0$, d'Alembert's solution with $a = 1$ is

$$u(x,t) = \frac{1}{2}[f(x+t) + f(x-t)].$$

Sample plots are shown below.

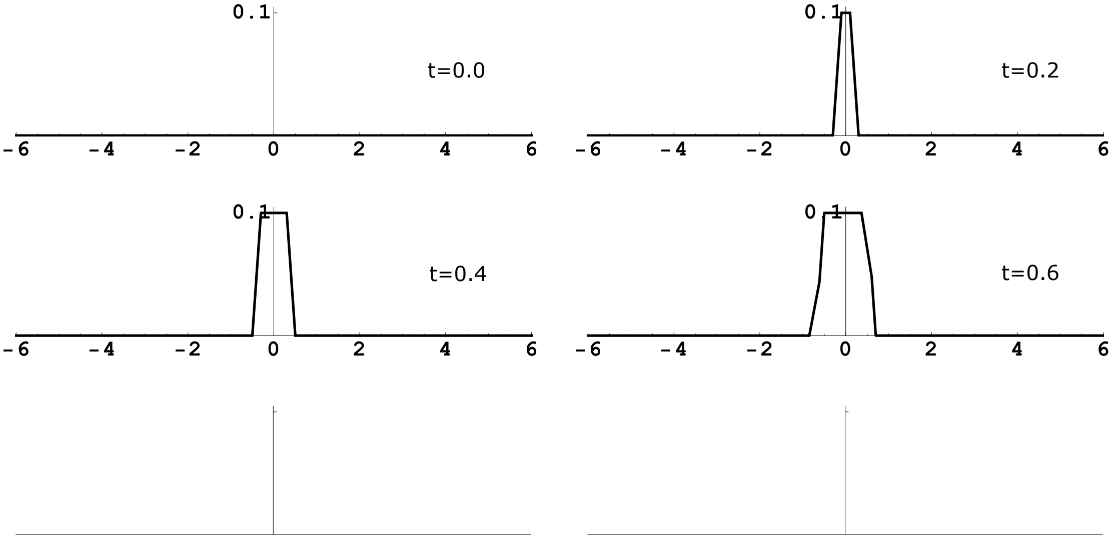


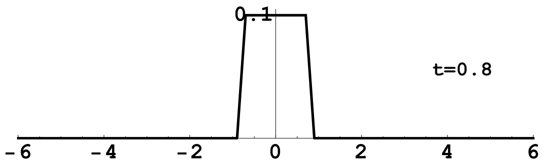
**(c)** The single peaked wave disolves into two peaks moving outward.

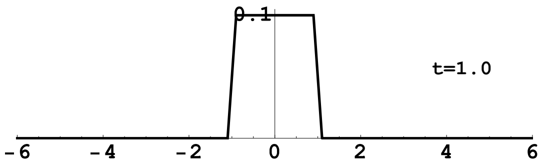
**27. (a)** With $a = 1$, d'Alembert's solution is

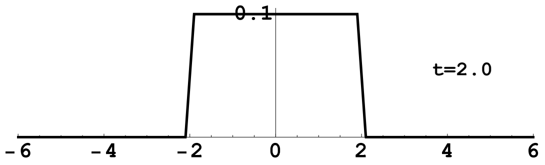
$$u(x,t) = \frac{1}{2}\int_{x-t}^{x+t} g(s)\,ds \qquad \text{where} \qquad g(s) = \begin{cases} 1, & |s| \le 0.1 \\ 0, & |s| > 0.1. \end{cases}$$

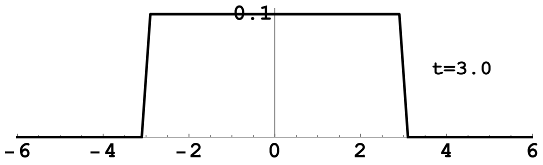
Sample plots are shown below.

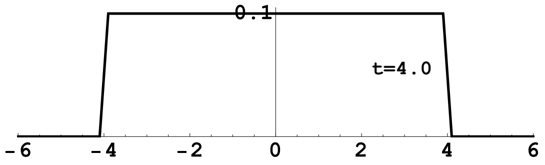


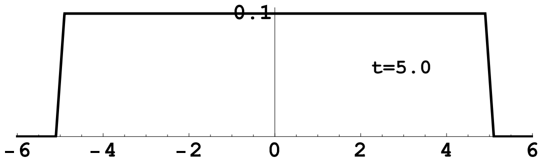












**(b)** Some frames of the movie are shown in part (**a**), The string has a roughly rectangular shape with the base on the $x$-axis increasing in length.

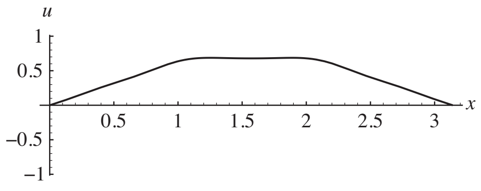
**28. (a)-(b)** With the given parameters, the solution is

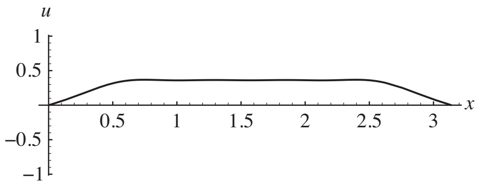
$$u(x,t)=\frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\cos nt\sin nx.$$

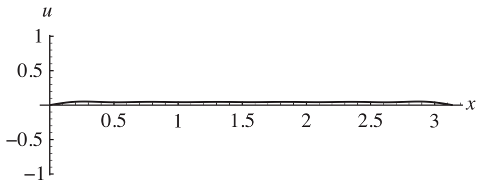
For $n$ even, $\sin(n\pi/2)=0$, so the first six nonzero terms correspond to $n=1$, 3, 5, 7, 9, 11. In this case $\sin(n\pi/2)=\sin(2p-1)/2=(-1)^{p+1}$ for $p=1$, 2, 3, 4, 5, 6, and

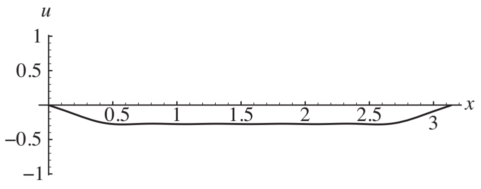
$$u(x,t)=\frac{8}{\pi^2}\sum_{p=1}^{\infty}\frac{(-1)^{p+1}}{(2p-1)^2}\cos(2p-1)t\sin(2p-1)x.$$

Frames of the movie corresponding to $t=0.5$, 1, 1.5, and 2 are shown.









## 12.5 Laplace's Equation

**1.** Using $u = XY$ and $-\lambda$ as a separation constant we obtain

$$X'' + \lambda X = 0,$$
$$X(0) = 0,$$
$$X(a) = 0,$$

and

$$Y'' - \lambda Y = 0,$$
$$Y(0) = 0.$$

With $\lambda = \alpha^2 > 0$ the solutions of the differential equations are

$$X = c_1 \cos \alpha x + c_2 \sin \alpha x \qquad \text{and} \qquad Y = c_3 \cosh \alpha y + c_4 \sinh \alpha y$$

The boundary and initial conditions imply

$$X = c_2 \sin \frac{n\pi}{a} x \qquad \text{and} \qquad Y = c_4 \sinh \frac{n\pi}{a} y$$

for $n = 1, 2, 3, \ldots$ so that

$$u = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{a} x \sinh \frac{n\pi}{a} y.$$

Imposing

$$u(x, b) = f(x) = \sum_{n=1}^{\infty} A_n \sinh \frac{n\pi b}{a} \sin \frac{n\pi}{a} x$$

gives

$$A_n \sinh \frac{n\pi b}{a} = \frac{2}{a} \int_0^a f(x) \sin \frac{n\pi}{a} x\, dx$$

so that

$$u(x, y) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{a} x \sinh \frac{n\pi}{a} y$$

where

$$A_n = \frac{2}{a} \operatorname{csch} \frac{n\pi b}{a} \int_0^a f(x) \sin \frac{n\pi}{a} x\, dx.$$

**2.** Using $u = XY$ and $-\lambda$ as a separation constant we obtain

$$X'' + \lambda X = 0,$$
$$X(0) = 0,$$
$$X(a) = 0,$$

and

$$Y'' - \lambda Y = 0,$$

$$Y'(0) = 0.$$

With $\lambda = \alpha^2 > 0$ the solutions of the differential equations are

$$X = c_1 \cos \alpha x + c_2 \sin \alpha x \qquad \text{and} \qquad Y = c_3 \cosh \alpha y + c_4 \sinh \alpha y$$

The boundary and initial conditions imply

$$X = c_2 \sin \frac{n\pi}{a} x \qquad \text{and} \qquad Y = c_3 \cosh \frac{n\pi}{a} y$$

for $n = 1, 2, 3, \ldots$ so that

$$u = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{a} x \cosh \frac{n\pi}{a} y.$$

Imposing

$$u(x,b) = f(x) = \sum_{n=1}^{\infty} A_n \cosh \frac{n\pi b}{a} \sin \frac{n\pi}{a} x$$

gives

$$A_n \cosh \frac{n\pi b}{a} = \frac{2}{a} \int_0^a f(x) \sin \frac{n\pi}{a} x \, dx$$

so that

$$u(x,y) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{a} x \cosh \frac{n\pi}{a} y$$

where

$$A_n = \frac{2}{a} \operatorname{sech} \frac{n\pi b}{a} \int_0^a f(x) \sin \frac{n\pi}{a} x \, dx.$$

**3.** Using $u = XY$ and $-\lambda$ as a separation constant we obtain

$$X'' + \lambda X = 0,$$

$$X(0) = 0,$$

$$X(a) = 0,$$

and

$$Y'' - \lambda Y = 0,$$

$$Y(b) = 0.$$

With $\lambda = \alpha^2 > 0$ the solutions of the differential equations are

$$X = c_1 \cos \alpha x + c_2 \sin \alpha x \qquad \text{and} \qquad Y = c_3 \cosh \alpha y + c_4 \sinh \alpha y$$

The boundary and initial conditions imply

$$X = c_2 \sin \frac{n\pi}{a} x \qquad \text{and} \qquad Y = c_2 \cosh \frac{n\pi}{a} y - c_2 \frac{\cosh \frac{n\pi b}{a}}{\sinh \frac{n\pi b}{a}} \sinh \frac{n\pi}{a} y$$

for $n = 1, 2, 3, \ldots$ so that

$$u = \sum_{n=1}^{\infty} A_n \left( \cosh \frac{n\pi}{a} y - \frac{\cosh \frac{n\pi b}{a}}{\sinh \frac{n\pi b}{a}} \sinh \frac{n\pi}{a} y \right) \sin \frac{n\pi}{a} x.$$

Imposing

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{a} x$$

gives

$$A_n = \frac{2}{a} \int_0^a f(x) \sin \frac{n\pi}{a} x \, dx$$

so that

$$u(x,y) = \frac{2}{a} \sum_{n=1}^{\infty} \left( \int_0^a f(x) \sin \frac{n\pi}{a} x \, dx \right) \left( \cosh \frac{n\pi}{a} y - \frac{\cosh \frac{n\pi b}{a}}{\sinh \frac{n\pi b}{a}} \sinh \frac{n\pi}{a} y \right) \sin \frac{n\pi}{a} x.$$

**4.** Using $u = XY$ and $-\lambda$ as a separation constant we obtain

$$X'' + \lambda X = 0,$$
$$X'(0) = 0,$$
$$X'(a) = 0,$$

and

$$Y'' - \lambda Y = 0,$$
$$Y(b) = 0.$$

With $\lambda = \alpha^2 > 0$ the solutions of the differential equations are

$$X = c_1 \cos \alpha x + c_2 \sin \alpha x \qquad \text{and} \qquad Y = c_3 \cosh \alpha y + c_4 \sinh \alpha y$$

The boundary and initial conditions imply

$$X = c_1 \cos \frac{n\pi}{a} x \qquad \text{and} \qquad Y = c_3 \cosh \frac{n\pi}{a} y - c_3 \frac{\cosh \frac{n\pi b}{a}}{\sinh \frac{n\pi b}{a}} \sinh \frac{n\pi}{a} y$$

for $n = 1, 2, 3, \ldots$. Since $\lambda = 0$ is an eigenvalue for both differential equations with corresponding eigenfunctions 1 and $y - b$, respectively we have

$$u = A_0(y-b) + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi}{a} x \left( \cosh \frac{n\pi}{a} y - \frac{\cosh \frac{n\pi b}{a}}{\sinh \frac{n\pi b}{a}} \sinh \frac{n\pi}{a} y \right).$$

Imposing

$$u(x,0) = x = -A_0 b + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi}{a}x$$

gives

$$-A_0 b = \frac{1}{a}\int_0^a x\,dx = \frac{1}{2}a$$

and

$$A_n = \frac{2}{a}\int_0^a x\cos\frac{n\pi}{a}x\,dx = \frac{2a}{n^2\pi^2}[(-1)^n - 1]$$

so that

$$u(x,y) = \frac{a}{2b}(b-y) + \frac{2a}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n^2}\cos\frac{n\pi}{a}x\left(\cosh\frac{n\pi}{a}y - \frac{\cosh\frac{n\pi b}{a}}{\sinh\frac{n\pi b}{a}}\sinh\frac{n\pi}{a}y\right).$$

**5.** Using $u = XY$ and $-\lambda$ as a separation constant we obtain

$$X'' + \lambda X = 0,$$

$$X'(0) = 0,$$

$$X'(a) = 0,$$

and

$$Y'' - \lambda Y = 0,$$

$$Y(b) = 0.$$

With $\lambda = -\alpha^2 < 0$ the solutions of the differential equations are

$$X = c_1\cosh\alpha x + c_2\sinh\alpha x \qquad \text{and} \qquad Y = c_3\cos\alpha y + c_4\sin\alpha y$$

for $n = 1, 2, 3 \ldots.$ The boundary and initial conditions imply

$$X = c_2\sinh n\pi x \qquad \text{and} \qquad Y = c_3\cos n\pi y$$

for $n = 1, 2, 3, \ldots.$ Since $\lambda = 0$ is an eigenvalue for the differential equation in $X$ with corresponding eigenfunction $x$ we have

$$u = A_0 x + \sum_{n=1}^{\infty} A_n \sinh n\pi x\,\cos n\pi y.$$

Imposing

$$u(1,y) = 1 - y = A_0 + \sum_{n=1}^{\infty} A_n \sinh n\pi\,\cos n\pi y$$

gives

$$A_0 = \int_0^1 (1-y)\,dy$$

and

$$A_n \sinh n\pi = 2\int_0^1 (1-y)\cos n\pi y\,dy = \frac{2[1-(-1)^n]}{{}^2\pi^2}$$

for $n = 1, 2, 3, \ldots$ so that

$$u(x,y) = \frac{1}{2}x + \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^2\sinh n\pi}\sinh n\pi x\,\cos n\pi y.$$

**6.** Using $u = XY$ and $-\lambda$ as a separation constant we obtain

$$X'' + \lambda X = 0,$$

$$X'(1) = 0$$

and

$$Y'' - \lambda Y = 0,$$

$$Y'(0) = 0,$$

$$Y'(\pi) = 0.$$

With $\lambda = \alpha^2 < 0$ the solutions of the differential equations are

$$X = c_1\cosh\alpha x + c_2\sinh\alpha x \qquad \text{and} \qquad Y = c_3\cos\alpha y + c_4\sin\alpha y$$

The boundary and initial conditions imply

$$X = c_1\cosh nx - c_1\frac{\sinh n}{\cosh n}\sinh nx \qquad \text{and} \qquad Y = c_3\cos ny$$

for $n = 1, 2, 3, \ldots$. Since $\lambda = 0$ is an eigenvalue for both differential equations with corresponding eigenfunctions 1 and 1 we have

$$u = A_0 + \sum_{n=1}^{\infty} A_n\left(\cosh nx - \frac{\sinh n}{\cosh n}\sinh nx\right)\cos ny.$$

Imposing

$$u(0,y) = g(y) = A_0 + \sum_{n=1}^{\infty} A_n\cos ny$$

gives

$$A_0 = \frac{1}{\pi}\int_0^{\pi} g(y)\,dy \qquad \text{and} \qquad A_n = \frac{2}{\pi}\int_0^{\pi} g(y)\cos ny\,dy$$

for $n = 1, 2, 3, \ldots$ so that

$$u(x,y) = \frac{1}{\pi}\int_0^{\pi} g(y)\,dy + \sum_{n=1}^{\infty}\left(\frac{2}{\pi}\int_0^{\pi} g(y)\cos ny\,dy\right)\left(\cosh nx - \frac{\sinh n}{\cosh n}\sinh nx\right)\cos ny.$$

**7.** Using $u = XY$ and $-\lambda$ as a separation constant we obtain

$$X'' + \lambda X = 0,$$
$$X'(0) = X(0)$$

and

$$Y'' - \lambda Y = 0,$$
$$Y(0) = 0,$$
$$Y(\pi) = 0.$$

With $\lambda = \alpha^2 < 0$ the solutions of the differential equations are

$$X = c_1 \cosh \alpha x + c_2 \sinh \alpha x \qquad \text{and} \qquad Y = c_3 \cos \alpha y + c_4 \sin \alpha y$$

The boundary and initial conditions imply

$$Y = c_4 \sin ny \qquad \text{and} \qquad X = c_2(n \cosh nx + \sinh nx)$$

for $n = 1, 2, 3, \ldots$ so that

$$u = \sum_{n=1}^{\infty} A_n(n \cosh nx + \sinh nx) \sin ny.$$

Imposing

$$u(\pi, y) = 1 = \sum_{n=1}^{\infty} A_n(n \cosh n\pi + \sinh n\pi) \sin ny$$

gives

$$A_n(n \cosh n\pi + \sinh n\pi) = \frac{2}{\pi} \int_0^{\pi} \sin ny \, dy = \frac{2[1 - (-1)^n]}{n\pi}$$

for $n = 1, 2, 3, \ldots$ so that

$$u(x, y) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n} \frac{n \cosh nx + \sinh nx}{n \cosh n\pi + \sinh n\pi} \sin ny.$$

**8.** Using $u = XY$ and $-\lambda$ as a separation constant we obtain

$$X'' + \lambda X = 0,$$
$$X(0) = 0,$$
$$X(1) = 0,$$

and

$$Y'' - \lambda Y = 0,$$
$$Y'(0) = Y(0).$$

With $\lambda = \alpha^2 > 0$ the solutions of the differential equations are

$$X = c_1 \cos \alpha x + c_2 \sin \alpha x \qquad \text{and} \qquad Y = c_3 \cosh \alpha y + c_4 \sinh \alpha y$$

The boundary and initial conditions imply

$$X = c_2 \sin n\pi x \qquad \text{and} \qquad Y = c_4(n\pi \cosh n\pi y + \sinh n\pi y)$$

for $n = 1, 2, 3, \ldots$ so that

$$u = \sum_{n=1}^{\infty} A_n(n\pi \cosh n\pi y + \sinh n\pi y) \sin n\pi x.$$

Imposing

$$u(x,1) = f(x) = \sum_{n=1}^{\infty} A_n(n\pi \cosh n\pi + \sinh n\pi) \sin n\pi x$$

gives

$$A_n(n\pi \cosh n\pi + \sinh n\pi) = \frac{2}{\pi} \int_0^{\pi} f(x) \sin n\pi x \, dx$$

for $n = 1, 2, 3, \ldots$ so that

$$u(x,y) = \sum_{n=1}^{\infty} A_n(n\pi \cosh n\pi y + \sinh n\pi y) \sin n\pi x$$

where

$$A_n = \frac{2}{n\pi \cosh n\pi + \pi \sinh n\pi} \int_0^1 f(x) \sin n\pi x \, dx.$$

**9.** This boundary-value problem has the form of Problem 1 from the text of this section, with $a = b = 1$, $f(x) = 100$, and $g(x) = 200$. The solution, then, is

$$u(x,y) = \sum_{n=1}^{\infty} (A_n \cosh n\pi y + B_n \sinh n\pi y) \sin n\pi x,$$

where

$$A_n = 2\int_0^1 100 \sin n\pi x \, dx = 200 \left( \frac{1 - (-1)^n}{n\pi} \right)$$

and

$$\begin{aligned} B_n &= \frac{1}{\sinh n\pi} \left[ 2\int_0^1 200 \sin n\pi x \, dx - A_n \cosh n\pi \right] \\ &= \frac{1}{\sinh n\pi} \left[ 400 \left( \frac{1 - (-1)^n}{n\pi} \right) - 200 \left( \frac{1 - (-1)^n}{n\pi} \right) \cosh n\pi \right] \\ &= 200 \left[ \frac{1 - (-1)^n}{n\pi} \right] [2 \operatorname{csch} n\pi - \coth n\pi]. \end{aligned}$$

**10.** This boundary-value problem has the form of Problem 2 from the text of this section, with $a = 1$ and $b = 1$. Thus, the solution has the form

$$u(x, y) = \sum_{n=1}^{\infty} (A_n \cosh n\pi x + B_n \sinh n\pi x) \sin n\pi y.$$

The boundary condition $u(0, y) = 10y$ implies

$$10y = \sum_{n=1}^{\infty} A_n \sin n\pi y$$

and

$$A_n = \frac{2}{1} \int_0^1 10y \sin n\pi y \, dy = \frac{20}{n\pi}(-1)^{n+1}.$$

The boundary condition $u_x(1, y) = -1$ implies

$$-1 = \sum_{n=1}^{\infty} (n\pi A_n \sinh n\pi + n\pi B_n \cosh n\pi) \sin n\pi y$$

and

$$\begin{aligned} n\pi A_n \sinh n\pi + n\pi B_n \cosh n\pi &= \frac{2}{1} \int_0^1 (-\sin n\pi y) \, dy \\ A_n \sinh n\pi + B_n \cosh n\pi &= -\frac{2}{n\pi} \left[1 - (-1)^n\right] \\ B_n &= \frac{2}{n\pi} \left[(-1)^n - 1\right] \operatorname{sech} n\pi - \frac{20}{n\pi}(-1)^{n+1} \tanh n\pi. \end{aligned}$$

**11.** Using $u = XY$ and $-\lambda$ as a separation constant we obtain

$$\begin{aligned} X'' + \lambda X &= 0, \\ X(0) &= 0, \\ X(\pi) &= 0, \end{aligned}$$

and

$$Y'' - \lambda Y = 0.$$

With $\lambda = \alpha^2 > 0$ the solutions of the differential equations are

$$X = c_1 \cos \alpha x + c_2 \sin \alpha x \qquad \text{and} \qquad Y = c_3 e^{\alpha y} + c_4 e^{-\alpha y}$$

Then the boundedness of $u$ as $y \to \infty$ implies $c_3 = 0$, so $Y = c_4 e^{-ny}$. The boundary conditions at $x = 0$ and $x = \pi$ imply $c_1 = 0$ so $X = c_2 \sin nx$ for $n = 1, 2, 3, \ldots$ and

$$u = \sum_{n=1}^{\infty} A_n e^{-ny} \sin nx.$$

Imposing

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} A_n \sin nx$$

gives

$$A_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

so that

$$u(x,y) = \sum_{n=1}^{\infty}\left(\frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx\right)e^{-ny}\sin nx.$$

**12.** Using $u = XY$ and $-\lambda$ as a separation constant we obtain

$$X'' + \lambda X = 0,$$

$$X'(0) = 0,$$

$$X'(\pi) = 0,$$

and

$$Y'' - \lambda Y = 0.$$

With $\lambda = \alpha^2 > 0$ the solutions of the differential equations are

$$X = c_1\cos\alpha x + c_2\sin\alpha x \qquad \text{and} \qquad Y = c_3 e^{\alpha y} + c_4 e^{-\alpha y}$$

The boundary conditions at $x = 0$ and $x = \pi$ imply $c_2 = 0$ so $X = c_1\cos nx$ for $n = 1, 2, 3, \ldots$. Now the boundedness of $u$ as $y \to \infty$ implies $c_3 = 0$, so $Y = c_4 e^{-ny}$. In this problem $\lambda = 0$ is also an eigenvalue with corresponding eigenfunction 1 so that

$$u = A_0 + \sum_{n=1}^{\infty} A_n e^{-ny}\cos nx.$$

Imposing

$$u(x,0) = f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos nx$$

gives

$$A_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx \qquad \text{and} \qquad A_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

so that

$$u(x,y) = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx + \sum_{n=1}^{\infty}\left(\frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx\right)e^{-ny}\cos nx.$$

**13.** Since the boundary conditions at $y = 0$ and $y = b$ are functions of $x$ we choose to separate Laplace's equation as

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda$$

so that

$$X'' + \lambda X = 0$$

$$Y'' - \lambda Y = 0.$$

Then with $\lambda = \alpha^2$ we have

$$X(x) = c_1 \cos \alpha x + c_2 \sin \alpha x$$

$$Y(y) = c_3 \cosh \alpha y + c_4 \sinh \alpha y.$$

Now $X(0) = 0$ gives $c_1 = 0$ and $X(a) = 0$ implies $\sin \alpha a = 0$ or $\alpha = n\pi/a$ for $n = 1, 2, 3, \ldots$. Thus

$$u_n(x, y) = XY = \left(A_n \cosh \frac{n\pi}{a} y + B_n \sinh \frac{n\pi}{a} y\right) \sin \frac{n\pi}{a} x$$

and

$$u(x, y) = \sum_{n=1}^{\infty} \left(A_n \cosh \frac{n\pi}{a} y + B_n \sinh \frac{n\pi}{a} y\right) \sin \frac{n\pi}{a} x. \qquad (\mathbf{1})$$

At $y = 0$ we then have

$$f(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{a} x$$

and consequently

$$A_n = \frac{2}{a} \int_0^a f(x) \sin \frac{n\pi}{a} x \, dx. \qquad (\mathbf{2})$$

At $y = b$,

$$g(y) = \sum_{n=1}^{\infty} \left(A_n \cosh \frac{n\pi}{a} b + B_n \sinh \frac{n\pi}{b} a\right) \sin \frac{n\pi}{a} x$$

indicates that the entire expression in the parentheses is given by

$$A_n \cosh \frac{n\pi}{a} b + B_n \sinh \frac{n\pi}{a} b = \frac{2}{a} \int_0^a g(x) \sin \frac{n\pi}{a} x \, dx.$$

We can now solve for $B_n$:

$$B_n \sinh \frac{n\pi}{a} b = \frac{2}{a} \int_0^a g(x) \sin \frac{n\pi}{a} x \, dx - A_n \cosh \frac{n\pi}{a} b$$

$$B_n = \frac{1}{\sinh \frac{n\pi}{a} b} \left(\frac{2}{a} \int_0^a g(x) \sin \frac{n\pi}{a} x \, dx - A_n \cosh \frac{n\pi}{a} b\right). \qquad (\mathbf{3})$$

A solution to the given boundary-value problem consists of the series (**1**) with coefficients $A_n$ and $B_n$ given in (**2**) and (**3**), respectively.

**14.** Since the boundary conditions at $x = 0$ and $x = a$ are functions of $y$ we choose to separate Laplace's equation as

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda$$

so that

$$X'' + \lambda X = 0$$

$$Y'' - \lambda Y = 0.$$

Then with $\lambda = -\alpha^2$ we have

$$X(x) = c_1 \cosh \alpha x + c_2 \sinh \alpha x$$

$$Y(y) = c_3 \cos \alpha y + c_4 \sin \alpha y.$$

Now $Y(0) = 0$ gives $c_3 = 0$ and $Y(b) = 0$ implies $\sin \alpha b = 0$ or $\alpha = n\pi/b$ for $n = 1, 2, 3, \ldots$ . Thus

$$u_n(x, y) = XY = \left(A_n \cosh \frac{n\pi}{b} x + B_n \sinh \frac{n\pi}{b} x\right) \sin \frac{n\pi}{b} y$$

and

$$u(x, y) = \sum_{n=1}^{\infty} \left(A_n \cosh \frac{n\pi}{b} x + B_n \sinh \frac{n\pi}{b} x\right) \sin \frac{n\pi}{b} y. \qquad \textbf{(4)}$$

At $x = 0$ we then have

$$F(y) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{b} y$$

and consequently

$$A_n = \frac{2}{b} \int_0^b F(y) \sin \frac{n\pi}{b} y \, dy. \qquad \textbf{(5)}$$

At $x = a$,

$$G(y) = \sum_{n=1}^{\infty} \left(A_n \cosh \frac{n\pi}{b} a + B_n \sinh \frac{n\pi}{b} a\right) \sin \frac{n\pi}{b} y$$

indicates that the entire expression in the parentheses is given by

$$A_n \cosh \frac{n\pi}{b} a + B_n \sinh \frac{n\pi}{b} a = \frac{2}{b} \int_0^b G(y) \sin \frac{n\pi}{b} y \, dy.$$

We can now solve for $B_n$:

$$B_n \sinh \frac{n\pi}{b} a = \frac{2}{b} \int_0^b G(y) \sin \frac{n\pi}{b} y \, dy - A_n \cosh \frac{n\pi}{b} a$$

$$B_n = \frac{1}{\sinh \frac{n\pi}{b} a} \left( \frac{2}{b} \int_0^b G(y) \sin \frac{n\pi}{b} y \, dy - A_n \cosh \frac{n\pi}{b} a \right). \qquad \textbf{(6)}$$

A solution to the given boundary-value problem consists of the series (**4**) with coefficients $A_n$ and $B_n$ given in (**5**) and (**6**), respectively.

**15.** Referring to the discussion in this section of the text we identify $a = b = \pi$, $f(x) = 0$, $g(x) = 1$, $F(y) = 1$, and $G(y) = 1$. Then $A_n = 0$ and

$$u_1(x, y) = \sum_{n=1}^{\infty} B_n \sinh ny \sin nx$$

where

$$B_n = \frac{2}{\pi \sinh n\pi} \int_0^{\pi} \sin nx \, dx = \frac{2[1 - (-1)^n]}{n\pi \sinh n\pi}.$$

Next

$$u_2(x, y) = \sum_{n=1}^{\infty} (A_n \cosh nx + B_n \sinh nx) \sin ny$$

where

$$A_n = \frac{2}{\pi} \int_0^{\pi} \sin ny \, dy = \frac{2[1 - (-1)^n]}{n\pi}$$

and

$$\begin{aligned} B_n &= \frac{1}{\sinh n\pi} \left( \frac{2}{\pi} \int_0^{\pi} \sin ny \, dy - A_n \cosh n\pi \right) \\ &= \frac{1}{\sinh n\pi} \left( \frac{2[1 - (-1)^n]}{n\pi} - \frac{2[1 - (-1)^n]}{n\pi} \cosh n\pi \right) \\ &= \frac{2[1 - (-1)^n]}{n\pi \sinh n\pi} (1 - \cosh n\pi). \end{aligned}$$

Now

$$\begin{aligned} A_n \cosh nx + B_n \sinh nx &= \frac{2[1 - (-1)^n]}{n\pi} \left[ \cosh nx + \frac{\sinh nx}{\sinh n\pi} (1 - \cosh n\pi) \right] \\ &= \frac{2[1 - (-1)^n]}{n\pi \sinh n\pi} [\cosh nx \sinh n\pi + \sinh nx - \sinh nx \cosh n\pi] \\ &= \frac{2[1 - (-1)^n]}{n\pi \sinh n\pi} [\sinh nx + \sinh n(\pi - x)] \end{aligned}$$

and

$$\begin{aligned} u(x, y) = u_1 + u_2 &= \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n \sinh n\pi} \sinh ny \sin nx \\ &\quad + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{[1 - (-1)^n][\sinh nx + \sinh n(\pi - x)]}{n \sinh n\pi} \sin ny. \end{aligned}$$

**16.** Referring to the discussion in this section of the text we identify $a = b = 2$, $f(x) = 0$,

$$g(x) = \begin{cases} x, & 0 < x < 1 \\ 2 - x, & 1 < x < 2, \end{cases}$$

$F(y) = 0$, and $G(y) = y(2 - y)$. Then $A_n = 0$ and

$$u_1(x, y) = \sum_{n=1}^{\infty} B_n \sinh \frac{n\pi}{2}\, y \sin \frac{n\pi}{2}\, x$$

where

$$\begin{aligned} B_n &= \frac{1}{\sinh n\pi} \int_0^2 g(x) \sin \frac{n\pi}{2}\, x\, dx \\ &= \frac{1}{\sinh n\pi} \left( \int_0^1 x \sin \frac{n\pi}{2}\, x\, dx + \int_1^2 (2 - x) \sin \frac{n\pi}{2}\, x\, dx \right) \\ &= \frac{8 \sin \frac{n\pi}{2}}{n^2 \pi^2 \sinh n\pi}\,. \end{aligned}$$

Next, since $A_n = 0$ in $u_2$, we have

$$u_2(x, y) = \sum_{n=1}^{\infty} B_n \sinh \frac{n\pi}{2}\, x \sin \frac{n\pi}{2}$$

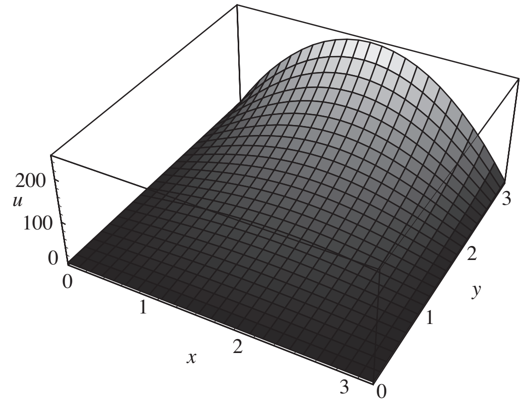
where

$$B_n = \frac{1}{\sinh n\pi} \int_0^2 y(2 - y) \sin \frac{n\pi}{2}\, y\, dy = \frac{16[1 - (-1)^n]}{n^3 \pi^3 \sinh n\pi}\,.$$

Thus

$$\begin{aligned} u(x, y) = u_1 + u_2 &= \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{\sin \frac{n\pi}{2}}{n^2 \sinh n\pi} \sinh \frac{n\pi}{2}\, y \sin \frac{n\pi}{2}\, x \\ &\quad + \frac{16}{\pi^3} \sum_{n=1}^{\infty} \frac{[1 - (-1)^n]}{n^3 \sinh n\pi} \sinh \frac{n\pi}{2}\, x \sin \frac{n\pi}{2}\, y. \end{aligned}$$

**17. (a)**



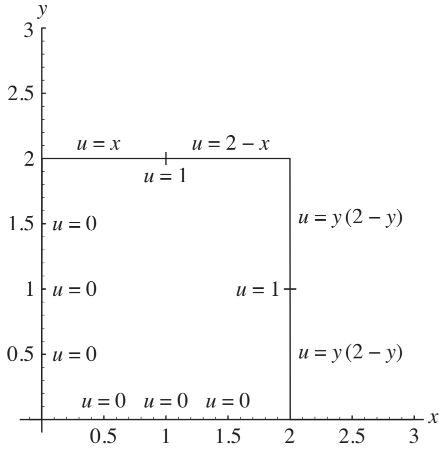
**(b)** The maximum value occurs at $(\pi/2, \pi)$ and is $f(\pi/2) = 25\pi^2$.

**(c)** The coefficients are

$$\begin{aligned} A_n &= \frac{2}{\pi}\operatorname{csch} n\pi \int_0^{\pi} 100x(\pi - x)\sin nx\,dx \\ &= \frac{200\operatorname{csch} n\pi}{\pi}\left[\frac{200}{n^3}\left(1-(-1)^n\right)\right] = \frac{400}{n^3\pi}\left[1-(-1)^n\right]\operatorname{csch} n\pi. \end{aligned}$$

See part (**a**) for the graph.

**18.** From the figure showing the boundary conditions we see that the maximum value of the temperature is 1 at $(1, 2)$ and $(2, 1)$.



**19.** Assuming $u(x, y) = X(x)Y(y)$ and substituting into the partial differential equationwe get $X''Y + XY'' = 0$. Separating variables and using $\lambda = \alpha^2$ we get

$$X'' - \alpha^2 X = 0, \quad X'(0) = 0,$$

which implies $X(x) = c_3 \cosh \alpha x$. From

$$Y'' + \alpha^2 Y = 0, \quad Y'(0) = 0, \quad Y'(b) = 0$$

we get $Y(y) = c_1 \cos \alpha y + c_2 \sin \alpha y$ and eigenvalues $\lambda_n = n^2\pi^2/b^2$, $n = 1, 2, 3, \ldots$ . The corresponding eigenfunctions are $Y(y) = c_1 \cos(n\pi y/b)$. For $\lambda = 0$ the boundary conditions applied to $X(x) = c_3 + c_4 x$ and $Y(y) = c_1 + c_2 y$ imply $X = c_3$ and $Y = c_1$. Forming products and using the superposition principle then gives

$$u(x, y) = A_0 + \sum_{n=1}^{\infty} A_n \cosh \frac{n\pi}{b} x \cos \frac{n\pi}{b} y.$$

The remaining boundary condition, $u_x(a) = g(y)$ implies

$$g(y) = \left.\frac{\partial u}{\partial x}\right|_{x=a} = \sum_{n=1}^{\infty} A_n \frac{n\pi}{b} \sinh \frac{n\pi}{b} x \cos \frac{n\pi}{b} y,$$

and so $A_0$ remains arbitrary. In order that the series expression for $g(y)$ be a cosine series, the constant term in the series, $a_0/2$, must be 0. Thus, from Section 11.3 in the text,

$$a_0 = \frac{2}{b}\int_0^b g(y)\,dy = 0 \quad \text{so} \quad \int_0^b g(y)\,dy = 0.$$

Also,

$$A_n \frac{n\pi}{b} \sinh \frac{n\pi}{b} a = \frac{2}{b}\int_0^b g(y)\cos\frac{n\pi}{b}y\,dy$$

and

$$A_n = \frac{2}{n\pi \sinh n\pi a/b}\int_0^b g(y)\cos\frac{n\pi}{b}y\,dy.$$

The solution is then

$$u(x,y) = A_0 + \sum_{n=1}^{\infty} A_n \cosh\frac{n\pi}{b}x \cos\frac{n\pi}{b}y,$$

where the $A_n$ are defined above and $A_0$ is arbitrary. In general, Neumann problems do not have unique solutions.

For a physical interpretation of the compatibility condition $\int_0^b g(y)dy = 0$ see the texts *Elementary Partial Differential Equations* by Paul Berg and James McGregor (Holden-Day) and *Partial Differential Equations of Mathematical Physics* by Tyn Myint-U (North Holland).

**20.** Separating variables in the boundary-value problem leads to

$$Y'' + \lambda Y = 0, \quad Y'(0) = 0,\ Y'(\pi) = 0$$
$$X'' - \lambda X = 0.$$

The boundary conditions yield

$$\lambda_0 = 0 \quad \text{so} \quad Y = c_1 \quad \text{and} \quad X = c_3 + c_4 x.$$

Also

$$\lambda_n = n^2,\ n > 0 \quad \text{so} \quad Y(y) = c_5 \cos ny \quad \text{and} \quad X(x) = c_7 \cosh nx + c_8 \sinh nx.$$

Thus product solutions are

$$u(x,y) = A_0 + B_0 x + \sum_{n=1}^{\infty} (A_n \cosh nx + B_n \sinh nx)\cos ny.$$

The last part of this problem involves matching coefficients.

At $x = 0$,

$$u_0 \cos y = A_0 + \sum_{n=1}^{\infty} A_n \cos ny \quad \text{implies} \quad A_0 = 0,\ A_1 = u_0,\ \text{and}\ A_n = 0,\ \text{for}\ n \ge 2.$$

Therefore

$$u(x,y) = B_0x + u_0 \cosh x \cos y + B_1 \sinh x \cos y + B_2 \sinh 2x \cos 2y + \sum_{n=3}^{\infty} B_n \sinh nx \cos ny.$$

At $x = 1$,

$$u_0(1 + \cos 2y) = B_0 + u_0 \cosh 1 \cos y + B_1 \sinh 1 \cos y + B_2 \sinh 2 \cos 2y + \sum_{n=3}^{\infty} B_n \sinh n \cos ny,$$

so

$$u_0 - u_0 \cosh 1 \cos y + u_0 \cos 2y = B_0 + B_1 \sinh 1 \cos y + B_2 \sinh 2 \cos 2y + \sum_{n=3}^{\infty} B_n \sinh n \cos ny.$$

Then

$$B_0 = u_0, \;\; B_1 = -u_0 \frac{\cosh 1}{\sinh 1}, \;\; B_2 = \frac{u_0}{\sinh 2}, \;\; \text{and } B_n = 0 \text{ for } n \geq 3.$$

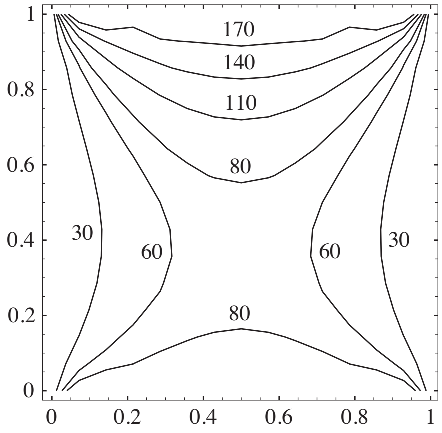
Therefore

$$\begin{aligned} u(x,y) &= u_0x + u_0 \cosh x \cos y - u_0 \frac{\cosh 1}{\sinh 1} \sinh x \cos y + \frac{u_0}{\sinh 2} \sinh 2x \cos 2y \\ &= u_0x + u_0 \left( \frac{\sinh 1 \cosh x - \cosh 1 \sinh x}{\sinh 1} \right) \cos y + \frac{u_0}{\sinh 2} \sinh 2x \cos 2y \end{aligned}$$

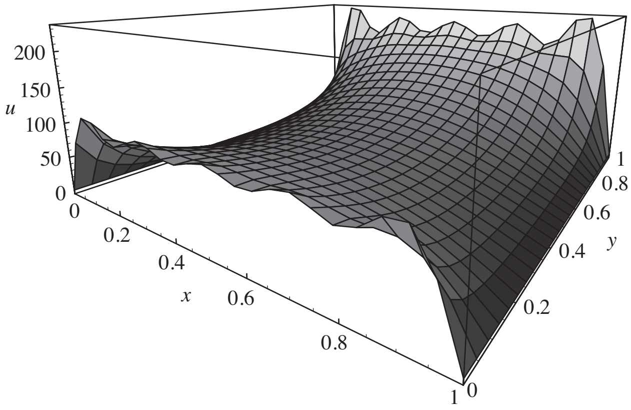
or

$$u(x,y) = u_0x + u_0 \frac{\sinh(1-x)}{\sinh 1} \cos y + \frac{u_0}{\sinh 2} \sinh 2x \cos 2y.$$

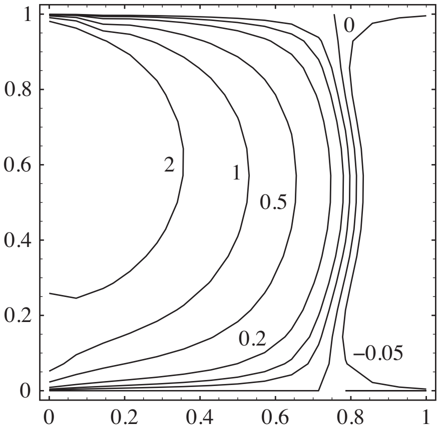
**21. (a)**



**(b)**



**22.**



## 12.6 Nonhomogeneous Boundary-Value Problems

**1.** Using $v(x,t) = u(x,t) - 100$ we wish to solve $kv_{xx} = v_t$ subject to $v(0,t) = 0$, $v(1,t) = 0$, and $v(x,0) = -100$. Let $v = XT$ and use $-\lambda$ as a separation constant so that

$$X'' + \lambda X = 0,$$

$$X(0) = 0,$$

$$X(1) = 0,$$

and

$$T' + \lambda kT = 0.$$

This leads to

$$X = c_2 \sin(n\pi x) \qquad \text{and} \qquad T = c_3 e^{-kn^2\pi^2 t}$$

for $n = 1, 2, 3, \ldots$ so that

$$v(x,t) = \sum_{n=1}^{\infty} A_n e^{-kn^2\pi^2 t} \sin n\pi x.$$

Imposing

$$v(x,0) = -100 = \sum_{n=1}^{\infty} A_n \sin n\pi x$$

gives

$$A_n = 2\int_0^1 (-100)\sin n\pi x\,dx = \frac{-200}{n\pi}[1 - (-1)^n]$$

so that

$$u(x,t) = v(x,t) + 100 = 100 + \frac{200}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n} e^{-kn^2\pi^2 t} \sin n\pi x.$$

**2.** Letting $u(x,t) = v(x,t) + \psi(x)$ and proceeding as in Example 1 in the text we find $\psi(x) = u_0 - u_0x$. Then $v(x,t) = u(x,t) + u_0x - u_0$ and we wish to solve $kv_{xx} = v_t$ subject to $v(0,t) = 0$, $v(1,t) = 0$, and $v(x,0) = f(x) + u_0x - u_0$. Let $v = XT$ and use $-\lambda$ as a separation constant so that

$$X'' + \lambda X = 0,$$

$$X(0) = 0,$$

$$X(1) = 0,$$

and

$$T' + \lambda kT = 0.$$

Then

$$X = c_2 \sin n\pi x \qquad \text{and} \qquad T = c_3 e^{-kn^2\pi^2t}$$

for $n = 1, 2, 3, \ldots$ so that

$$v = \sum_{n=1}^{\infty} A_n e^{-kn^2\pi^2t} \sin n\pi x.$$

Imposing

$$v(x,0) = f(x) + u_0x - u_0 = \sum_{n=1}^{\infty} A_n \sin n\pi x$$

gives

$$A_n = 2\int_0^1 (f(x) + u_0x - u_0) \sin n\pi x \, dx$$

so that

$$u(x,t) = v(x,t) + u_0 - u_0x = u_0 - u_0x + \sum_{n=1}^{\infty} A_n e^{-kn^2\pi^2t} \sin n\pi x.$$

**3.** If we let $u(x,t) = v(x,t) + \psi(x)$, then we obtain as in Example 1 in the text

$$k\psi'' + r = 0$$

or

$$\psi(x) = -\frac{r}{2k}x^2 + c_1x + c_2.$$

The boundary conditions become

$$u(0,t) = v(0,t) + \psi(0) = u_0$$

$$u(1,t) = v(1,t) + \psi(1) = u_0.$$

Letting $\psi(0) = \psi(1) = u_0$ we obtain homogeneous boundary conditions in $v$:

$$v(0,t) = 0 \qquad \text{and} \qquad v(1,t) = 0.$$

Now $\psi(0) = \psi(1) = u_0$ implies $c_2 = u_0$ and $c_1 = r/2k$. Thus

$$\psi(x) = -\frac{r}{2k}x^2 + \frac{r}{2k}x + u_0 = u_0 - \frac{r}{2k}x(x-1).$$

To determine $v(x,t)$ we solve

$$k\frac{\partial^2 v}{\partial x^2} = \frac{\partial v}{dt}, \quad 0 < x < 1,\ t > 0$$

$$v(0,t) = 0, \quad v(1,t) = 0,$$

$$v(x,0) = \frac{r}{2k}x(x-1) - u_0.$$

Separating variables, we find

$$v(x,t) = \sum_{n=1}^{\infty} A_n e^{-kn^2\pi^2 t} \sin n\pi x,$$

where

$$A_n = 2\int_0^1 \left[\frac{r}{2k}x(x-1) - u_0\right] \sin n\pi x\, dx = 2\left[\frac{u_0}{n\pi} + \frac{r}{kn^3\pi^3}\right][(-1)^n - 1]. \qquad (\mathbf{1})$$

Hence, a solution of the original problem is

$$u(x,t) = \psi(x) + v(x,t) = u_0 - \frac{r}{2k}x(x-1) + \sum_{n=1}^{\infty} A_n e^{-kn^2\pi^2 t} \sin n\pi x,$$

where $A_n$ is defined in (**1**).

**4.** If we let $u(x,t) = v(x,t) + \psi(x)$, then we obtain as in Example 1 in the text

$$k\psi'' + r = 0.$$

Integrating gives

$$\psi(x) = -\frac{r}{2k}x^2 + c_1 x + c_2.$$

The boundary conditions become

$$u(0,t) = v(0,t) + \psi(0) = u_0$$

$$u(1,t) = v(1,t) + \psi(1) = u_1.$$

Letting $\psi(0) = u_0$ and $\psi(1) = u_1$ we obtain homogeneous boundary conditions in $v$:

$$v(0,t) = 0 \qquad \text{and} \qquad v(1,t) = 0.$$

Now $\psi(0) = u_0$ and $\psi(1) = u_1$ imply $c_2 = u_0$ and $c_1 = u_1 - u_0 + r/2k$. Thus

$$\psi(x) = -\frac{r}{2k}x^2 + \left(u_1 - u_0 + \frac{r}{2k}\right)x + u_0.$$

To determine $v(x,t)$ we solve

$$k\frac{\partial^2 v}{\partial x^2} = \frac{\partial v}{dt}, \quad 0 < x < 1,\ t > 0$$

$$v(0,t) = 0, \quad v(1,t) = 0,$$

$$v(x,0) = f(x) - \psi(x).$$

Separating variables, we find

$$v(x,t) = \sum_{n=1}^{\infty} A_n e^{-kn^2\pi^2 t} \sin n\pi x,$$

where

$$A_n = 2\int_0^1 [f(x) - \psi(x)] \sin n\pi x \, dx. \qquad \textbf{(2)}$$

Hence, a solution of the original problem is

$$u(x,t) = \psi(x) + v(x,t) = -\frac{r}{2k}x^2 + \left(u_1 - u_0 + \frac{r}{2k}\right)x + u_0 + \sum_{n=1}^{\infty} A_n e^{-kn^2\pi^2 t} \sin n\pi x,$$

where $A_n$ is defined in (**2**).

**5.** Substituting $u(x,t) = v(x,t) + \psi(x)$ into the partial differential equation gives

$$k\,\frac{\partial^2 v}{\partial x^2} + k\psi'' + Ae^{-\beta x} = \frac{\partial v}{\partial t}\,.$$

This equation will be homogeneous provided $\psi$ satisfies

$$k\psi'' + Ae^{-\beta x} = 0.$$

The solution of this differential equation is obtained by successive integrations:

$$\psi(x) = -\frac{A}{\beta^2 k}e^{-\beta x} + c_1 x + c_2.$$

From $\psi(0) = 0$ and $\psi(1) = 0$ we find

$$c_1 = \frac{A}{\beta^2 k}(e^{-\beta} - 1) \qquad \text{and} \qquad c_2 = \frac{A}{\beta^2 k}\,.$$

Hence

$$\psi(x) = -\frac{A}{\beta^2 k}e^{-\beta x} + \frac{A}{\beta^2 k}(e^{-\beta} - 1)x + \frac{A}{\beta^2 k} = \frac{A}{\beta^2 k}\left[1 - e^{-\beta x} + (e^{-\beta} - 1)x\right].$$

Now the new problem is

$$k\frac{\partial^2 v}{\partial x^2} = \frac{\partial v}{\partial t} \quad, \; 0 < x < 1, \quad t > 0,$$

$$v(0,t) = 0, \quad v(1,t) = 0, \quad t > 0,$$

$$v(x,0) = f(x) - \psi(x), \quad 0 < x < 1.$$

Identifying this as the heat equation solved in Section 12.3 in the text with $L = 1$ we obtain

$$v(x,t) = \sum_{n=1}^{\infty} A_n e^{-kn^2\pi^2 t} \sin n\pi x$$

where

$$A_n = 2\int_0^1 [f(x) - \psi(x)] \sin n\pi x \, dx.$$

Thus

$$u(x,t) = \frac{A}{\beta^2 k}\left[1 - e^{-\beta x} + (e^{-\beta} - 1)x\right] + \sum_{n=1}^{\infty} A_n e^{-kn^2\pi^2 t} \sin n\pi x.$$

**6.** Substituting $u(x,t) = v(x,t) + \psi(x)$ into the partial differential equation gives

$$k\frac{\partial^2 v}{\partial x^2} + k\psi'' - hv - h\psi = \frac{\partial v}{\partial t}\,.$$

This equation will be homogeneous provided $\psi$ satisfies

$$k\psi'' - h\psi = 0.$$

Since $k$ and $h$ are positive, the general solution of this latter linear second-order equation is

$$\psi(x) = c_1 \cosh\sqrt{\frac{h}{k}}\,x + c_2 \sinh\sqrt{\frac{h}{k}}\,x.$$

From $\psi(0) = 0$ and $\psi(\pi) = u_0$ we find $c_1 = 0$ and $c_2 = u_0/\sinh\sqrt{h/k}\,\pi$. Hence

$$\psi(x) = u_0\frac{\sinh\sqrt{h/k}\,x}{\sinh\sqrt{h/k}\,\pi}\,.$$

Now the new problem is

$$k\frac{\partial^2 v}{\partial x^2} - hv = \frac{\partial v}{\partial t}\,, \quad 0 < x < \pi, \; t > 0$$

$$v(0,t) = 0, \quad v(\pi,t) = 0, \quad t > 0$$

$$v(x,0) = -\psi(x), \quad 0 < x < \pi.$$

If we let $v = XT$ then

$$\frac{X''}{X} = \frac{T' + hT}{kT} = -\lambda.$$

With $\lambda = \alpha^2 > 0$, the separated differential equations

$$X'' + \alpha^2 X = 0 \qquad \text{and} \qquad T' + \left(h + k\alpha^2\right) T = 0.$$

have the respective solutions

$$X(x) = c_3 \cos \alpha x + c_4 \sin \alpha x$$

$$T(t) = c_5 e^{-\left(h + k\alpha^2\right)t}.$$

From $X(0) = 0$ we get $c_3 = 0$ and from $X(\pi) = 0$ we find $\alpha = n$ for $n = 1, 2, 3, \ldots$ . Consequently, it follows that

$$v(x,t) = \sum_{n=1}^{\infty} A_n e^{-\left(h+kn^2\right)t} \sin nx$$

where

$$A_n = -\frac{2}{\pi}\int_0^{\pi} \psi(x) \sin nx \, dx.$$

Hence a solution of the original problem is

$$u(x,t) = u_0 \frac{\sinh \sqrt{h/k}\, x}{\sinh \sqrt{h/k}\, \pi} + e^{-ht} \sum_{n=1}^{\infty} A_n e^{-kn^2 t} \sin nx$$

where

$$A_n = -\frac{2}{\pi}\int_0^{\pi} u_0 \frac{\sinh \sqrt{h/k}\, x}{\sinh \sqrt{h/k}\, \pi} \sin nx \, dx.$$

Using the exponential definition of the hyperbolic sine and integration by parts we find

$$A_n = \frac{2u_0 nk(-1)^n}{\pi\left(h + kn^2\right)}.$$

**7.** Substituting $u(x,t) = v(x,t) + \psi(x)$ into the partial differential equation gives

$$k\frac{\partial^2 v}{\partial x^2} + k\psi'' - hv - h\psi + hu_0 = \frac{\partial v}{\partial t}.$$

This equation will be homogeneous provided $\psi$ satisfies

$$k\psi'' - h\psi + hu_0 = 0 \qquad \text{or} \qquad k\psi'' - h\psi = -hu_0.$$

This non-homogeneous, linear, second-order, differential equation has solution

$$\psi(x) = c_1 \cosh \sqrt{\frac{h}{k}}\, x + c_2 \sinh \sqrt{\frac{h}{k}}\, x + u_0,$$

where we assume $h > 0$ and $k > 0$. From $\psi(0) = u_0$ and $\psi(1) = 0$ we find $c_1 = 0$ and $c_2 = -u_0/\sinh\sqrt{h/k}$. Thus, the steady-state solution is

$$\psi(x) = -\frac{u_0}{\sinh\sqrt{\frac{h}{k}}}\sinh\sqrt{\frac{h}{k}}\,x + u_0 = u_0\left(1 - \frac{\sinh\sqrt{\frac{h}{k}}\,x}{\sinh\sqrt{\frac{h}{k}}}\right).$$

**8.** The partial differential equation is

$$k\frac{\partial^2 u}{\partial x^2} - hu = \frac{\partial u}{\partial t}\,.$$

Substituting $u(x,t) = v(x,t) + \psi(x)$ gives

$$k\frac{\partial^2 v}{\partial x^2} + k\psi'' - hv - h\psi = \frac{\partial v}{\partial t}\,.$$

This equation will be homogeneous provided $\psi$ satisfies

$$k\psi'' - h\psi = 0.$$

Assuming $h > 0$ and $k > 0$, we have

$$\psi = c_1 e^{\sqrt{h/k}\,x} + c_2 e^{-\sqrt{h/k}\,x},$$

where we have used the exponential form of the solution since the rod is infinite. Now, in order that the steady-state temperature $\psi(x)$ be bounded as $x \to \infty$, we require $c_1 = 0$. Then

$$\psi(x) = c_2 e^{-\sqrt{h/k}\,x}$$

and $\psi(0) = u_0$ implies $c_2 = u_0$. Thus

$$\psi(x) = u_0 e^{-\sqrt{h/k}\,x}.$$

**9.** Substituting $u(x,t) = v(x,t) + \psi(x)$ into the partial differential equation gives

$$a^2\frac{\partial^2 v}{\partial x^2} + a^2\psi'' + Ax = \frac{\partial^2 v}{\partial t^2}\,.$$

This equation will be homogeneous provided $\psi$ satisfies

$$a^2\psi'' + Ax = 0.$$

The solution of this differential equation is

$$\psi(x) = -\frac{A}{6a^2}x^3 + c_1 x + c_2.$$

From $\psi(0) = 0$ we obtain $c_2 = 0$, and from $\psi(1) = 0$ we obtain $c_1 = A/6a^2$. Hence

$$\psi(x) = \frac{A}{6a^2}(x - x^3).$$

Now the new problem is

$$a^2\frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 v}{\partial t^2}$$

$$v(0,t) = 0, \quad v(1,t) = 0, \quad t > 0,$$

$$v(x,0) = -\psi(x), \quad v_t(x,0) = 0, \quad 0 < x < 1.$$

Identifying this as the wave equation solved in Section 12.4 in the text with $L = 1$, $f(x) = -\psi(x)$, and $g(x) = 0$ we obtain

$$v(x,t) = \sum_{n=1}^{\infty} A_n \cos n\pi at \sin n\pi x$$

where

$$A_n = 2\int_0^1 [-\psi(x)] \sin n\pi x\, dx = \frac{A}{3a^2}\int_0^1 (x^3 - x) \sin n\pi x\, dx = \frac{2A(-1)^n}{a^2\pi^3 n^3}\,.$$

Thus

$$u(x,t) = \frac{A}{6a^2}(x - x^3) + \frac{2A}{a^2\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\cos n\pi at \sin n\pi x.$$

**10.** We solve

$$a^2\frac{\partial^2 u}{\partial x^2} - g = \frac{\partial^2 u}{\partial t^2}\,, \quad 0 < x < 1,\ t > 0$$

$$u(0,t) = 0, \quad u(1,t) = 0, \quad t > 0$$

$$u(x,0) = 0, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0, \quad 0 < x < 1.$$

The partial differential equation is nonhomogeneous. The substitution $u(x,t) = v(x,t) + \psi(x)$ yields a homogeneous partial differential equation provided $\psi$ satisfies

$$a^2\psi'' - g = 0.$$

By integrating twice we find

$$\psi(x) = \frac{g}{2a^2}x^2 + c_1 x + c_2.$$

The imposed conditions $\psi(0) = 0$ and $\psi(1) = 0$ then lead to $c_2 = 0$ and $c_1 = -g/2a^2$. Hence

$$\psi(x) = \frac{g}{2a^2}\left(x^2 - x\right).$$

The new problem is now

$$a^2\frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 v}{\partial t^2}\,, \quad 0 < x < 1,\ t > 0$$

$$v(0,t) = 0, \quad v(1,t) = 0$$

$$v(x,0) = \frac{g}{2a^2}\left(x - x^2\right), \quad \left.\frac{\partial v}{\partial t}\right|_{t=0} = 0.$$

Substituting $v = XT$ we find in the usual manner

$$X'' + \alpha^2 X = 0$$

$$T'' + a^2\alpha^2 T = 0$$

with solutions

$$X(x) = c_3\cos\alpha x + c_4\sin\alpha x$$

$$T(t) = c_5\cos a\alpha t + c_6\sin a\alpha t.$$

The conditions $X(0) = 0$ and $X(1) = 0$ imply in turn that $c_3 = 0$ and $\alpha = n\pi$ for $n = 1, 2, 3, \ldots$ . The condition $T'(0) = 0$ implies $c_6 = 0$. Hence, by the superposition principle

$$v(x,t) = \sum_{n=1}^{\infty} A_n\cos(an\pi t)\sin(n\pi x).$$

At $t = 0$,

$$\frac{g}{2a^2}\left(x - x^2\right) = \sum_{n=1}^{\infty} A_n\sin(n\pi x)$$

and so

$$A_n = \frac{g}{a^2}\int_0^1\left(x - x^2\right)\sin(n\pi x)\,dx = \frac{2g}{a^2n^3\pi^3}\left[1 - (-1)^n\right].$$

Thus the solution to the original problem is

$$u(x,t) = \psi(x) + v(x,t) = \frac{g}{2a^2}\left(x^2 - x\right) + \frac{2g}{a^2\pi^3}\sum_{n=1}^{\infty}\frac{1 - (-1)^n}{n^3}\cos(an\pi t)\sin(n\pi x).$$

**11.** Substituting $u(x,y) = v(x,y) + \psi(y)$ into Laplace's equation we obtain

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \psi''(y) = 0.$$

This equation will be homogeneous provided $\psi$ satisfies $\psi(y) = c_1y + c_2$. Considering

$$u(x,0) = v(x,0) + \psi(0) = u_1$$

$$u(x,1) = v(x,1) + \psi(1) = u_0$$

$$u(0,y) = v(0,y) + \psi(y) = 0$$

we require that $\psi(0) = u_1$, $\psi_1 = u_0$ and $v(0, y) = -\psi(y)$. Then $c_1 = u_0 - u_1$ and $c_2 = u_1$. The new boundary-value problem is

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

$$v(x, 0) = 0, \quad v(x, 1) = 0,$$

$$v(0, y) = -\psi(y), \quad 0 < y < 1,$$

where $v(x, y)$ is bounded at $x \to \infty$. This problem is similar to Problem 11 in Section 12.5. The solution is

$$\begin{aligned} v(x, y) &= \sum_{n=1}^{\infty} \left( 2 \int_0^1 [-\psi(y) \sin n\pi y] \, dy \right) e^{-n\pi x} \sin n\pi y \\ &= 2 \sum_{n=1}^{\infty} \left[ (u_1 - u_0) \int_0^1 y \sin n\pi y \, dy - u_1 \int_0^1 \sin n\pi y \, dy \right] e^{-n\pi x} \sin n\pi y \\ &= \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{u_0(-1)^n - u_1}{n} e^{-n\pi x} \sin n\pi y. \end{aligned}$$

Thus

$$u(x, y) = v(x, y) + \psi(y) = (u_0 - u_1)y + u_1 + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{u_0(-1)^n - u_1}{n} e^{-n\pi x} \sin n\pi y.$$

**12.** Substituting $u(x, y) = v(x, y) + \psi(x)$ into Poisson's equation we obtain

$$\frac{\partial^2 v}{\partial x^2} + \psi''(x) + h + \frac{\partial^2 v}{\partial y^2} = 0.$$

The equation will be homogeneous provided $\psi$ satisfies $\psi''(x) + h = 0$ or $\psi(x) = -\frac{h}{2}x^2 + c_1 x + c_2$. From $\psi(0) = 0$ we obtain $c_2 = 0$. From $\psi(\pi) = 1$ we obtain

$$c_1 = \frac{1}{\pi} + \frac{h\pi}{2}.$$

Then

$$\psi(x) = \left( \frac{1}{\pi} + \frac{h\pi}{2} \right) x - \frac{h}{2} x^2.$$

The new boundary-value problem is

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

$$v(0, y) = 0, \quad v(\pi, y) = 0,$$

$$v(x, 0) = -\psi(x), \quad 0 < x < \pi.$$

This is Problem 11 in Section 12.5. The solution is

$$v(x,y)=\sum_{n=1}^{\infty}A_ne^{-ny}\sin nx$$

where

$$A_n=\frac{2}{\pi}\int_0^{\pi}[-\psi(x)\sin nx]\,dx=\frac{2(-1)^n}{n}\left(\frac{1}{\pi}+\frac{h\pi}{2}\right)-h(-1)^n\left(\frac{\pi}{n}+\frac{2}{n^2}\right).$$

Thus

$$u(x,y)=v(x,y)+\psi(x)=\left(\frac{1}{\pi}+\frac{h\pi}{2}\right)x-\frac{h}{2}x^2+\sum_{n=1}^{\infty}A_ne^{-ny}\sin nx.$$

**13.** From (13) and (14) we have

$$\psi(x,t)=\sin t+x[0-\sin t]=(1-x)\sin t,$$

Then the substitution

$$u(x,t)=v(x,t)+\psi(x,t)=v(x,t)+(1-x)\sin t$$

leads to the boundary-value problem

$$\frac{\partial^2 v}{\partial x^2}+(x-1)\cos t=\frac{\partial v}{\partial t},\quad 0<x<1,\quad t>0$$

$$v(0,t)=0,\quad v(1,t)=0,\quad t>0$$

$$v(x,0)=0,\quad 0<x<1.$$

The eigenvalues and eigenfunctions of the Sturm-Liouville problem

$$X''+\lambda X=0,\quad X(0)=0,\quad X(1)=0$$

are $\lambda_n=\alpha_n^2=n^2\pi^2$ and $\sin n\pi x$, $n=1, 2, 3, \ldots$ . With $G(x,t)=(x-1)\cos t$ we assume for fixed $t$ that $v$ and $G$ can be written as Fourier sine series:

$$v(x,t)=\sum_{n=1}^{\infty}v_n(t)\sin n\pi x \qquad\text{and}\qquad G(x,t)=\sum_{n=1}^{\infty}G_n(t)\sin n\pi x.$$

By treating $t$ as a parameter, the coefficients $G_n$ can be computed:

$$G_n(t)=\frac{2}{1}\int_0^1(x-1)\cos t\,\sin n\pi x\,dx=2\cos t\int_0^1(x-1)\sin n\pi x\,dx=-\frac{2}{n\pi}\cos t.$$

Hence

$$(x-1)\cos t=\sum_{n=1}^{\infty}\frac{-2\cos t}{n\pi}\sin n\pi x.$$

Now, using the series representation for $v(x,t)$, we have

$$\frac{\partial^2 v}{\partial x^2}=\sum_{n=1}^{\infty}v_n(t)(-n^2\pi^2)\sin n\pi x \quad\text{and}\quad \frac{\partial v}{\partial t}=\sum_{n=1}^{\infty}v_n'(t)\sin n\pi x.$$

Writing the partial differential equation as $v_t-v_{xx}=(x-1)\cos t$ and using the above results we have

$$\sum_{n=1}^{\infty}[v_n'(t)+n^2\pi^2v_n(t)]\sin n\pi x=\sum_{n=1}^{\infty}\frac{-2\cos t}{n\pi}\sin n\pi x.$$

Equating coefficients we get

$$v_n'(t)+n^2\pi^2v_n(t)=-\frac{2\cos t}{n\pi}.$$

For each $n$ this is a linear first-order differential equation whose general solution is

$$v_n(t)=-\frac{2}{n\pi}\left[\frac{n^2\pi^2\cos t+\sin t}{n^4\pi^4+1}\right]+C_ne^{-n^2\pi^2t}.$$

Thus

$$v(x,t)=\sum_{n=1}^{\infty}\left[-\frac{2n^2\pi^2\cos t+2\sin t}{n\pi(n^4\pi^4+1)}+C_ne^{-n^2\pi^2t}\right]\sin n\pi x.$$

The initial condition $v(x,0)=0$ implies

$$\sum_{n=1}^{\infty}\left[-\frac{2n\pi}{n^4\pi^4+1}+C_n\right]\sin n\pi x=0$$

so that $C_n=2n\pi/(n^4\pi^4+1)$. Therefore

$$\begin{aligned}v(x,t)&=\sum_{n=1}^{\infty}\left[-\frac{2n^2\pi^2\cos t+2\sin t}{n\pi(n^4\pi^4+1)}+\frac{2n\pi}{n^4\pi^4+1}e^{-n^2\pi^2t}\right]\sin n\pi x\\&=\frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{n^2\pi^2e^{-n^2\pi^2t}-n^2\pi^2\cos t-\sin t}{n(n^4\pi^4+1)}\right]\sin n\pi x\end{aligned}$$

and

$$u(x,t)=v(x,t)+\psi(x,t)=(1-x)\sin t+\frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{n^2\pi^2e^{-n^2\pi^2t}-n^2\pi^2\cos t-\sin t}{n(n^4\pi^4+1)}\right]\sin n\pi x.$$

**14.** From (13) and (14) we have

$$\psi(x,t)=u_0(t)+\frac{X}{L}[u_1(t)-u_0(t)]=x+(1-x)t^2,$$

Then the substitution

$$u(x,t)=v(x,t)+\psi(x,t)=v(x,t)+x+(1-x)t^2$$

leads to the boundary-value problem

$$\frac{\partial^2 v}{\partial x^2} + 5tx = \frac{\partial v}{\partial t}, \quad 0 < x < 1, \quad t > 0$$

$$v(0,t) = 0, \quad v(1,t) = 0, \quad t > 0$$

$$v(x,0) = x^2 - x, \quad 0 < x < 1.$$

The eigenvalues and eigenfunctions of the Sturm-Liouville problem

$$X'' + \lambda X = 0, \quad X(0) = 0, \quad X(1) = 0$$

are $\lambda_n = \alpha_n^2 = n^2\pi^2$ and $\sin n\pi x$, $n = 1, 2, 3, \ldots$ . With $G(x,t) = 5tx$ we assume for fixed $t$ that $v$ and $G$ can be written as Fourier sine series:

$$v(x,t) = \sum_{n=1}^{\infty} v_n(t) \sin n\pi x \qquad \text{and} \qquad G(x,t) = \sum_{n=1}^{\infty} G_n(t) \sin n\pi x.$$

By treating $t$ as a parameter, the coefficients $G_n$ can be computed:

$$G_n(t) = \frac{2}{1} \int_0^1 5tx \, \sin n\pi x \, dx = 10t \int_0^1 x \, \sin n\pi x \, dx = \frac{10t}{n\pi} (-1)^{n+1}.$$

Hence

$$5tx = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{10t}{n\pi} \, \sin n\pi x.$$

Now, using the series representation for $v(x,t)$, we have

$$\frac{\partial^2 v}{\partial x^2} = \sum_{n=1}^{\infty} v_n(t)(-n^2\pi^2) \sin n\pi x \quad \text{and} \quad \frac{\partial v}{\partial t} = \sum_{n=1}^{\infty} v_n'(t) \sin n\pi x.$$

Writing the partial differential equation as $v_t - v_{xx} = 5tx$ and using the above results we have

$$\sum_{n=1}^{\infty} [v_n'(t) + n^2\pi^2 v_n(t)] \sin n\pi x = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{10t}{n\pi} \, \sin n\pi x.$$

Equating coefficients we get

$$v_n'(t) + n^2\pi^2 v_n(t) = (-1)^{n+1} \frac{10t}{n\pi}.$$

For each $n$ this is a linear first-order differential equation whose general solution is

$$v_n(t) = 10(-1)^{n+1} \frac{n^2\pi^2 t - 1}{n^5\pi^5} + C_n e^{-n^2\pi^2 t}.$$

Thus

$$v(x,t) = \sum_{n=1}^{\infty} \left[ 10(-1)^{n+1} \frac{n^2\pi^2 t - 1}{n^5\pi^5} + C_n e^{-n^2\pi^2 t} \right] \sin n\pi x.$$

The initial condition $v(x,0)=x^2-x$ implies

$$\sum_{n=1}^{\infty}\left[10(-1)^{n+1}\frac{-1}{n^5\pi^5}+C_n\right]\sin n\pi x=x^2-x.$$

Thinking of $x^2-x$ as a Fourier sine series with coefficients

$$2\int_0^1 (x^2-x)\sin n\pi x\,dx=[4(-1)^n-4]/n^3\pi^3$$

we equate coefficients to obtain

$$\frac{10(-1)^n}{n^5\pi^5}+C_n=\frac{4(-1)^n-4}{n^3\pi^3}$$

so

$$C_n=\frac{4(-1)^n-4}{n^3\pi^3}-\frac{10(-1)^n}{n^5\pi^5}.$$

Therefore

$$v(x,t)=\sum_{n=1}^{\infty}\left[10(-1)^{n+1}\frac{n^2\pi^2t-1}{n^5\pi^5}+\left(\frac{4(-1)^n-4}{n^3\pi^3}-\frac{10(-1)^n}{n^5\pi^5}\right)e^{-n^2\pi^2t}\right]\sin n\pi x$$

and

$$\begin{aligned}u(x,t)&=v(x,t)+\psi(x,t)\\&=x+(1-x)t^2+\sum_{n=1}^{\infty}\left[10(-1)^{n+1}\frac{n^2\pi^2t-1}{n^5\pi^5}+\left(\frac{4(-1)^n-4}{n^3\pi^3}-\frac{10(-1)^n}{n^5\pi^5}\right)e^{-n^2\pi^2t}\right]\sin n\pi x.\end{aligned}$$

**15.** From (13) and (14) we have

$$\psi(x,t)=u_0(t)+\frac{X}{L}\,[u_1(t)-u_0(t)]=x\sin t,$$

Then the substitution

$$u(x,t)=v(x,t)+\psi(x,t)=x\sin t$$

leads to the boundary-value problem

$$\frac{\partial^2 v}{\partial x^2}+x\sin t=\frac{\partial^2 v}{\partial t^2},\quad 0<x<1,\quad t>0$$

$$v(0,t)=0,\quad v(1,t)=0,\quad t>0$$

$$v(x,0)=0,\quad \left.\frac{\partial v}{\partial t}\right|_{t=0}=-x,\quad 0<x<1.$$

The eigenvalues and eigenfunctions of the Sturm-Liouville problem

$$X''+\lambda X=0,\quad X(0)=0,\quad X(1)=0$$

are $\lambda_n = \alpha_n^2 = n^2\pi^2$ and $\sin n\pi x$, $n = 1, 2, 3, \ldots$ . With $G(x,t) = x \sin t$ we assume for fixed $t$ that $v$ and $G$ can be written as Fourier sine series:

$$v(x,t) = \sum_{n=1}^{\infty} v_n(t) \sin n\pi x \qquad \text{and} \qquad G(x,t) = \sum_{n=1}^{\infty} G_n(t) \sin n\pi x.$$

By treating $t$ as a parameter, the coefficients $G_n$ can be computed:

$$G_n(t) = \frac{2}{1} \int_0^1 x \sin t \, \sin n\pi x \, dx = \frac{2}{1} \sin t \int_0^1 x \, \sin n\pi x \, dx = -2 \frac{(-1)^n \sin t}{n\pi}.$$

Hence

$$x \sin t = \sum_{n=1}^{\infty} -2 \frac{(-1)^n \sin t}{n\pi} \, \sin n\pi x.$$

Now, using the series representation for $v(x,t)$, we have

$$\frac{\partial^2 v}{\partial x^2} = \sum_{n=1}^{\infty} v_n(t)(-n^2\pi^2) \sin n\pi x \quad \text{and} \quad \frac{\partial v}{\partial t} = \sum_{n=1}^{\infty} v_n'(t) \sin n\pi x.$$

Writing the partial differential equation as $v_{tt} - v_{xx} = x \sin t$ and using the above results we have

$$\sum_{n=1}^{\infty} \left[v_n''(t) + n^2\pi^2 v_n(t)\right] \sin n\pi x = \sum_{n=1}^{\infty} -2 \frac{(-1)^n \sin t}{n\pi} \, \sin n\pi x.$$

Equating coefficients we get

$$v_n''(t) + n^2\pi^2 v_n(t) = -2 \frac{(-1)^n \sin t}{n\pi}.$$

For each $n$ the general solution is

$$v_n(t) = A_n \cos n\pi t + B_n \sin n\pi t - 2 \frac{(-1)^n}{n\pi \, (n^2\pi^2 - 1)} \sin t$$

Thus

$$v(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos n\pi t + B_n \sin n\pi t - 2 \frac{(-1)^n}{n\pi \, (n^2\pi^2 - 1)} \sin t\right] \sin n\pi x.$$

The initial condition $v(x,0) = 0$ implies

$$\sum_{n=1}^{\infty} A_n \sin n\pi x = 0.$$

or $A_n = 0$ for $n = 1, 2, 3, \ldots$ . So

$$v(x,t) = \sum_{n=1}^{\infty} \left[B_n \sin n\pi t - 2 \frac{(-1)^n}{n\pi \, (n^2\pi^2 - 1)} \sin t\right] \sin n\pi x.$$

and

$$\left.\frac{\partial v}{\partial t}\right|_{t=0} = -x = \sum_{n=1}^{\infty}\left[n\pi B_n - 2\frac{(-1)^n}{n\pi\,(n^2\pi^2-1)}\right]\sin n\pi x.$$

Thinking of $-x$ as a Fourier sine series with coefficients

$$-2\int_0^1 x\sin n\pi x\,dx = 2\frac{(-1)^n}{n\pi}$$

we equate coefficients to obtain

$$n\pi B_n - 2\frac{(-1)^n}{n\pi\,(n^2\pi^2-1)} = 2\frac{(-1)^n}{n\pi}$$

so

$$B_n = 2\frac{(-1)^n}{n\pi} + 2\frac{(-1)^n}{n\pi\,(n^2\pi^2-1)}.$$

Therefore

$$v(x,t) = \sum_{n=1}^{\infty}\left[\left(2\frac{(-1)^n}{n\pi} + 2\frac{(-1)^n}{n\pi\,(n^2\pi^2-1)}\right)\sin n\pi t - 2\frac{(-1)^n}{n\pi\,(n^2\pi^2-1)}\sin t\right]\sin n\pi x$$

and

$$\begin{aligned} u(x,t) &= v(x,t) + \psi(x,t) \\ &= x\sin t + 2\sum_{n=1}^{\infty}\left[\left(\frac{(-1)^n}{n\pi} + \frac{(-1)^n}{n\pi\,(n^2\pi^2-1)}\right)\sin n\pi t - \frac{(-1)^n}{n\pi\,(n^2\pi^2-1)}\sin t\right]\sin n\pi x. \end{aligned}$$

**16.** From (13) and (14) we have

$$\psi(x,t) = u_0(t) + \frac{X}{L}\left[u_1(t) - u_0(t)\right] = 1 - e^{-t},$$

Then the substitution

$$u(x,t) = v(x,t) + \psi(x,t) = v(x,t) + 1 - e^{-t}$$

leads to the boundary-value problem

$$\frac{\partial^2 v}{\partial x^2} - e^{-t} = \frac{\partial v}{\partial t}, \quad 0 < x < 1, \quad t > 0$$

$$v(0,t) = 0, \quad v(1,t) = 0, \quad t > 0$$

$$v(x,0) = 0, \quad 0 < x < 1.$$

The eigenvalues and eigenfunctions of the Sturm-Liouville problem

$$X'' + \lambda X = 0, \quad X(0) = 0, \quad X(1) = 0$$

are $\lambda_n = \alpha_n^2 = n^2\pi^2$ and $\sin n\pi x$, $n = 1, 2, 3, \ldots$ . With $G(x,t) = -e^{-t}$ we assume for fixed $t$ that $v$ and $G$ can be written as Fourier sine series:

$$v(x,t) = \sum_{n=1}^{\infty} v_n(t) \sin n\pi x \qquad \text{and} \qquad G(x,t) = \sum_{n=1}^{\infty} G_n(t) \sin n\pi x.$$

By treating $t$ as a parameter, the coefficients $G_n$ can be computed:

$$G_n(t) = \frac{2}{1} e^{-t} \int_0^1 \sin n\pi x \, dx = 2e^{-t} \frac{(-1)^n - 1}{n\pi}.$$

Hence

$$-e^{-t} = 2e^{-t} \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n\pi} \sin n\pi x.$$

Now, using the series representation for $v(x,t)$, we have

$$\frac{\partial^2 v}{\partial x^2} = \sum_{n=1}^{\infty} v_n(t)(-n^2\pi^2) \sin n\pi x \quad \text{and} \quad \frac{\partial v}{\partial t} = \sum_{n=1}^{\infty} v_n'(t) \sin n\pi x.$$

Writing the partial differential equation as $v_t - v_{xx} = -e^{-t}$ and using the above results we have

$$\sum_{n=1}^{\infty} [v_n'(t) + n^2\pi^2 v_n(t)] \sin n\pi x = 2e^{-t} \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n\pi} \sin n\pi x.$$

Equating coefficients we get

$$v_n'(t) + n^2\pi^2 v_n(t) = 2 \frac{(-1)^n - 1}{n\pi} e^{-t}.$$

For each $n$ this is a linear first-order differential equation whose general solution is

$$v_n(t) = 2 \frac{(-1)^n - 1}{n\pi\,(n^2\pi^2 - 1)} e^{-t} + C_n e^{-n^2\pi^2 t}.$$

Thus

$$v(x,t) = \sum_{n=1}^{\infty} \left[ 2 \frac{(-1)^n - 1}{n\pi\,(n^2\pi^2 - 1)} e^{-t} + C_n e^{-n^2\pi^2 t} \right] \sin n\pi x.$$

The initial condition $v(x,0) = 0$ implies

$$C_n = -2 \frac{(-1)^n - 1}{n\pi\,(n^2\pi^2 - 1)}.$$

Therefore

$$v(x,t) = 2 \sum_{n=1}^{\infty} \left[ \frac{(-1)^n - 1}{n\pi\,(n^2\pi^2 - 1)} \left( e^{-t} - e^{-n^2\pi^2 t} \right) \right] \sin n\pi x$$

and

$$u(x,t) = v(x,t) + \psi(x,t) = 1 - e^{-t} + \frac{2}{\pi} \sum_{n=1}^{\infty} \left[ \frac{(-1)^n - 1}{n\,(n^2\pi^2 - 1)} \left( e^{-t} - e^{-n^2\pi^2 t} \right) \right] \sin n\pi x.$$

**17.** Identifying $k = 1$ and $L = \pi$ we see that the eigenfunctions of $X'' + \lambda X = 0$, $X(0) = 0$, $X(\pi) = 0$ are $\sin nx$, $n = 1, 2, 3, \ldots$ . Assuming that $u(x,t) = \sum_{n=1}^{\infty} u_n(t)\sin nx$, the formal partial derivatives of $u$ are

$$\frac{\partial^2 u}{\partial x^2} = \sum_{n=1}^{\infty} u_n(t)(-n^2)\sin nx \qquad \text{and} \qquad \frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} u_n'(t)\sin nx.$$

Assuming that $xe^{-3t} = \sum_{n=1}^{\infty} F_n(t)\sin nx$ we have

$$F_n(t) = \frac{2}{\pi}\int_0^{\pi} xe^{-3t}\sin nx\,dx = \frac{2e^{-3t}}{\pi}\int_0^{\pi} x\sin nx\,dx = \frac{2e^{-3t}(-1)^{n+1}}{n}.$$

Then

$$xe^{-3t} = \sum_{n=1}^{\infty} \frac{2e^{-3t}(-1)^{n+1}}{n}\sin nx$$

and

$$u_t - u_{xx} = \sum_{n=1}^{\infty}[u_n'(t) + n^2u_n(t)]\sin nx = xe^{-3t} = \sum_{n=1}^{\infty}\frac{2e^{-3t}(-1)^{n+1}}{n}\sin nx.$$

Equating coefficients we obtain

$$u_n'(t) + n^2u_n(t) = \frac{2e^{-3t}(-1)^{n+1}}{n}.$$

This is a linear first-order differential equation whose solution is

$$u_n(t) = \frac{2(-1)^{n+1}}{n(n^2-3)}e^{-3t} + C_ne^{-n^2t}.$$

Thus

$$u(x,t) = \sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n(n^2-3)}e^{-3t}\sin nx + \sum_{n=1}^{\infty} C_ne^{-n^2t}\sin nx$$

and $u(x,0) = 0$ implies

$$\sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n(n^2-3)}\sin nx + \sum_{n=1}^{\infty} C_n\sin nx = 0$$

so that $C_n = 2(-1)^n/n(n^2-3)$. Therefore

$$u(x,t) = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n(n^2-3)}e^{-3t}\sin nx + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n(n^2-3)}e^{-n^2t}\sin nx.$$

**18.** Identifying $k = 1$ and $L = \pi$ we see that the eigenfunctions of $X'' + \lambda X = 0$, $X(0) = 0$, $X'(\pi) = 0$ are 1, $\cos nx$, $n = 1, 2, 3, \ldots$ . Assuming that $u(x,t) = \frac{1}{2}u_0(t) + \sum_{n=1}^{\infty} u_n(t)\cos nx$, the formal partial derivatives of $u$ are

$$\frac{\partial^2 u}{\partial x^2} = \sum_{n=1}^{\infty} u_n(t)(-n^2)\cos nx \qquad \text{and} \qquad \frac{\partial u}{\partial t} = \frac{1}{2}u_0' + \sum_{n=1}^{\infty} u_n'(t)\cos nx.$$

Assuming that $xe^{-3t} = \frac{1}{2}F_0(t) + \sum_{n=1}^{\infty} F_n(t)\cos nx$ we have

$$F_0(t) = \frac{2e^{-3t}}{\pi}\int_0^{\pi} x\,dx = \pi e^{-3t}$$

and

$$F_n(t) = \frac{2e^{-3t}}{\pi}\int_0^{\pi} x\cos nx\,dx = \frac{2e^{-3t}[(-1)^n - 1]}{\pi n^2}.$$

Then

$$xe^{-3t} = \frac{\pi}{2}e^{-3t} + \sum_{n=1}^{\infty} \frac{2e^{-3t}[(-1)^n - 1]}{\pi n^2}\cos nx$$

and

$$\begin{aligned} u_t - u_{xx} &= \frac{1}{2}u_0'(t) + \sum_{n=1}^{\infty}[u_n'(t) + n^2 u_n(t)]\cos nx \\ &= xe^{-3t} = \frac{\pi}{2}e^{-3t} + \sum_{n=1}^{\infty} \frac{2e^{-3t}[(-1)^n - 1]}{\pi n^2}\cos nx. \end{aligned}$$

Equating coefficients, we obtain

$$u_0'(t) = \pi e^{-3t} \qquad \text{and} \qquad u_n'(t) + n^2 u_n(t) = \frac{2e^{-3t}[(-1)^n - 1]}{\pi n^2}\cos nx.$$

The first equation yields $u_0(t) = -(\pi/3)e^{-3t} + C_0$ and the second equation, which is a linear first-order differential equation, yields

$$u_n(t) = \frac{2[(-1)^n - 1]}{\pi n^2(n^2 - 3)}\,e^{-3t} + C_n e^{-n^2 t}.$$

Thus

$$u(x,t) = -\frac{\pi}{3}e^{-3t} + C_0 + \sum_{n=1}^{\infty} \frac{2[(-1)^n - 1]}{\pi n^2(n^2 - 3)}\,e^{-3t}\cos nx + \sum_{n=1}^{\infty} C_n e^{-n^2 t}\cos nx$$

and $u(x,0) = 0$ implies

$$-\frac{\pi}{3} + C_0 + \sum_{n=1}^{\infty} \frac{2[(-1)^n - 1]}{\pi n^2(n^2 - 3)}\cos nx + \sum_{n=1}^{\infty} C_n\cos nx = 0$$

so that $C_0 = \pi/3$ and $C_n = 2[(-1)^n - 1]/\pi n^2(n^2-3)$. Therefore

$$u(x,t) = \frac{\pi}{3}(1-e^{-3t}) + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n-1}{n^2(n^2-3)}\,e^{-3t}\cos nx + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^2(n^2-3)}\,e^{-n^2t}\cos nx.$$

**19.** Identifying $k=1$ and $L=1$ we see that the eigenfunctions of $X''+\lambda X=0$, $X(0)=0$, $X(1)=0$ are $\sin n\pi x$, $n=1, 2, 3, \ldots$. Assuming that $u(x,t)=\sum_{n=1}^{\infty}u_n(t)\sin n\pi x$, the formal partial derivatives of $u$ are

$$\frac{\partial^2 u}{\partial x^2} = \sum_{n=1}^{\infty}u_n(t)(-n^2\pi^2)\sin n\pi x \qquad \text{and} \qquad \frac{\partial u}{\partial t} = \sum_{n=1}^{\infty}u_n'(t)\sin n\pi x.$$

Assuming that $-1+x-x\cos t = \sum_{n=1}^{\infty}F_n(t)\sin n\pi x$ we have

$$F_n(t) = \frac{2}{1}\int_0^1(-1+x-x\cos t)\sin n\pi x\,dx = \frac{2[-1+(-1)^n\cos t]}{n\pi}.$$

Then

$$-1+x-x\cos t = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{-1+(-1)^n\cos t}{n}\sin n\pi x$$

and

$$\begin{aligned} u_t - u_{xx} &= \sum_{n=1}^{\infty}[u_n'(t)+n^2\pi^2u_n(t)]\sin n\pi x \\ &= -1+x-x\cos t = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{-1+(-1)^n\cos t}{n}\sin n\pi x. \end{aligned}$$

Equating coefficients we obtain

$$u_n'(t)+n^2\pi^2u_n(t) = \frac{2[-1+(-1)^n\cos t]}{n\pi}.$$

This is a linear first-order differential equation whose solution is

$$u_n(t) = \frac{2}{n\pi}\left[-\frac{1}{n^2\pi^2}+(-1)^n\frac{n^2\pi^2\cos t+\sin t}{n^4\pi^4+1}\right]+C_ne^{-n^2\pi^2t}.$$

Thus

$$u(x,t) = \sum_{n=1}^{\infty}\frac{2}{n\pi}\left[-\frac{1}{n^2\pi^2}+(-1)^n\frac{n^2\pi^2\cos t+\sin t}{n^4\pi^4+1}\right]\sin n\pi x + \sum_{n=1}^{\infty}C_ne^{-n^2\pi^2t}\sin n\pi x$$

and $u(x,0)=x(1-x)$ implies

$$\sum_{n=1}^{\infty}\frac{2}{n\pi}\left[-\frac{1}{n^2\pi^2}+(-1)^n\frac{n^2\pi^2}{n^4\pi^4+1}+C_n\right]\sin n\pi x = x(1-x).$$

Hence

$$\frac{2}{n\pi}\left[-\frac{1}{n^2\pi^2}+(-1)^n\frac{n^2\pi^2}{n^4\pi^4+1}+C_n\right]=\frac{2}{1}\int_0^1 x(1-x)\sin n\pi x\,dx=2\left[\frac{1-(-1)^n}{n^3\pi^3}\right]$$

and

$$C_n=\frac{4-2(-1)^n}{n^3\pi^3}-(-1)^n\frac{2n\pi}{n^4\pi^4+1}\,.$$

Therefore

$$u(x,t)=\sum_{n=1}^{\infty}\frac{2}{n\pi}\left[-\frac{1}{n^2\pi^2}+(-1)^n\frac{n^2\pi^2\cos t+\sin t}{n^4\pi^4+1}\right]\sin n\pi x$$
$$+\sum_{n=1}^{\infty}\left[\frac{4-2(-1)^n}{n^3\pi^3}-(-1)^n\frac{2n\pi}{n^4\pi^4+1}\right]e^{-n^2\pi^2t}\sin n\pi x.$$

**20.** Identifying $k=1$ and $L=\pi$ we see that the eigenfunctions of $X''+\lambda X=0$, $X(0)=0$, $X(\pi)=0$ are $\sin nx$, $n=1,2,3,\ldots$ . Assuming that $u(x,t)=\sum_{n=1}^{\infty}u_n(t)\sin nx$, the formal partial derivatives of $u$ are

$$\frac{\partial^2u}{\partial x^2}=\sum_{n=1}^{\infty}u_n(t)(-n^2)\sin nx \qquad\text{and}\qquad \frac{\partial^2u}{\partial t^2}=\sum_{n=1}^{\infty}u_n''(t)\sin nx.$$

Then

$$u_{tt}-u_{xx}=\sum_{n=1}^{\infty}[u_n''(t)+n^2u_n(t)]\sin nx=\cos t\sin x.$$

Equating coefficients, we obtain $u_1''(t)+u_1(t)\cos t$ and $u_n''(t)+n^2u_n(t)=0$ for $n=2,3,4,\ldots$. Solving the first differential equation we obtain $u_1(t)=A_1\cos t+B_1\sin t+\frac{1}{2}t\sin t$. From the second differential equation we obtain $u_n(t)=A_n\cos nt+B_n\sin nt$ for $n=2,3,4,\ldots$. Thus

$$u(x,t)=\left(A_1\cos t+B_1\sin t+\frac{1}{2}t\sin t\right)\sin x+\sum_{n=2}^{\infty}(A_n\cos nt+B_n\sin nt)\sin nx.$$

From

$$u(x,0)=A_1\sin x+\sum_{n=2}^{\infty}A_n\sin nx=0$$

we see that $A_n=0$ for $n=1,2,3,\ldots$ . Thus

$$u(x,t)=\left(B_1\sin t+\frac{1}{2}t\sin t\right)\sin x+\sum_{n=2}^{\infty}B_n\sin nt\sin nx$$

and

$$\frac{\partial u}{\partial t}=\left(B_1\cos t+\frac{1}{2}t\cos t+\frac{1}{2}\sin t\right)\sin x+\sum_{n=2}^{\infty}nB_n\cos nt\sin nx,$$

so

$$\left.\frac{\partial u}{\partial t}\right|_{t=0}=B_1\sin x+\sum_{n=2}^{\infty}nB_n\sin nx=0.$$

We see that $B_n=0$ for all $n$ so $u(x,t)=\frac{1}{2}t\sin t\sin x$.

## 12.7 Orthogonal Series Expansions

**1.** Referring to Example 1 in the text we have

$$X(x) = c_1 \cos \alpha x + c_2 \sin \alpha x \qquad \text{and} \qquad T(t) = c_3 e^{-k\alpha^2 t}.$$

From $X'(0) = 0$ (since the left end of the rod is insulated), we find $c_2 = 0$. Then $X(x) = c_1 \cos \alpha x$ and the other boundary condition $X'(1) = -hX(1)$ implies

$$-\alpha \sin \alpha + h \cos \alpha = 0 \qquad \text{or} \qquad \cot \alpha = \frac{\alpha}{h}.$$

Denoting the consecutive positive roots of this latter equation by $\alpha_n$ for $n = 1, 2, 3, \ldots$, we have

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-k\alpha_n^2 t} \cos \alpha_n x.$$

From the initial condition $u(x,0) = 1$ we obtain

$$1 = \sum_{n=1}^{\infty} A_n \cos \alpha_n x$$

and

$$A_n = \frac{\displaystyle\int_0^1 \cos \alpha_n x \, dx}{\displaystyle\int_0^1 \cos^2 \alpha_n x \, dx} = \frac{\sin \alpha_n / \alpha_n}{\frac{1}{2}\left[1 + \frac{1}{2\alpha_n} \sin 2\alpha_n\right]} = \frac{2 \sin \alpha_n}{\alpha_n \left[1 + \frac{1}{\alpha_n} \sin \alpha_n \cos \alpha_n\right]}$$

$$= \frac{2 \sin \alpha_n}{\alpha_n \left[1 + \frac{1}{h\alpha_n} \sin \alpha_n (\alpha_n \sin \alpha_n)\right]} = \frac{2h \sin \alpha_n}{\alpha_n [h + \sin^2 \alpha_n]}.$$

The solution is

$$u(x,t) = 2h \sum_{n=1}^{\infty} \frac{\sin \alpha_n}{\alpha_n (h + \sin^2 \alpha_n)} e^{-k\alpha_n^2 t} \cos \alpha_n x.$$

**2.** Substituting $u(x,t) = v(x,t) + \psi(x)$ into the partial differential equation gives

$$k \frac{\partial^2 v}{\partial x^2} + k\psi'' = \frac{\partial v}{\partial t}.$$

This equation will be homogeneous if $\psi''(x) = 0$ or $\psi(x) = c_1 x + c_2$. The boundary condition $u(0,t) = 0$ implies $\psi(0) = 0$ which implies $c_2 = 0$. Thus $\psi(x) = c_1 x$. Using the second boundary condition we obtain

$$-\left(\frac{\partial v}{\partial x} + \psi'\right)\bigg|_{x=1} = -h[v(1,t) + \psi(1) - u_0],$$

which will be homogeneous when

$$-\psi'(1) = -h\psi(1) + hu_0.$$

Since $\psi(1) = \psi'(1) = c_1$ we have $-c_1 = -hc_1 + hu_0$ and $c_1 = hu_0/(h-1)$. Thus

$$\psi(x) = \frac{hu_0}{h-1}x.$$

The new boundary-value problem is

$$k\,\frac{\partial^2 v}{\partial x^2} = \frac{\partial v}{\partial t}\,, \quad 0 < x < 1, \quad t > 0$$

$$v(0,t) = 0, \quad \left.\frac{\partial v}{\partial x}\right|_{x=1} = -hv(1,t), \quad h > 0, \quad t > 0$$

$$v(x,0) = f(x) - \frac{hu_0}{h-1}\,x, \quad 0 < x < 1.$$

Referring to Example 1 in the text we see that

$$v(x,t) = \sum_{n=1}^{\infty} A_n e^{-k\alpha_n^2 t} \sin\alpha_n x$$

and

$$u(x,t) = v(x,t) + \psi(x) = \frac{hu_0}{h-1}x + \sum_{n=1}^{\infty} A_n e^{-k\alpha_n^2 t} \sin\alpha_n x$$

where

$$f(x) - \frac{hu_0}{h-1}x = \sum_{n=1}^{\infty} A_n \sin\alpha_n x$$

and $\alpha_n$ is a solution of $\alpha_n\cos\alpha_n = -h\sin\alpha_n$. The coefficients are

$$A_n = \frac{\displaystyle\int_0^1 [f(x) - hu_0x/(h-1)]\sin\alpha_n x\,dx}{\displaystyle\int_0^1 \sin^2\alpha_n x\,dx} = \frac{\displaystyle\int_0^1 [f(x) - hu_0x/(h-1)]\sin\alpha_n x\,dx}{\frac{1}{2}\left[1 - \frac{1}{2\alpha_n}\sin 2\alpha_n\right]}$$

$$= \frac{2\displaystyle\int_0^1 [f(x) - hu_0x/(h-1)]\sin\alpha_n x\,dx}{1 - \frac{1}{\alpha_n}\sin\alpha_n\cos\alpha_n} = \frac{2\displaystyle\int_0^1 [f(x) - hu_0x/(h-1)]\sin\alpha_n x\,dx}{1 - \frac{1}{h\alpha_n}(h\sin\alpha_n)\cos\alpha_n}$$

$$= \frac{2\int_0^1 [f(x) - hu_0x/(h-1)]\sin\alpha_n x\,dx}{1 - \frac{1}{h\alpha_n}(-\alpha_n\cos\alpha_n)\cos\alpha_n} = \frac{2h}{h + \cos^2\alpha_n}\int_0^1 \left[f(x) - \frac{hu_0}{h-1}x\right]\sin\alpha_n x\,dx.$$

**3.** Separating variables in Laplace's equation gives

$$X'' + \alpha^2 X = 0$$

$$Y'' - \alpha^2 Y = 0$$

and

$$X(x) = c_1 \cos \alpha x + c_2 \sin \alpha x$$
$$Y(y) = c_3 \cosh \alpha y + c_4 \sinh \alpha y.$$

From $u(0, y) = 0$ we obtain $X(0) = 0$ and $c_1 = 0$. From $u_x(a, y) = -hu(a, y)$ we obtain $X'(a) = -hX(a)$ and

$$\alpha \cos \alpha a = -h \sin \alpha a \qquad \text{or} \qquad \tan \alpha a = -\frac{\alpha}{h}.$$

Let $\alpha_n$, where $n = 1, 2, 3, \ldots$, be the consecutive positive roots of this equation. From $u(x, 0) = 0$ we obtain $Y(0) = 0$ and $c_3 = 0$. Thus

$$u(x, y) = \sum_{n=1}^{\infty} A_n \sinh \alpha_n y \sin \alpha_n x.$$

Now

$$f(x) = \sum_{n=1}^{\infty} A_n \sinh \alpha_n b \sin \alpha_n x$$

and

$$A_n \sinh \alpha_n b = \frac{\displaystyle\int_0^a f(x) \sin \alpha_n x \, dx}{\int_0^a \sin^2 \alpha_n x \, dx}.$$

Since

$$\begin{aligned}\int_0^a \sin^2 \alpha_n x \, dx &= \frac{1}{2}\left[a - \frac{1}{2\alpha_n}\sin 2\alpha_n a\right] = \frac{1}{2}\left[a - \frac{1}{\alpha_n}\sin \alpha_n a \cos \alpha_n a\right] \\ &= \frac{1}{2}\left[a - \frac{1}{h\alpha_n}(h \sin \alpha_n a)\cos \alpha_n a\right] \\ &= \frac{1}{2}\left[a - \frac{1}{h\alpha_n}(-\alpha_n \cos \alpha_n a)\cos \alpha_n a\right] = \frac{1}{2h}\left[ah + \cos^2 \alpha_n a\right],\end{aligned}$$

we have

$$A_n = \frac{2h}{\sinh \alpha_n b[ah + \cos^2 \alpha_n a]} \int_0^a f(x) \sin \alpha_n x \, dx.$$

**4.** Letting $u(x, y) = X(x)Y(y)$ and separating variables gives

$$X''Y + XY'' = 0.$$

The boundary conditions

$$\left.\frac{\partial u}{\partial y}\right|_{y=0} = 0 \qquad \text{and} \qquad \left.\frac{\partial u}{\partial y}\right|_{y=1} = -hu(x, 1)$$

correspond to

$$X(x)Y'(0) = 0 \qquad \text{and} \qquad X(x)Y'(1) = -hX(x)Y(1)$$

or

$$Y'(0) = 0 \qquad \text{and} \qquad Y'(1) = -hY(1).$$

Since these homogeneous boundary conditions are in terms of $Y$, we separate the differential equation as

$$\frac{X''}{X} = -\frac{Y''}{Y} = \alpha^2.$$

Then

$$Y'' + \alpha^2 Y = 0$$

and

$$X'' - \alpha^2 X = 0$$

have solutions

$$Y(y) = c_1 \cos \alpha y + c_2 \sin \alpha y$$

and

$$X(x) = c_3 e^{-\alpha x} + c_4 e^{\alpha x}.$$

We use exponential functions in the solution of $X(x)$ since the interval over which $X$ is defined is infinite. (See the informal rule given in Section 11.5 of the text that discusses when to use the exponential form and when to use the hyperbolic form of the solution of $y'' - \alpha^2 y = 0$.) Now, $Y'(0) = 0$ implies $c_2 = 0$, so $Y(y) = c_1 \cos \alpha y$. Since $Y'(y) = -c_1 \alpha \sin \alpha y$, the boundary condition $Y'(1) = -hY(1)$ implies

$$-c_1 \alpha \sin \alpha = -hc_1 \cos \alpha \qquad \text{or} \qquad \cot \alpha = \frac{\alpha}{h}\,.$$

Consideration of the graphs of $f(\alpha) = \cot \alpha$ and $g(\alpha) = \alpha/h$ show that $\cos \alpha = \alpha h$ has an infinite number of roots. The consecutive positive roots $\alpha_n$ for $n = 1, 2, 3, \ldots$ , are the eigenvalues of the problem. The corresponding eigenfunctions are $Y_n(y) = c_1 \cos \alpha_n y$. The condition $\lim_{x\to\infty} u(x, y) = 0$ is equivalent to $\lim_{x\to\infty} X(x) = 0$. Thus $c_4 = 0$ and $X(x) = c_3 e^{-\alpha x}$. Therefore

$$u_n(x, y) = X_n(x)Y_n(x) = A_n e^{-\alpha_n x} \cos \alpha_n y$$

and by the superposition principle

$$u(x, y) = \sum_{n=1}^{\infty} A_n e^{-\alpha_n x} \cos \alpha_n y.$$

[It is easily shown that there are no eigenvalues corresponding to $\alpha = 0$.] Finally, the condition $u(0, y) = u_0$ implies

$$u_0 = \sum_{n=1}^{\infty} A_n \cos \alpha_n y.$$

This is not a Fourier cosine series since the coefficients $\alpha_n$ of $y$ are not integer multiples of $\pi/p$, where $p = 1$ in this problem. The functions $\cos\alpha_n y$ are however orthogonal since they are eigenfunctions of the Sturm-Lionville problem

$$Y'' + \alpha^2 Y = 0,$$

$$Y'(0) = 0$$

$$Y'(1) + hY(1) = 0,$$

with weight function $p(x) = 1$. Thus we find

$$A_n = \frac{\displaystyle\int_0^1 u_0 \cos\alpha_n y\, dy}{\displaystyle\int_0^1 \cos^2\alpha_n y\, dy}.$$

Now

$$\int_0^1 u_0 \cos\alpha_n y\, dy = \frac{u_0}{\alpha_n}\sin\alpha_n y\bigg|_0^1 = \frac{u_0}{\alpha_n}\sin\alpha_n$$

and

$$\int_0^1 \cos^2\alpha_n y\, dy = \frac{1}{2}\int_0^1 (1+\cos 2\alpha_n y)\, dy = \frac{1}{2}\left[y + \frac{1}{2\alpha_n}\sin 2\alpha_n y\right]_0^1$$

$$= \frac{1}{2}\left[1 + \frac{1}{2\alpha_n}\sin 2\alpha_n\right] = \frac{1}{2}\left[1 + \frac{1}{\alpha_n}\sin\alpha_n\cos\alpha_n\right].$$

Since $\cot\alpha = \alpha/h$,

$$\frac{\cos\alpha}{\alpha} = \frac{\sin\alpha}{h}$$

and

$$\int_0^1 \cos^2\alpha_n y\, dy = \frac{1}{2}\left[1 + \frac{\sin^2\alpha_n}{h}\right].$$

Then

$$A_n = \frac{\frac{u_0}{\alpha_n}\sin\alpha_n}{\frac{1}{2}\left[1+\frac{1}{h}\sin^2\alpha_n\right]} = \frac{2hu_0\sin\alpha_n}{\alpha_n\left(h+\sin^2\alpha_n\right)}$$

and

$$u(x,y) = 2hu_0\sum_{n=1}^{\infty}\frac{\sin\alpha_n}{\alpha_n\left(h+\sin^2\alpha_n\right)}e^{-\alpha_n x}\cos\alpha_n y$$

where $\alpha_n$ for $n = 1, 2, 3, \ldots$ are the consecutive positive roots of $\cot\alpha = \alpha/h$.

**5.** The boundary-value problem is

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad 0 < x < L, \quad t > 0,$$

$$u(0,t) = 0, \quad \left.\frac{\partial u}{\partial x}\right|_{x=L} = 0, \quad t > 0,$$

$$u(x,0) = f(x), \quad 0 < x < L.$$

Separation of variables leads to

$$X'' + \alpha^2 X = 0$$
$$T' + k\alpha^2 T = 0$$

and

$$X(x) = c_1 \cos \alpha x + c_2 \sin \alpha x$$
$$T(t) = c_3 e^{-k\alpha^2 t}.$$

From $X(0) = 0$ we find $c_1 = 0$. From $X'(L) = 0$ we obtain $\cos \alpha L = 0$ and

$$\alpha = \frac{\pi(2n-1)}{2L}, \ n = 1, 2, 3, \ldots \ .$$

Thus

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-k(2n-1)^2\pi^2 t/4L^2} \sin\left(\frac{2n-1}{2L}\right)\pi x$$

where

$$A_n = \frac{\displaystyle\int_0^L f(x) \sin\left(\frac{2n-1}{2L}\right)\pi x\, dx}{\displaystyle\int_0^L \sin^2\left(\frac{2n-1}{2L}\right)\pi x\, dx} = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{2n-1}{2L}\right)\pi x\, dx.$$

**6.** Substituting $u(x,t) = v(x,t) + \psi(x)$ into the partial differential equation gives

$$a^2 \frac{\partial^2 v}{\partial x^2} + \psi''(x) = \frac{\partial^2 v}{\partial t^2}.$$

This equation will be homogeneous if $\psi''(x) = 0$ or $\psi(x) = c_1 x + c_2$. The boundary condition $u(0,t) = 0$ implies $\psi(0) = 0$ which implies $c_2 = 0$. Thus $\psi(x) = c_1 x$. Using the second boundary condition, we obtain

$$E\left(\frac{\partial v}{\partial x} + \psi'\right)\bigg|_{x=L} = F_0,$$

which will be homogeneous when

$$E\psi'(L) = F_0.$$

Since $\psi'(x) = c_1$ we conclude that $c_1 = F_0/E$ and

$$\psi(x) = \frac{F_0}{E}x.$$

The new boundary-value problem is

$$a^2\frac{\partial^2 v}{\partial x^2}=\frac{\partial^2 v}{\partial t^2}\,,\quad 0<x<L,\quad t>0$$

$$v(0,t)=0,\quad \left.\frac{\partial v}{\partial x}\right|_{x=L}=0,\quad t>0,$$

$$v(x,0)=-\frac{F_0}{E}x,\quad \left.\frac{\partial v}{\partial t}\right|_{t=0}=0,\quad 0<x<L.$$

Referring to Example 2 in the text we see that

$$v(x,t)=\sum_{n=1}^{\infty}A_n\cos a\left(\frac{2n-1}{2L}\right)\pi t\,\sin\left(\frac{2n-1}{2L}\right)\pi x$$

where

$$-\frac{F_0}{E}x=\sum_{n=1}^{\infty}A_n\sin\left(\frac{2n-1}{2L}\right)\pi x$$

and

$$A_n=\frac{-F_0\displaystyle\int_0^L x\sin\left(\frac{2n-1}{2L}\right)\pi x\,dx}{E\displaystyle\int_0^L\sin^2\left(\frac{2n-1}{2L}\right)\pi x\,dx}=\frac{8F_0L(-1)^n}{E\pi^2(2n-1)^2}\,.$$

Thus

$$u(x,t)=v(x,t)+\psi(x)=\frac{F_0}{E}x+\frac{8F_0L}{E\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^2}\cos a\left(\frac{2n-1}{2L}\right)\pi t\,\sin\left(\frac{2n-1}{2L}\right)\pi x.$$

**7.** Separation of variables leads to

$$Y''+\alpha^2Y=0$$

$$X''-\alpha^2X=0$$

and

$$Y(y)=c_1\cos\alpha y+c_2\sin\alpha y$$

$$X(x)=c_3\cosh\alpha x+c_4\sinh\alpha x.$$

From $Y(0)=0$ we find $c_1=0$. From $Y'(1)=0$ we obtain $\cos\alpha=0$ and

$$\alpha=\frac{\pi(2n-1)}{2}\,,\ n=1,2,3,\dots\ .$$

Thus

$$Y(y)=c_2\sin\left(\frac{2n-1}{2}\right)\pi y.$$

From $X'(0)=0$ we find $c_4=0$. Then

$$u(x,y)=\sum_{n=1}^{\infty}A_n\cosh\left(\frac{2n-1}{2}\right)\pi x\,\sin\left(\frac{2n-1}{2}\right)\pi y$$

where

$$u_0=u(1,y)=\sum_{n=1}^{\infty}A_n\cosh\left(\frac{2n-1}{2}\right)\pi\,\sin\left(\frac{2n-1}{2}\right)\pi y$$

and

$$A_n\cosh\left(\frac{2n-1}{2}\right)\pi=\frac{\displaystyle\int_0^1 u_0\sin\left(\frac{2n-1}{2}\right)\pi y\,dy}{\displaystyle\int_0^1\sin^2\left(\frac{2n-1}{2}\right)\pi y\,dy}=\frac{4u_0}{(2n-1)\pi}\,.$$

Thus

$$u(x,y)=\frac{4u_0}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)\cosh\left(\frac{2n-1}{2}\right)\pi}\cosh\left(\frac{2n-1}{2}\right)\pi x\,\sin\left(\frac{2n-1}{2}\right)\pi y.$$

**8.** Using $u=XT$ and separation of variables leads to

$$X''+\lambda X=0$$

$$T'+(1+\lambda)T=0$$

with

$$X'(0)=0\quad\text{and}\quad X(1)=0$$

This is a regular Sturm-Liouville problem. For $\lambda=0$ we have $X''=0$ or $X=c_1+c_2x$. From $X'(0)=0$ we find $c_2=0$, so $X=c_1$. From $X(1)=0$ we find $c_1=0$. Thus $X=0$.

For $\lambda=\alpha^2$ the general solution is $X=c_1\cos a\alpha x+c_2\sin\alpha x$. From $X'(0)=0$ we find $c_2=0$, so $X=c_1\cos\alpha x$. From $X(1)=0$ we find $\cos\alpha=0$ or $\alpha=\left(\frac{2n-1}{2}\right)\pi$ for $n=1,\,2,\,3,\,\ldots\,.$ Therefore

$$X=c_1\cos\left(\frac{2n-1}{2}\right)\pi x\qquad\text{and}\qquad T=c_3e^{-\left[(2n-1)^2\pi^2t/4+1\right]}$$

Thus

$$u(x,t)=e^{-t}\sum_{n=1}^{\infty}A_ne^{-(2n-1)^2\pi^2t/4}\cos\left(\frac{2n-1}{2}\right)\pi x.$$

where

$$A_n=\frac{\displaystyle\int_0^1(1-x^2)\cos\left(\frac{2n-1}{2}\right)\pi x\,dx}{\displaystyle\int_0^1\cos^2\left(\frac{2n-1}{2}\right)\pi x\,dx}=\frac{\dfrac{-16\cos n\pi}{(2n-1)^3\pi^3}}{1/2}=\frac{-32(-1)^n}{(2n-1)^3\pi^3}.$$

**9.** The boundary-value problem is

$$k\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t}\,, \quad 0<x<1, \quad t>0$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=0}=hu(0,t), \quad \left.\frac{\partial u}{\partial x}\right|_{x=1}=-hu(1,t), \quad h>0, \quad t>0,$$

$$u(x,0)=f(x), \quad 0<x<1.$$

Referring to Example 1 in the text we have

$$X(x)=c_1\cos\alpha x+c_2\sin\alpha x \qquad \text{and} \qquad T(t)=c_3e^{-k\alpha^2 t}.$$

Applying the boundary conditions, we obtain

$$X'(0)=hX(0) \quad \text{and} \quad X'(1)=-hX(1)$$

or

$$\alpha c_2=hc_1$$

$$-\alpha c_1\sin\alpha+\alpha c_2\cos\alpha=-hc_1\cos\alpha-hc_2\sin\alpha.$$

Choosing $c_1=\alpha$ and $c_2=h$ (to satisfy the first equation above) we obtain

$$-\alpha^2\sin\alpha+h\alpha\cos\alpha=-h\alpha\cos\alpha-h^2\sin\alpha$$

$$2h\alpha\cos\alpha=(\alpha^2-h^2)\sin\alpha.$$

The eigenvalues $\alpha_n$ are the consecutive positive roots of

$$\tan\alpha=\frac{2h\alpha}{\alpha^2-h^2}\,.$$

Then

$$u(x,t)=\sum_{n=1}^{\infty}A_ne^{-k\alpha_n^2 t}(\alpha_n\cos\alpha_nx+h\sin\alpha_nx)$$

where

$$f(x)=u(x,0)=\sum_{n=1}^{\infty}A_n(\alpha_n\cos\alpha_nx+h\sin\alpha_nx)$$

and

$$A_n=\frac{\displaystyle\int_0^1 f(x)(\alpha_n\cos\alpha_nx+h\sin\alpha_nx)\,dx}{\displaystyle\int_0^1(\alpha_n\cos\alpha_nx+h\sin\alpha_nx)^2\,dx}$$

$$=\frac{2}{\alpha_n^2+2h+h^2}\int_0^1 f(x)(\alpha_n\cos\alpha_nx+h\sin\alpha_nx)\,dx.$$

[Note: the evaluation and simplification of the integral in the denominator requires the use of the relationship $(\alpha^2-h^2)\sin\alpha=2h\alpha\cos\alpha$.]

**10.** The eigenfunctions of the associated homogeneous boundary-value problem are $\sin \alpha_n x$, $n = 1, 2, 3, \ldots$, where the $\alpha_n$ are the consecutive positive roots of $\tan \alpha = -\alpha$. We assume that

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) \sin \alpha_n x \qquad \text{and} \qquad xe^{-2t} = \sum_{n=1}^{\infty} F_n(t) \sin \alpha_n x.$$

Then

$$F_n(t) = \frac{e^{-2t} \displaystyle\int_0^1 x \sin \alpha_n x \, dx}{\displaystyle\int_0^1 \sin^2 \alpha_n x \, dx}.$$

Since $\alpha_n \cos \alpha_n = -\sin \alpha_n$ and

$$\int_0^1 \sin^2 \alpha_n x \, dx = \frac{1}{2}\left[1 - \frac{1}{2\alpha_n} \sin 2\alpha_n\right],$$

we have

$$e^{-2t} \int_0^1 x \sin \alpha_n x \, dx = e^{-2t}\left(\frac{\sin \alpha_n - \alpha_n \cos \alpha_n}{\alpha_n^2}\right) = \frac{2 \sin \alpha_n}{\alpha_n^2} e^{-2t}$$

$$\int_0^1 \sin^2 \alpha_n x \, dx = \frac{1}{2}[1 + \cos^2 \alpha_n]$$

and so

$$F_n(t) = \frac{4 \sin \alpha_n}{\alpha_n^2 (1 + \cos^2 \alpha_n)} \, e^{-2t}.$$

Substituting the assumptions into $u_t - ku_{xx} = xe^{-2t}$ and equating coefficients leads to the linear first-order differential equation

$$u_n'(t) + k\alpha_n^2 u(t) = \frac{4 \sin \alpha_n}{\alpha_n^2 (1 + \cos^2 \alpha_n)} \, e^{-2t}$$

whose solution is

$$u_n(t) = \frac{4 \sin \alpha_n}{\alpha_n^2 (1 + \cos^2 \alpha_n)(k\alpha_n^2 - 2)} \, e^{-2t} + C_n e^{-k\alpha_n^2 t}.$$

From

$$u(x,t) = \sum_{n=1}^{\infty} \left[\frac{4 \sin \alpha_n}{\alpha_n^2 (1 + \cos^2 \alpha_n)(k\alpha_n^2 - 2)} \, e^{-2t} + C_n e^{-k\alpha_n^2 t}\right] \sin \alpha_n x$$

and the initial condition $u(x,0) = 0$ we see

$$C_n = -\frac{4 \sin \alpha_n}{\alpha_n^2 (1 + \cos^2 \alpha_n)(k\alpha_n^2 - 2)}.$$

The formal solution of the original problem is then

$$u(x,t) = \sum_{n=1}^{\infty} \frac{4 \sin \alpha_n}{\alpha_n^2 (1 + \cos^2 \alpha_n)(k\alpha_n^2 - 2)} (e^{-2t} - e^{-k\alpha_n^2 t}) \sin \alpha_n x.$$

**11.** Using $u = XT$ and separation constant $-\lambda = \alpha^4$ we find

$$X^{(4)} - \alpha^4 X = 0 \qquad \text{and} \qquad X(x) = c_1 \cos \alpha x + c_2 \sin \alpha x + c_3 \cosh \alpha x + c_4 \sinh \alpha x.$$

Since $u = XT$ the boundary conditions become

$$X(0) = 0, \quad X'(0) = 0, \quad X''(1) = 0, \quad X'''(1) = 0.$$

Now $X(0) = 0$ implies $c_1 + c_3 = 0$, while $X'(0) = 0$ implies $c_2 + c_4 = 0$. Thus

$$X(x) = c_1 \cos \alpha x + c_2 \sin \alpha x - c_1 \cosh \alpha x - c_2 \sinh \alpha x.$$

The boundary condition $X''(1) = 0$ implies

$$-c_1 \cos \alpha - c_2 \sin \alpha - c_1 \cosh \alpha - c_2 \sinh \alpha = 0$$

while the boundary condition $X'''(1) = 0$ implies

$$c_1 \sin \alpha - c_2 \cos \alpha - c_1 \sinh \alpha - c_2 \cosh \alpha = 0.$$

We then have the system of two equations in two unknowns

$$(\cos \alpha + \cosh \alpha)c_1 + (\sin \alpha + \sinh \alpha)c_2 = 0$$

$$(\sin \alpha - \sinh \alpha)c_1 - (\cos \alpha + \cosh \alpha)c_2 = 0.$$

This homogeneous system will have nontrivial solutions for $c_1$ and $c_2$ provided

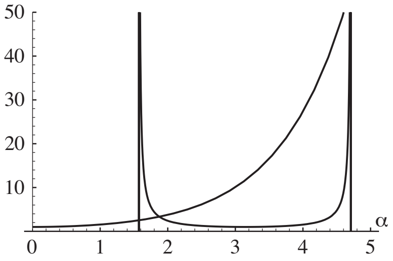
$$\begin{vmatrix} \cos \alpha + \cosh \alpha & \sin \alpha + \sinh \alpha \\ \sin \alpha - \sinh \alpha & -\cos \alpha - \cosh \alpha \end{vmatrix} = 0$$

or

$$-2 - 2 \cos \alpha \cosh \alpha = 0.$$

Thus, the eigenvalues are determined by the equation $\cos \alpha \cosh \alpha = -1$.

Using a computer to graph $\cosh \alpha$ and $-1/\cos \alpha = -\sec \alpha$ we see that the first two positive eigenvalues occur near 1.9 and 4.7. Applying Newton's method with these initial values we find that the eigenvalues are $\alpha_1 = 1.8751$ and $\alpha_2 = 4.6941$.



**12.** **(a)** In this case the boundary conditions are

$$u(0,t) = 0, \qquad \left.\frac{\partial u}{\partial x}\right|_{x=0} = 0$$

$$u(1,t) = 0, \qquad \left.\frac{\partial u}{\partial x}\right|_{x=1} = 0.$$

Separating variables leads to

$$X(x) = c_1 \cos \alpha x + c_2 \sin \alpha x + c_3 \cosh \alpha x + c_4 \sinh \alpha x$$

subject to

$$X(0) = 0, \quad X'(0) = 0, \quad X(1) = 0, \qquad \text{and} \qquad X'(1) = 0.$$

Now $X(0) = 0$ implies $c_1 + c_3 = 0$ while $X'(0) = 0$ implies $c_2 + c_4 = 0$. Thus

$$X(x) = c_1 \cos \alpha x + c_2 \sin \alpha x - c_1 \cosh \alpha x - c_2 \sinh \alpha x.$$

The boundary condition $X(1) = 0$ implies

$$c_1 \cos \alpha + c_2 \sin \alpha - c_1 \cosh \alpha - c_2 \sinh \alpha = 0$$

while the boundary condition $X'(1) = 0$ implies

$$-c_1 \sin \alpha + c_2 \cos \alpha - c_1 \sinh \alpha - c_2 \cosh \alpha = 0.$$

We then have the system of two equations in two unknowns

$$\begin{aligned}(\cos \alpha - \cosh \alpha)c_1 + (\sin \alpha - \sinh \alpha)c_2 &= 0\\ -(\sin \alpha + \sinh \alpha)c_1 + (\cos \alpha - \cosh \alpha)c_2 &= 0.\end{aligned}$$

This homogeneous system will have nontrivial solutions for $c_1$ and $c_2$ provided

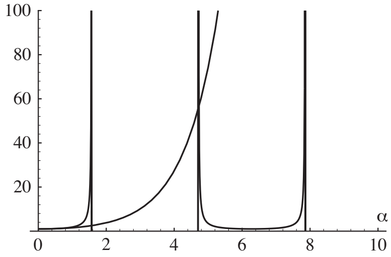
$$\begin{vmatrix} \cos \alpha - \cosh \alpha & \sin \alpha - \sinh \alpha \\ -\sin \alpha - \sinh \alpha & \cos \alpha - \cosh \alpha \end{vmatrix} = 0$$

or

$$2 - 2 \cos \alpha \cosh \alpha = 0.$$

Thus, the eigenvalues are determined by the equation $\cos \alpha \cosh \alpha = 1$.

**(b)** Using a computer to graph $\cosh \alpha$ and $1/\cos \alpha = \sec \alpha$ we see that the first two positive eigenvalues occur near the vertical asymptotes of $\sec \alpha$, at $3\pi/2$ and $5\pi/2$. Applying Newton's method with these initial values we find that the eigenvalues are $\alpha_1 = 4.7300$ and $\alpha_2 = 7.8532$.



## 12.8 Higher Dimensional Problems

**1.** This boundary-value problem was solved in Example 1 in the text. Identifying $b = c = \pi$ and $f(x, y) = u_0$ we have

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}e^{-k(m^2+n^2)t}\sin mx \sin ny$$

where

$$\begin{aligned} A_{mn} &= \frac{4}{\pi^2}\int_0^\pi\int_0^\pi u_0 \sin mx \sin ny\,dx\,dy = \frac{4u_0}{\pi^2}\int_0^\pi \sin mx\,dx\int_0^\pi \sin ny\,dy \\ &= \frac{4u_0}{mn\pi^2}[1-(-1)^m][1-(-1)^n]. \end{aligned}$$

**2.** As shown in Example 1 in the text, separation of variables leads to

$$X(x) = c_1\cos\alpha x + c_2\sin\alpha x$$

$$Y(y) = c_3\cos\beta y + c_4\sin\beta y$$

and

$$T(t) + c_5e^{-k(\alpha^2+\beta^2)t}.$$

The boundary conditions

$$\left.\begin{aligned} u_x(0,y,t) = 0, \quad u_x(1,y,t) = 0 \\ u_y(x,0,t) = 0, \quad u_y(x,1,t) = 0 \end{aligned}\right\} \quad \text{imply} \quad \left\{\begin{aligned} X'(0) = 0, \quad X'(1) = 0 \\ Y'(0) = 0, \quad Y'(1) = 0 \end{aligned}\right.$$

Applying these conditions to

$$X'(x) = -\alpha c_1\sin\alpha x + \alpha c_2\cos\alpha x$$

and

$$Y'(y) = -\beta c_3\sin\beta y + \beta c_4\cos\beta y$$

gives $c_2 = c_4 = 0$ and $\sin\alpha = \sin\beta = 0$. Then

$$\alpha = m\pi,\ m = 0,1,2,\ldots \qquad \text{and} \qquad \beta = n\pi,\ n = 0,1,2,\ldots\ .$$

By the superposition principle

$$\begin{aligned} u(x,y,t) = A_{00} &+ \sum_{m=1}^{\infty} A_{m0}e^{-km^2\pi^2t}\cos m\pi x + \sum_{n=1}^{\infty} A_{0n}e^{-kn^2\pi^2t}\cos n\pi y \\ &+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}e^{-k(m^2+n^2)\pi^2t}\cos m\pi x\cos n\pi y. \end{aligned}$$

We now compute the coefficients of the double cosine series: Identifying $b = c = 1$ and $f(x,y) = xy$ we have

$$A_{00} = \int_0^1 \int_0^1 xy\,dx\,dy = \int_0^1 \frac{1}{2}x^2y\Big|_0^1\,dy = \frac{1}{2}\int_0^1 y\,dy = \frac{1}{4}\,,$$

$$\begin{aligned} A_{m0} &= 2\int_0^1 \int_0^1 xy\cos m\pi x\,dx\,dy = 2\int_0^1 \frac{1}{m^2\pi^2}(\cos m\pi x + m\pi x\sin m\pi x)\Big|_0^1\,y\,dy \\ &= 2\int_0^1 \frac{\cos m\pi - 1}{m^2\pi^2}y\,dy = \frac{\cos m\pi - 1}{m^2\pi^2} = \frac{(-1)^m - 1}{m^2\pi^2}\,, \\ A_{0n} &= 2\int_0^1 \int_0^1 xy\cos n\pi y\,dx\,dy = \frac{(-1)^n - 1}{n^2\pi^2}\,, \end{aligned}$$

and

$$\begin{aligned} A_{mn} &= 4\int_0^1 \int_0^1 xy\cos m\pi x\cos n\pi y\,dx\,dy = 4\int_0^1 x\cos m\pi x\,dx \int_0^1 y\cos n\pi y\,dy \\ &= 4\left(\frac{(-1)^m - 1}{m^2\pi^2}\right)\left(\frac{(-1)^n - 1}{n^2\pi^2}\right). \end{aligned}$$

*In Problems 3 and 4 we need to solve the partial differential equation*

$$a^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = \frac{\partial^2 u}{\partial t^2}\,.$$

*To separate this equation we try $u(x,y,t) = X(x)Y(y)T(t)$:*

$$a^2(X''YT + XY''T) = XYT''$$

$$\frac{X''}{X} = -\frac{Y''}{Y} + \frac{T''}{a^2T} = -\alpha^2.$$

*Then*

$$X'' + \alpha^2 X = 0 \qquad (\mathbf{1})$$

$$\frac{Y''}{Y} = \frac{T''}{a^2T} + \alpha^2 = -\beta^2$$

$$Y'' + \beta^2 Y = 0 \qquad (\mathbf{2})$$

$$T'' + a^2\left(\alpha^2 + \beta^2\right)T = 0. \qquad (\mathbf{3})$$

*The general solutions of equations* (**1**), (**2**), *and* (**3**) *are, respectively,*

$$\begin{aligned}X(x) &= c_1\cos\alpha x + c_2\sin\alpha x\\ Y(y) &= c_3\cos\beta y + c_4\sin\beta y\\ T(t) &= c_5\cos a\sqrt{\alpha^2+\beta^2}\,t + c_6\sin a\sqrt{\alpha^2+\beta^2}\,t.\end{aligned}$$

**3.** The conditions $X(0)=0$ and $Y(0)=0$ give $c_1=0$ and $c_3=0$. The conditions $X(\pi)=0$ and $Y(\pi)=0$ yield two sets of eigenvalues:

$$\alpha=m,\ \ m=1,2,3,\dots \qquad \text{and} \qquad \beta=n,\ \ n=1,2,3,\dots\ .$$

A product solution of the partial differential equation that satisfies the boundary conditions is

$$u_{mn}(x,y,t)=\left(A_{mn}\cos a\sqrt{m^2+n^2}\,t+B_{mn}\sin a\sqrt{m^2+n^2}\,t\right)\sin mx\sin ny.$$

To satisfy the initial conditions we use the superposition principle:

$$u(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(A_{mn}\cos a\sqrt{m^2+n^2}\,t+B_{mn}\sin a\sqrt{m^2+n^2}\,t\right)\sin mx\sin ny.$$

The initial condition $u_t(x,y,0)=0$ implies $B_{mn}=0$ and

$$u(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}\cos a\sqrt{m^2+n^2}\,t\sin mx\sin ny.$$

At $t=0$ we have

$$xy(x-\pi)(y-\pi)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}\sin mx\sin ny.$$

Using (12) and (13) in the text, it follows that

$$\begin{aligned}A_{mn}&=\frac{4}{\pi^2}\int_0^{\pi}\int_0^{\pi}xy(x-\pi)(y-\pi)\sin mx\sin ny\,dx\,dy\\ &=\frac{4}{\pi^2}\int_0^{\pi}x(x-\pi)\sin mx\,dx\int_0^{\pi}y(y-\pi)\sin ny\,dy\\ &=\frac{16}{m^3n^3\pi^2}[(-1)^m-1][(-1)^n-1].\end{aligned}$$

**4.** The conditions $X(0)=0$ and $Y(0)=0$ give $c_1=0$ and $c_3=0$. The conditions $X(b)=0$ and $Y(c)=0$ yield two sets of eigenvalues

$$\alpha=m\pi/b,\ \ m=1,\ 2,\ 3,\ \dots \qquad \text{and} \qquad \beta=n\pi/c,\ \ n=1,\ 2,\ 3,\ \dots\ .$$

A product solution of the partial differential equation that satisfies the boundary conditions is

$$u_{mn}(x,y,t)=(A_{mn}\cos a\omega_{mn}t+B_{mn}\sin a\omega_{mn}t)\sin\left(\frac{m\pi}{b}x\right)\sin\left(\frac{n\pi}{c}y\right),$$

where $\omega_{mn} = \sqrt{(m\pi/b)^2 + (n\pi/c)^2}$ . To satisfy the initial conditions we use the superposition principle:

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(A_{mn}\cos a\omega_{mn}t + B_{mn}\sin a\omega_{mn}t)\sin\left(\frac{m\pi}{b}\,x\right)\sin\left(\frac{n\pi}{c}\,y\right).$$

At $t = 0$ we have

$$f(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}\sin\left(\frac{m\pi}{b}\,x\right)\sin\left(\frac{n\pi}{c}\,y\right)$$

and

$$g(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}B_{mn}a\omega_{mn}\sin\left(\frac{m\pi}{b}\,x\right)\sin\left(\frac{n\pi}{c}\,y\right).$$

Using (12) and (13) in the text, it follows that

$$A_{mn} = \frac{4}{bc}\int_0^c\int_0^b f(x,y)\sin\left(\frac{m\pi}{b}\,x\right)\sin\left(n\pi c\, y\right)dx\,dy$$

$$B_{mn} = \frac{4}{abc\,\omega_{mn}}\int_0^c\int_0^b g(x,y)\sin\left(\frac{m\pi}{b}\,x\right)\sin\left(\frac{n\pi}{c}\,y\right)dx\,dy.$$

*In Problems 5 and 6 we try $u(x,y,z) = X(x)Y(y)Z(z)$ to separate Laplace's equation in three dimensions:*

$$X''YZ + XY''Z + XYZ'' = 0$$

$$\frac{X''}{X} = -\frac{Y''}{Y} - \frac{Z''}{Z} = -\alpha^2.$$

*Then*

$$X'' + \alpha^2 X = 0 \tag{4}$$

$$\frac{Y''}{Y} = -\frac{Z''}{Z} + \alpha^2 = -\beta^2$$

$$Y'' + \beta^2 Y = 0 \tag{5}$$

$$Z'' - (\alpha^2 + \beta^2)Z = 0. \tag{6}$$

*The general solutions of equations* (**4**), (**5**), *and* (**6**) *are, respectively*

$$\begin{aligned}X(x) &= c_1\cos\alpha x + c_2\sin\alpha x\\ Y(y) &= c_3\cos\beta y + c_4\sin\beta y\\ Z(z) &= c_5\cosh\sqrt{\alpha^2+\beta^2}\,z + c_6\sinh\sqrt{\alpha^2+\beta^2}\,z.\end{aligned}$$

**5.** The boundary and initial conditions are

$$\begin{aligned} u(0,y,z) &= 0, \quad u(a,y,z) = 0 \\ u(x,0,z) &= 0, \quad u(x,b,z) = 0 \\ u(x,y,0) &= 0, \quad u(x,y,c) = f(x,y). \end{aligned}$$

The conditions $X(0) = Y(0) = Z(0) = 0$ give $c_1 = c_3 = c_5 = 0$. The conditions $X(a) = 0$ and $Y(b) = 0$ yield two sets of eigenvalues:

$$\alpha = \frac{m\pi}{a}, \ m = 1, 2, 3, \ldots \qquad \text{and} \qquad \beta = \frac{n\pi}{b}, \ n = 1, 2, 3, \ldots .$$

By the superposition principle

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sinh \omega_{mn} z \sin \frac{m\pi}{a} x \sin \frac{n\pi}{b} y$$

where

$$\omega_{mn}^2 = \frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2}$$

and

$$A_{mn} = \frac{4}{ab \sinh \omega_{mn} c} \int_0^b \int_0^a f(x,y) \sin \frac{m\pi}{a} x \sin \frac{n\pi}{b} y \, dx \, dy.$$

**6.** The boundary and initial conditions are

$$\begin{aligned} u(0,y,z) &= 0, & u(a,y,z) &= 0, \\ u(x,0,z) &= 0, & u(x,b,z) &= 0, \\ u(x,y,0) &= f(x,y), & u(x,y,c) &= 0 \end{aligned}$$

The conditions $X(0) = Y(0) = 0$ give $c_1 = c_3 = 0$. The conditions $X(a) = Y(b) = 0$ yield two sets of eigenvalues:

$$\alpha = \frac{m\pi}{a}, \ m = 1, 2, 3, \ldots \qquad \text{and} \qquad \beta = \frac{n\pi}{b}, \ n = 1, 2, 3, \ldots .$$

Let

$$\omega_{mn}^2 = \frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2}.$$

Then the boundary condition $Z(c) = 0$ gives

$$c_5 \cosh c\,\omega_{mn} + c_6 \sinh c\,\omega_{mn} = 0$$

from which we obtain

$$\begin{aligned} Z(z) &= c_5 \left( \cosh w_{mn} z - \frac{\cosh c\,\omega_{mn}}{\sinh c\,\omega_{mn}} \sinh \omega z \right) \\ &= \frac{c_5}{\sinh c\,\omega_{mn}} (\sinh c\,\omega_{mn} \cosh \omega_{mn} z - \cosh c\,\omega_{mn} \sinh \omega_{mn} z) = c_{mn} \sinh \omega_{mn}(c - z). \end{aligned}$$

By the superposition principle

$$u(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}\sinh\omega_{mn}(c-z)\sin\frac{m\pi}{a}x\sin\frac{n\pi}{b}y$$

where

$$A_{mn}=\frac{4}{ab\sinh c\,\omega_{mn}}\int_0^b\int_0^a f(x,y)\sin\frac{m\pi}{a}x\sin\frac{n\pi}{b}y\,dx\,dy.$$

## Chapter 12 in Review

**1.** Letting $u(x,y)=X(x)+Y(y)$ we have $X'Y'=XY$ and

$$\frac{X'}{X}=YY'=-\lambda.$$

If $\lambda=0$ then $X'=0$ and $X(x)=c_1$. Also $Y(y)=0$ so $u=0$.

If $\lambda\neq 0$ then $X'+\lambda X=0$ and $Y+(1/\lambda)Y=0$. Thus $X(x)=c_1e^{-\lambda x}$ and $Y(y)=c_2e^{-y/\lambda}$ so

$$u(x,y)=Ae^{(-\lambda x-y/\lambda)}.$$

**2.** Letting $u=XY$ we have $X''Y+XY''+2X'Y+2XY'=0$ so that $(X''+2X')Y+X(Y''+2Y')=0$. Separating variables and using the separation constant $-\lambda$ we obtain

$$\frac{X''+2X'}{-X}=\frac{Y''+2Y'}{Y}=-\lambda$$

so that

$$X''+2X'-\lambda X=0 \qquad\text{and}\qquad Y''+2Y'+\lambda Y=0.$$

The corresponding auxiliary equations are $m^2+2m-\lambda=0$ and $m^2+2m+\lambda$ with solutions $m=-1\pm\sqrt{1+\lambda}$ and $m=-1\pm\sqrt{1-\lambda}$, respectively. We consider five cases:

**I.** $\lambda=-1$: In this case $X=c_1e^{-x}+c_2xe^{-x}$ and $Y=c_3e^{(-1+\sqrt{2})y}+c_4e^{(-1-\sqrt{2})y}$ so that

$$u=(c_1e^{x}+c_2xe^{-x})\left(c_3e^{\left(-1+\sqrt{2}\right)y}+c_4e^{\left(-1-\sqrt{2}\right)y}\right).$$

**II.** $\lambda=1$: In this case $X=c_5e^{\left(-1+\sqrt{2}\right)x}+c_6e^{\left(-1-\sqrt{2}\right)y}$ and $Y=c_7e^{-y}+c_8ye^{-y}$ so that

$$u=\left(c_5e^{\left(-1+\sqrt{2}\right)x}+c_6e^{\left(-1-\sqrt{2}\right)x}\right)\left(c_7e^{-y}+c_8ye^{-y}\right).$$

**III.** $-1<\lambda<1$: Here both $1+\lambda$ and $1-\lambda$ are positive so

$$u=\left(c_9e^{\left(-1+\sqrt{1+\lambda}\right)x}+c_{10}e^{\left(-1-\sqrt{1+\lambda}\right)x}\right)\left(c_{11}e^{\left(-1+\sqrt{1-\lambda}\right)y}+c_{12}e^{\left(-1-\sqrt{1-\lambda}\right)y}\right).$$

**IV.** $\lambda < -1$: Here $1 + \lambda < 0$ and $1 - \lambda > 0$ so

$$u = e^{-x}\left(c_{13}\cos\sqrt{-1-\lambda}\,x + c_{14}\sin\sqrt{-1-\lambda}\,x\right) + \left(c_{15}e^{(-1+\sqrt{1-\lambda})y} + c_{16}e^{(-1-\sqrt{1-\lambda})y}\right).$$

**V.** $\lambda > 1$: Here $1 + \lambda > 0$ and $1 - \lambda < 0$ so

$$u = \left(c_{17}e^{(-1+\sqrt{1+\lambda})x} + c_{18}e^{(-1-\sqrt{1+\lambda})x}\right) + e^{-x}\left(c_{19}\cos\sqrt{\lambda-1}\,y + c_{20}\sin\sqrt{\lambda-1}\,y\right).$$

We see from the above that it is not possible to choose $\lambda$ so that both $X$ and $Y$ are oscillatory.

**3.** Substituting $u(x,t) = v(x,t) + \psi(x)$ into the partial differential equation we obtain

$$k\,\frac{\partial^2 v}{\partial x^2} + k\psi''(x) = \frac{\partial v}{\partial t}.$$

This equation will be homogeneous provided $\psi$ satisfies

$$k\psi'' = 0 \qquad \text{or} \qquad \psi = c_1 x + c_2.$$

Considering

$$u(0,t) = v(0,t) + \psi(0) = u_0$$

we set $\psi(0) = u_0$ so that $\psi(x) = c_1 x + u_0$. Now

$$-\left.\frac{\partial u}{\partial x}\right|_{x=\pi} = -\left.\frac{\partial v}{\partial x}\right|_{x=\pi} - \psi'(x) = v(\pi,t) + \psi(\pi) - u_1$$

is equivalent to

$$\left.\frac{\partial v}{\partial x}\right|_{x=\pi} + v(\pi,t) = u_1 - \psi'(x) - \psi(\pi) = u_1 - c_1 - (c_1\pi + u_0),$$

which will be homogeneous when

$$u_1 - c_1 - c_1\pi - u_0 = 0 \qquad \text{or} \qquad c_1 = \frac{u_1 - u_0}{1+\pi}.$$

The steady-state solution is

$$\psi(x) = \left(\frac{u_1 - u_0}{1+\pi}\right)x + u_0.$$

**4.** The solution of the problem represents the heat of a thin rod of length $\pi$. The left boundary $x = 0$ is kept at constant temperature $u_0$ for $t > 0$. Heat is lost from the right end of the rod by being in contact with a medium that is held at constant temperature $u_1$. Moreover, intially the entire rod is at temperature zero.

**5.** The boundary-value problem is

$$a^2\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial t^2},\quad 0<x<1,\quad t>0,$$

$$u\,(0,t)=0,\quad u=(1,t)=0,\quad t>0,$$

$$u\,(x,0)=0,\quad \left.\frac{\partial u}{\partial t}\right|_{t=0}=g(x),\quad 0<x<1.$$

From Section 12.4 in the text we see that $A_n=0$,

$$B_n=\frac{2}{n\pi a}\iint_0^1 g\,(x)\sin n\pi\,dx=\frac{2}{n\pi a}\iint_{1/4}^{3/4} h\sin n\pi x\,dx$$

$$=\frac{2h}{n\pi a}\left(-\frac{1}{n\pi}\cos n\pi x\right)\Bigg|_{1/4}^{3/4}=\frac{2h}{n^2\pi^2 a}\left(\cos\frac{n\pi}{4}-\cos\frac{3n\pi}{4}\right)$$

and

$$u\,(x,t)=\sum_{n=1}^{\infty}B_n\sin n\pi at\,\sin n\pi x.$$

**6.** The boundary-value problem is

$$\frac{\partial^2 u}{\partial x^2}+x^2=\frac{\partial^2 u}{\partial t^2},\quad 0<x<1,\quad t>0,$$

$$u(0,t)=1,\quad u(1,t)=0,\quad t>0,$$

$$u(x,0)=f(x),\quad u_t(x,0)=0,\quad 0<x<1.$$

Substituting $u(x,t)=v(x,t)+\psi(x)$ into the partial differential equation gives

$$\frac{\partial^2 v}{\partial x^2}+\psi''(x)+x^2=\frac{\partial^2 v}{\partial t^2}\,.$$

This equation will be homogeneous provided $\psi''(x)+x^2=0$ or

$$\psi(x)=-\frac{1}{12}x^4+c_1x+c_2.$$

From $\psi(0)=1$ and $\psi(1)=0$ we obtain $c_1=-11/12$ and $c_2=1$. The new problem is

$$\frac{\partial^2 v}{\partial x^2}=\frac{\partial^2 v}{\partial t^2}\,,\quad 0<x<1,\quad t>0,$$

$$v(0,t)=0,\quad v(1,t)=0,\quad t>0,$$

$$v(x,0)=f(x)-\psi(x),\quad v_t(x,0)=0,\quad 0<x<1.$$

From Section 12.4 in the text we see that $B_n = 0$,

$$A_n = 2\int_0^1 [f(x) - \psi(x)]\sin n\pi x\,dx = 2\int_0^1 \left[f(x) + \frac{1}{12}x^4 + \frac{11}{12}x - 1\right]\sin n\pi x\,dx,$$

and

$$v(x,t) = \sum_{n=1}^{\infty} A_n \cos n\pi t \sin n\pi x.$$

Thus

$$u(x,t) = v(x,t) + \psi(x) = -\frac{1}{12}x^4 - \frac{11}{12}x + 1 + \sum_{n=1}^{\infty} A_n \cos n\pi t \sin n\pi x.$$

**7.** Using $u = XY$ and $-\lambda$ as a separation constant leads to

$$X'' - \lambda X = 0,$$
$$X(0) = 0,$$

and

$$Y'' + \lambda Y = 0,$$
$$Y(0) = 0,$$
$$Y(\pi) = 0.$$

This leads to

$$Y = c_4 \sin ny \qquad \text{and} \qquad X = c_2 \sinh nx$$

for $n = 1, 2, 3, \ldots$ so that

$$u = \sum_{n=1}^{\infty} A_n \sinh nx \sin ny.$$

Imposing

$$u(\pi, y) = 50 = \sum_{n=1}^{\infty} A_n \sinh n\pi \sin ny$$

gives

$$A_n = \frac{100}{n\pi}\,\frac{1 - (-1)^n}{\sinh n\pi}$$

so that

$$u(x,y) = \frac{100}{\pi}\sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n \sinh n\pi} \sinh nx \sin ny.$$

**8.** Using $u = XY$ and $-\lambda$ as a separation constant leads to

$$X'' - \lambda X = 0,$$

and

$$\begin{aligned} Y'' + \lambda Y &= 0, \\ Y'(0) &= 0, \\ Y'(\pi) &= 0. \end{aligned}$$

This leads to

$$Y = c_3 \cos ny \quad \text{and} \quad X = c_2 e^{-nx}$$

for $n = 1, 2, 3, \ldots$. In this problem we also have $\lambda = 0$ is an eigenvalue with corresponding eigenfunctions 1 and 1. Thus

$$u = A_0 + \sum_{n=1}^{\infty} A_n e^{-nx} \cos ny.$$

Imposing

$$u(0, y) = 50 = A_0 + \sum_{n=1}^{\infty} A_n \cos ny$$

gives

$$A_0 = \frac{1}{\pi} \int_0^{\pi} 50 \, dy = 50$$

and

$$A_n \frac{2}{\pi} \int_0^{\pi} 50 \cos ny \, dy = 0$$

for $n = 1, 2, 3, \ldots$ so that

$$u(x, y) = 50.$$

**9.** Using $u = XY$ and $-\lambda$ as a separation constant leads to

$$X'' - \lambda X = 0,$$

and

$$\begin{aligned} Y'' + \lambda Y &= 0, \\ Y(0) &= 0, \\ Y(\pi) &= 0. \end{aligned}$$

Then

$$X = c_1 e^{nx} + c_2 e^{-nx} \qquad \text{and} \qquad Y = c_3 \cos ny + c_4 \sin ny$$

for $n = 1, 2, 3, \ldots$. Since $u$ must be bounded as $x \to \infty$, we define $c_1 = 0$. Also $Y(0) = 0$ implies $c_3 = 0$ so

$$u = \sum_{n=1}^{\infty} A_n e^{-nx} \sin ny.$$

Imposing

$$u(0,y) = 50 = \sum_{n=1}^{\infty} A_n \sin ny$$

gives

$$A_n = \frac{2}{\pi}\int_0^{\pi} 50 \sin ny\, dy = \frac{100}{n\pi}[1-(-1)^n]$$

so that

$$u(x,y) = \sum_{n=1}^{\infty} \frac{100}{n\pi}[1-(-1)^n]e^{-nx}\sin ny.$$

**10.** The boundary-value problem is

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad -L < x < L, \quad t > 0,$$

$$u(-L,t) = 0, \quad u(L,t) = 0, \quad t > 0,$$

$$u(x,0) = u_0, \quad -L < x < L.$$

Referring to Section 12.3 in the text we have

$$X(x) = c_1 \cos ax + c_2 \sin ax$$

and

$$T(t) = c_3 e^{-ka^2 t}.$$

Using the boundary conditions $u(-L,0) = X(-L)T(0) = 0$ and $u(L,0) = X(L)T(0) = 0$ we obtain $X(-L) = 0$ and $X(L) = 0$. Thus

$$c_1 \cos(-\alpha L) + c_2 \sin(-\alpha L) = 0$$

$$c_1 \cos \alpha L + c_2 \sin \alpha L = 0$$

or

$$c_1 \cos \alpha L - c_2 \sin \alpha L = 0$$

$$c_1 \cos \alpha L + c_2 \sin \alpha L = 0.$$

Adding, we find $\cos \alpha L = 0$ which gives the eigenvalues

$$\alpha = \frac{2n-1}{2L}\pi, \quad n = 1, 2, 3, \ldots$$

Thus

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-[(2n-1)\pi/2L]^2 kt} \cos\left(\frac{2n-1}{2L}\right)\pi x.$$

From

$$u(x,0) = u_0 = \sum_{n=1}^{\infty} A_n \cos\left(\frac{2n-1}{2L}\right)\pi x$$

we find

$$A_n = \frac{2\int_0^L u_0 \cos\left(\frac{2n-1}{2L}\right)\pi x\,dx}{2\int_0^L \cos^2\left(\frac{2n-1}{2L}\right)\pi x\,dx} = \frac{u_0\,(-1)^{n+1}\,2L/\pi\,(2n-1)}{L/2} = \frac{4u_0\,(-1)^{n+1}}{\pi\,(2n-1)}.$$

**11. (a)** The coefficients of the series

$$u\,(x,0) = \sum_{n=1}^{\infty} B_n \sin nx$$

are

$$B_n = frac2\pi \iint\limits_0^\pi \sin x \sin nx\,dx = \frac{2}{\pi}\iint\limits_0^\pi \frac{1}{2}\left[\cos\,(1-n)\,x - \cos\,(1+n)\,x\right]\,dx$$

$$= \frac{1}{\pi}\left[\left.\frac{\sin\,(1-n)\,x}{1-n}\right|_0^\pi - \left.\frac{\sin\,(1+n)\,x}{1+n}\right|_0^\pi\right] = 0 \text{ for } n \neq 1.$$

For $n = 1$,

$$B_1 = \frac{2}{\pi}\iint\limits_0^\pi \sin^2 x\,dx = \frac{1}{\pi}\iint\limits_0^\pi (1 - \cos 2x)\,dx = 1.$$

Thus

$$u\,(x,t) = \sum_{n=1}^{\infty} B_n e^{-n^2 t} \sin nx$$

reduces to $u\,(x,t) = e^{-t}\sin x$ for $n = 1$.

**(b)** This is like part (a), but in this case, for $n \neq 3$ and $n \neq 5$,

$$B_n = \frac{2}{\pi}\iint\limits_0^\pi (100\sin 3x - 30\sin 5x)\sin nx\,dx = 0;$$

while $B_3 = 100$ and $B_5 = -30$. Therefore

$$u\,(x,t) = 100e^{-9t}\sin 3x - 30e^{-25t}\sin 5x.$$

**12.** Substituting $u(x,t) = v(x,t) + \psi(x)$ into the partial differential equation results in $\psi'' = -\sin x$ and $\psi(x) = c_1 x + c_2 + \sin x$. The boundary conditions $\psi(0) = 400$ and $\psi(\pi) = 200$ imply $c_1 = -200/\pi$ and $c_2 = 400$ so

$$\psi(x) = -\frac{200}{\pi}x + 400 + \sin x.$$

Solving

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial v}{\partial t}, \qquad 0 < x < \pi, \quad t > 0$$

$$v(0,t) = 0, \qquad v(\pi,t) = 0, \quad t > 0$$

$$v(x,0) = 400 + \sin x - \left(-\frac{200}{\pi}x + 400 + \sin x\right) = \frac{200}{\pi}x, \qquad 0 < x < \pi$$

using separation of variables with separation constant $-\lambda$, where $\lambda = \alpha^2$, gives

$$X'' + \alpha^2 X = 0 \qquad \text{and} \qquad T' + \alpha^2 T = 0.$$

Using $X(0) = 0$ and $X(\pi) = 0$ we determine $\alpha^2 = n^2$, $X(x) = c_2 \sin nx$, and $T(t) = c_3 e^{-n^2 t}$. Then

$$v(x,t) = \sum_{n=1}^{\infty} A_n e^{-n^2 t} \sin nx$$

and

$$v(x,0) = \frac{200}{\pi}x = \sum_{n=1}^{\infty} A_n \sin nx$$

so

$$A_n = \frac{400}{\pi^2}\int_0^{\pi} x \sin nx \, dx = \frac{400}{n\pi}(-1)^{n+1}.$$

Thus

$$u(x,t) = -\frac{200}{\pi}x + 400 + \sin x + \frac{400}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} e^{-n^2 t} \sin nx.$$

**13.** Using $u = XT$ and $-\lambda$, where $\lambda = \alpha^2$, as a separation constant we find

$$X'' + 2X' + \alpha^2 X = 0 \quad \text{and} \quad T'' + 2T' + \left(1 + \alpha^2\right) T = 0.$$

Thus for $\alpha > 1$

$$\begin{aligned} X &= c_1 e^{-x} \cos \sqrt{\alpha^2 - 1}\, x + c_2 e^{-x} \sin \sqrt{\alpha^2 - 1}\, x \\ T &= c_3 e^{-t} \cos \alpha t + c_4 e^{-t} \sin \alpha t. \end{aligned}$$

For $0 \le \alpha \le 1$ we only obtain $X = 0$. Now the boundary conditions $X\left(0\right) = 0$ and $X\left(\pi\right) = 0$ give, in turn, $c_1 = 0$ and $\sqrt{\alpha^2 - 1}\pi = n\pi$ or $\alpha^2 = n^2 = n^2{+}1$, $n = 1, 2, 3, \ldots$. The corresponding solutions are $X = c_2 e^{-x} \sin nx$. The initial condition $T'\left(0\right) = 0$ implies $c_3 = \alpha c_4$ and so

$$T = c_4 e^{-t}\left[\sqrt{n^2+1}\cos\sqrt{n^2+1}\, t + \sin\sqrt{n^2+1}\, t\right].$$

Using $u = XT$ and the superposition principle, a formal series solution is

$$u\left(x,t\right) = e^{-(x+t)} \sum_{n=1}^{\infty} A_n \left[\sqrt{n^2+1}\cos\sqrt{n^2+1}\, t + \sin\sqrt{n^2+1}\, t\right] \sin nx.$$

**14.** Letting $c = XT$ and separating variables we obtain

$$\frac{kX'' - hX'}{X} = \frac{T'}{T} \qquad \text{or} \qquad \frac{X'' - aX'}{X} = \frac{T'}{kT} = -\lambda$$

where $a = h/k$. Setting $\lambda = \alpha^2$ leads to the separated differential equations

$$X'' - aX' + \alpha^2 X = 0 \qquad \text{and} \qquad T' + k\alpha^2 T = 0.$$

The solution of the second equation is

$$T(t) = c_3 e^{-k\alpha^2 t}.$$

For the first equation we have $m = \frac{1}{2}\left(a \pm \sqrt{a^2 - 4\alpha^2}\right)$, and we consider three cases using the boundary conditions $X(0) = X(1) = 0$:

$\boxed{a^2 > 4\alpha^2}$ The solution is $X = c_1 e^{m_1 x} + c_2 e^{m_2 x}$, where the boundary conditions imply $c_1 = c_2 = 0$, so $X = 0$. (Note in this case that if $\alpha = 0$, the solution is $X = c_1 + c_2 e^{ax}$ and the boundary conditions again imply $c_1 = c_2 = 0$, so $X = 0$.)

$\boxed{a^2 = 4\alpha^2}$ The solution is $X = c_1 e^{m_1 x} + c_2 x e^{m_1 x}$, where the boundary conditions imply $c_1 = c_2 = 0$, so $X = 0$.

$\boxed{a^2 < 4\alpha^2}$ The solution is

$$X(x) = c_1 e^{ax/2} \cos \frac{\sqrt{4\alpha^2 - a^2}}{2} x + c_2 e^{ax/2} \sin \frac{\sqrt{4\alpha^2 - a^2}}{2} x.$$

From $X(0) = 0$ we see that $c_1 = 0$. From $X(1) = 0$ we find

$$\frac{1}{2}\sqrt{4\alpha^2 - a^2} = n\pi \qquad \text{or} \qquad \alpha^2 = \frac{1}{4}\left(4n^2\pi^2 + a^2\right).$$

Thus

$$X(x) = c_2 e^{ax/2} \sin n\pi x,$$

and

$$c(x,t) = \sum_{n=1}^{\infty} A_n e^{ax/2} e^{-k\left(4n^2\pi^2 + a^2\right)t/4} \sin n\pi x.$$

The initial condition $c(x,0) = c_0$ implies

$$c_0 = \sum_{n=1}^{\infty} A_n e^{ax/2} \sin n\pi x.$$

From the self-adjoint form

$$\frac{d}{dx}\left[e^{-ax} X'\right] + \alpha^2 e^{-ax} X = 0$$

the eigenfunctions are orthogonal on $[0,1]$ with weight function $e^{-ax}$. That is

$$\iint\limits_0^1 e^{-ax}\left(e^{ax/2}\sin n\pi x\right)\left(e^{ax/2}\sin m\pi x\right)\,dx = 0, \quad n \neq m.$$

Multiplying (**1**) by $e^{-ax}e^{ax/2}\sin m\pi x$ and integrating we obtain

$$\iint\limits_0^1 c_0 e^{-ax}e^{ax/2}\sin m\pi x,\,dx = \sum_{n=1}^{\infty} A_n \iint\limits_0^1 e^{-ax}e^{ax/2}\left(\sin m\pi x\right)e^{ax/2}\sin n\pi x\,dx$$

$$c_0\int_0^1 e^{-ax/2}\sin n\pi x\,dx = A_n\int_0^1 \sin^2 n\pi x\,dx = \frac{1}{2}A_n$$

and

$$A_n = 2c_0\int_0^1 e^{-ax/2}\sin n\pi x\,dx = \frac{4c_0\left[2e^{a/2}n\pi - 2n\pi\left(-1\right)^n\right]}{e^{a/2}\left(a^2+4n^2\pi^2\right)} = \frac{8n\pi c_0\left[e^{a/2}-\left(-1\right)^n\right]}{e^{a/2}\left(a^2+4n^2\pi^2\right)}.$$

**15.** By the discussion in Section 12.6, we assume a solution in the form $u(x,t) = v(x,t) + \psi(x)$. Force this into the differential equation to get $\psi''(x) = 0$ together with the conditions $\psi(0) = 0$ and $\psi'(1) + \psi(1) = u_1$.

Integrating twice and imposing the two conditions leads to $\psi(x) = \frac{1}{2}(u_1 - u_0)x + u_0$. Now

$$\begin{gathered}
u(x,t) = v(x,t) + \psi(x) \\
u(0,t) = v(0,t) + \psi(0) = u_0 \\
v(0,t) + \left[\frac{1}{2}(u_1-u_0)(0) + u_0\right] = u_0 \\
v(0,t) = 0
\end{gathered}$$

Similarly we find that

$$\begin{gathered}
u_x(x,t) = v_x(x,t) + \psi'(x) \\
u_x(1,t) = v_x(1,t) + \psi'(1) = u_1 - u(1,t) \\
v_x(1,t) + \left[\frac{1}{2}(u_1-u_0)\right] = u_1 - [v(1,t) + \psi(1)] \\
v_x(1,t) = u_1 - [v(1,t) + \psi(1)] - \frac{1}{2}(u_1-u_0) \\
v_x(1,t) = -v(1,t)
\end{gathered}$$

So now we solve

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial v}{\partial t}, \quad 0 < x < 1,\ t > 0$$

$$v(0,t) = 0, \quad \left.\frac{\partial v}{\partial x}\right|_{x=1} = -v(1,t),\ t > 0$$

$$v(x,0) = \frac{1}{2}(u_0 - u_1)x,\ 0 < x < 1$$

Using the results of Example 1 from Section 12.7, $v(x,t) = \sum_{n=1}^{\infty} A_n e^{-\alpha_n^2 t} \sin \alpha_n x$ and the coefficients are given by

$$A_n = \frac{\frac{1}{2}(u_0 - u_1)\int_0^1 x \sin \alpha_n x\, dx}{\int_0^1 \sin^2 \alpha_n x\, dx} = \frac{1}{2}(u_0 - u_1)\frac{\dfrac{\sin \alpha_n - \alpha_n \cos \alpha_n}{\alpha_n^2}}{\frac{1}{2}\left(1 + \cos^2 \alpha_n\right)}$$

$$= (u_0 - u_1)\frac{\sin \alpha_n - \alpha_n \cos \alpha_n}{\alpha_n^2 \left(1 + \cos^2 \alpha_n\right)}.$$

Since $\tan \alpha_n = -\alpha_n$ we have $\sin \alpha_n = -\alpha_n \cos \alpha_n$ and so

$$A_n = 2(u_1 - u_0)\sum_{n=1}^{\infty} \frac{\cos \alpha_n}{\alpha_n \left(1 + \cos^2 \alpha_n\right)} e^{-\alpha_n^2 t} \sin\left(\alpha_n x\right)$$

and

$$u(x,t) = u_0 + \frac{1}{2}(u_1 - u_0)x + 2(u_1 - u_0)\sum_{n=1}^{\infty} \frac{\cos \alpha_n}{\alpha_n \left(1 + \cos^2 \alpha_n\right)} e^{-\alpha_n^2 t} \sin\left(\alpha_n x\right)$$

**16.** This is similar Problem 13 in Exercises 12.5 with $a = b = \pi$ and $f(x) = g(x) = x^2 - \pi x$. Thus

$$u(x,y) = \sum_{n=1}^{\infty} \left(A_n \cosh ny + B_n \sinh ny\right) \sin nx.$$

At $y = 0$ we then have

$$f(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{a} x$$

and consequently

$$A_n = \frac{2}{\pi}\int_0^{\pi} \left(x^2 - \pi x\right) \sin nx\, dx = \frac{4\left[(-1)^n - 1\right]}{\pi n^3}$$

At $y = \pi$,

$$g(x) = \sum_{n=1}^{\infty} \left(A_n \cosh n\pi + B_n \sinh n\pi\right) \sin nx$$

indicates that the entire expression in the parentheses is given by

$$A_n \cosh n\pi + B_n \sinh n\pi = \frac{2}{\pi}\int_0^{\pi} \left(x^2 - \pi x\right) \sin nx\, dx.$$

We can now solve for $B_n$:

$$B_n = \frac{1}{\sinh n\pi}\,\frac{2}{\pi}\int_0^{\pi} \left(x^2 - \pi x\right) \sin nx\, dx - \frac{\cosh n\pi}{\sinh n\pi} A_n$$

$$= \frac{4\left[(-1)^n - 1\right]}{\pi n^3 \sinh n\pi}\left(1 - \cosh n\pi\right) = A_n \frac{1 - \cosh n\pi}{\sinh n\pi}$$

Then

$$u(x,y)=\sum_{n=1}^{\infty}\frac{4\left[(-1)^n-1\right]}{\pi n^3}\left(\cosh ny+\frac{1-\cosh n\pi}{\sinh n\pi}\sinh ny\right)\sin nx$$

$$=\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{[(-1)^n-1]}{n^3}\left(\frac{\sinh n(\pi-y)+\sinh ny}{\sinh n\pi}\right)\sin nx.$$

**17.** Subtituting $u(x,y)=v(x,y)+\psi(x)$ into the boundary-value problem yields

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0.$$

The condition $\psi''(x)=-2$ has the general solution $\psi(x)=-x^2+c_1x+c_2$. The boundary conditions $x=0$ and $x=\pi$ will be homogeneous if we choose $\psi(0)=0$. The condition $\psi(\pi)=0$ gives $c_2=0$ and $c_1=\pi$. Thus

$$\psi(\pi)=-x^2+\pi x.$$

The boundary-value problem in $v$ is then

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0,\quad 0<x<\pi,\quad 0<y<\pi$$

$$v(0,y)=0,\quad v(\pi,y)=0,\quad 0<y<\pi$$

$$v(x,0)=-\psi(x),\quad v(x,\pi)=-\psi(x),\quad 0<y<\pi.$$

From Problem 16,

$$v(x,y)=\sum_{n=1}^{\infty}(A_n\cosh ny+B_n\sinh ny)\sin nx$$

$$=\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{[(-1)^n-1]}{n^3}\left(\frac{\sinh n(\pi-y)+\sinh ny}{\sinh n\pi}\right)\sin nx.$$

Therefore

$$u(x,y)=\psi(x)+v(x,y)=-x^2+\pi x+\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{[(-1)^n-1]}{n^3}\left(\frac{\sinh n(\pi-y)+\sinh ny}{\sinh n\pi}\right)\sin nx.$$

**18.** Using the subtitution $u(x,t)=v(x,t)+\psi(x)$ into the boundary-value problem yields

$$\frac{\partial^2 v}{\partial x^2}+\psi''+e^{-x}=\frac{\partial v}{\partial t}.$$

Now $u(0,t)=v(0,t)+\psi(0)=0$ and $u_x(\pi,t)=v_x(\pi,t)+\psi'(\pi)=0$. The condition $\psi(0)=0$ gives $-1+c_1=0$ or $c_1=1$. The condition $\psi'(\pi)=0$ gives $e^{-\pi}+c_2=0$ or $c_2=-e^{-\pi}$. Thus,

$$\psi(x)=-e^{-x}+1-e^{-\pi}x.$$

Now the homogeneous boundary-value problem for $v(x,t)$ is

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial v}{\partial t}, \quad 0 < x < \pi, \quad t > 0$$

$$v(0,t) = 0, \quad \left.\frac{\partial v}{\partial x}\right|_{x=\pi}, \quad t > 0$$

$$v(x,0) = f(x) + e^{-x} - 1 + e^{-\pi}x, \quad 0 < x < \pi.$$

Using $v = XT$ and separation of variables leads to

$$X'' + \lambda X = 0$$

$$T' + \lambda T = 0$$

with

$$X(0) = 0 \quad \text{and} \quad X'(\pi) = 0$$

This is a regular Sturm-Liouville problem. For $\lambda = \alpha^2$ the general solution is $X = c_3 \cos \alpha x + c_4 \sin \alpha x$. From $X(0) = 0$ we find $c_3 = 0$, so $X = c_4 \sin \alpha x$. From $X'(\pi) = 0$ we find $\cos \alpha\pi = 0$ or $\alpha\pi = \left(\frac{2n-1}{2}\right)\pi$ for $n = 1, 2, 3, \ldots$ . Therefore

$$X = c_1 \sin\left(\frac{2n-1}{2}\right)x \qquad \text{and} \qquad T = c_5 e^{-(2n-1)^2 t/4}$$

Thus

$$v(x,t) = \sum_{n=1}^{\infty} A_n e^{-(2n-1)^2 t/4} \sin\left(\frac{2n-1}{2}\right)x.$$

When $t = 0$,

$$f(x) + e^{-x} - 1 + e^{-\pi}x = \sum_{n=1}^{\infty} A_n \sin\left(\frac{2n-1}{2}\right)x$$

where

$$A_n = \frac{\displaystyle\int_0^{\pi} \left(f(x) + e^{-x} - 1 + e^{-\pi}x\right) \sin\left(\frac{2n-1}{2}\right)x\,dx}{\displaystyle\int_0^{\pi} \sin^2\left(\frac{2n-1}{2}\right)x}.$$

$$= \frac{2}{\pi}\int_0^{\pi} \left(f(x) + e^{-x} - 1 + e^{-\pi}x\right) \sin\left(\frac{2n-1}{2}\right)x\,dx.$$

The solutions is then

$$u(x,t) = -e^{-x} + 1 - e^{-\pi}x + \sum_{n=1}^{\infty} A_n e^{-(2n-1)^2 t/4} \sin\left(\frac{2n-1}{2}\right)x$$

where the coefficients $A_n$ are defined above.

**19.** For $w(x,y) = C\sin\dfrac{\pi x}{a}\sin\dfrac{\pi y}{b}$,

$$\frac{\partial^4 w}{\partial x^4} = C\left(\frac{\pi}{a}\right)^4 \sin\frac{\pi x}{a}\sin\frac{\pi y}{b}$$

$$\frac{\partial^4 w}{\partial x^2\partial y^2} = C\left(\frac{\pi}{a}\right)^2\left(\frac{\pi}{b}\right)^2 \sin\frac{\pi x}{a}\sin\frac{\pi y}{b}$$

$$\frac{\partial^4 w}{\partial y^4} = C\left(\frac{\pi}{b}\right)^4 \sin\frac{\pi x}{a}\sin\frac{\pi y}{b}$$

Thus

$$\begin{aligned}\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2\partial y^2} + \frac{\partial^4 w}{\partial y^4} &= C\left[\left(\frac{\pi}{a}\right)^4 + 2\left(\frac{\pi}{a}\right)^2\left(\frac{\pi}{b}\right)^2 + \left(\frac{\pi}{b}\right)^4\right]\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}\\ &= \frac{q_0}{D}\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}\end{aligned}$$

Hence

$$C = \frac{q_0/D}{\pi^4\left(\dfrac{1}{a^2}+\dfrac{1}{b^2}\right)^2} = \frac{q_0 a^4 b^4}{\pi^4 D\left(a^2+b^2\right)^2}$$

Therefore

$$w(x,y) = \frac{q_0 a^4 b^4}{\pi^4 D\left(a^2+b^2\right)^2}\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}.$$

**20.** Since $\sin 0 = 0$ and $\sin\pi = 0$, the first four boundary conditions are obvious:

$$w(0,y) = 0, \qquad w(a,y) = 0, \qquad w(x,0) = 0 \qquad w(x,b) = 0.$$

Also, from

$$\frac{\partial^2 w}{\partial x^2} = \frac{q_0 a^2 b^4}{\pi^4 D\left(a^2+b^2\right)^2}\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}$$

$$\frac{\partial^2 w}{\partial x^2} = \frac{q_0 a^4 b^2}{\pi^4 D\left(a^2+b^2\right)^2}\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}$$

and $\sin 0 = 0$ and $\sin\pi = 0$ we see that

$$\left.\frac{\partial^2 w}{\partial x^2}\right|_{x=0} = 0, \qquad \left.\frac{\partial^2 w}{\partial x^2}\right|_{x=a} = 0, \qquad \left.\frac{\partial^2 w}{\partial y^2}\right|_{y=0} = 0, \qquad \left.\frac{\partial^2 w}{\partial y^2}\right|_{y=b} = 0.$$